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ISOMORPHISMS AND AUTOMORPHISMS OF WITT RINGS

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ABSTRACT. For a field F, $\mathrm{char}(F) \neq 2$, let WF denote the Witt ring of quadratic forms of F and let $\langle F^* \rangle \subseteq WF$ denote the multiplicative group of 1-dimensional forms $\langle a \rangle$, $a \in F^*$. It follows from a construction of D. K. Harrison that if E, F are fields (both of characteristic $\neq 2$) and $\rho: WE \rightarrow WF$ is a ring isomorphism, then there exists a ring isomorphism $\bar{\rho}: WE \rightarrow WF$ which "preserves dimension" in the sense that $\bar{\rho}\langle E^* \rangle = \langle F^* \rangle$. In this paper, the relationship between ρ and $\bar{\rho}$ is clarified.

1. **Preliminaries.** Let R be an (abstract) Witt ring in the terminology of [6] and let G denote the distinguished group of units of R. For example, one could take $R = WF$ where F is some field, $\mathrm{char}(F) \neq 2$. In this case, $G = \langle F^* \rangle$.

One needs to know something of the structure of the full unit group R^*. If $u \in R^*$ then u decomposes uniquely as $u = a(1 + x)$ where $a \in G$ and $x \in I^2$. Here $I \subseteq R$ denotes the fundamental ideal. $a = d_{\pm}(u)$, the signed discriminant of u. Thus, it is enough to consider units of the form $u = 1 + x$, $x \in I^2$. Computing signatures this yields $\pm 1 = \sigma(u) = 1 + \sigma(x) \equiv 1 \pmod 4$ so $\sigma(x) = 0$ for all signatures σ of R. By Pfister's local-global principle, this implies x is nilpotent (i.e., 2-primary torsion). Conversely, if x is nilpotent then, from general ring theory, $1 + x$ is a unit.

For almost everything done here, the above will suffice. However to obtain certain refinements it is necessary to know the relationship between the additive order of x and the multiplicative order of $1 + x$. The first half of this is fairly easy:

1.1. PROPOSITION. *If* $x \in I$ *and* $2^k x = 0$ *then* $(1 + x)^{2^k} = 1$.

PROOF. $(1 + x)^{2^k} = (1 + 2x + x^2)^{2^{k-1}} = (1 + y)^{2^{k-1}}$ where $y = 2x + x^2$. By the Annihilator Theorem for Pfister forms, $x = \sum_i (1 - s_i)t_i$ where $t_i \in R$ and $s_i \in D(\langle 1, 1 \rangle^k)$. Here, $D(q)$ denotes the value set of the quadratic form q. Thus

$$\begin{aligned} x^2 &= \textstyle\sum_i (1 - s_i)^2 t_i^2 + \sum_{i \neq j} (1 - s_i)(1 - s_j)t_i t_j \\ &= \textstyle\sum_i 2(1 - s_i)t_i^2 + \sum_{i<j} 2(1 - s_i)(1 - s_j)t_i t_j. \end{aligned}$$

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Since $2^k(1 - s_i) = 0$, it follows that $2^{k-1}y = 0$. By induction on k this implies $(1 + y)^{2^{k-1}} = 1$.

The second half follows from the theory of logarithms and exponentials developed in [5]. This does not seem to have any simple proof:

1.2. PROPOSITION. *If* $x \in I^2$ *and* $(1 + x)^{2^k} = 1$ *then* $2^k x = 0$.

PROOF. See [5].

2. **Homomorphisms.** R can be described as the quotient of the integral group ring $\mathbf{Z}[G]$ obtained by factoring by the ideal generated by $1 + (-1)$ and all elements $(1 - a)(1 - b)$ where $a, b \in G$ satisfy $1 \in D\langle a, b\rangle$.

Let $\bar{R}$ be another Witt ring and let $\bar{G}$ be its distinguished group of units. From the presentation of R as a quotient of $\mathbf{Z}[G]$, specifying a (ring) homomorphism $\rho:R \to \bar{R}$ is equivalent to specifying a group homomorphism $\rho:G \to \bar{R}^*$ satisfying:

(1) $\rho(-1) = -1$ and

(2) $\forall a, b \in G, 1 \in D\langle a, b\rangle \Rightarrow (1 - \rho(a))(1 - \rho(b)) = 0$.

Since G may not be a ring invariant, one should not expect $\rho(G) \subseteq \bar{G}$ to hold in general. ρ will be referred to as a *scheme homomorphism* if $\rho(G) \subseteq \bar{G}$. For scheme homomorphisms, condition (2) can be replaced by the equivalent condition:

(2′) $\forall a, b \in G, 1 \in D\langle a, b\rangle \Rightarrow 1 \in \bar{D}\langle \rho(a), \rho(b)\rangle$.

2.1. PROPOSITION. *If either* $G = \{\pm 1\}$ *or* $\bar{I}^2$ *is torsion free then each homomorphism* $\rho:R \to \bar{R}$ *is a scheme homomorphism.*

PROOF. If $G = \{\pm 1\}$ then $\rho(G) = \rho(\{\pm 1\}) = \{\pm 1\} \subseteq \bar{G}$. If $\bar{I}^2$ is torsion free then, by results in section 1, $\bar{R}^* = \bar{G}$, so $\rho(G) \subseteq \bar{G}$ holds in this case too.

2.2. EXAMPLES. (i) $G = \{\pm 1\}$ holds if and only if $R = \mathbf{Z}$, $\mathbf{Z}/(2)$, or $\mathbf{Z}/(4)$. Specific realizations of these three types can be obtained by taking $R = WF$ where F is (respectively) $\mathbf{R}$, $\mathbf{C}$, or a finite field $\mathbf{F}_q$, $q \equiv 3 \pmod 4$. If $q \equiv 1 \pmod 4$, then $W\mathbf{F}_q = \mathbf{Z}/(2)[C_2]$ (the group ring over $\mathbf{Z}/(2)$ of the cyclic group C_2) so $G \neq \{\pm 1\}$ in this case. (ii) I (resp. I^2) is torsion free if and only if $D\langle 1, 1\rangle = 1$ (resp. $D\langle 1, -a\rangle = G$ for all $a \in D\langle 1, 1\rangle$). Thus, if $R = WF$, F a field, then I (resp. I^2) is torsion free if and only if F is Pythagorean (resp. Quasi-Pythagorean). Elementary examples: $\mathbf{R}$, $\mathbf{C}$ are Pythagorean; finite fields are Quasi-Pythagorean. (iii) If $R = WF$ where F is a global field or a local field $\neq \mathbf{R}, \mathbf{C}$ then I^2 is not torsion free but I^3 is torsion free.

To obtain Harrison's map $\rho \to \bar{\rho}$ (see [2], [3], and [6]) one needs to assume that $\bar{R}$ satisfies an additional property:

$$(*) \qquad \forall a, b \in \bar{G}, (1 - a)(1 - b) \in \bar{I}^3 \Rightarrow (1 - a)(1 - b) = 0.$$

This is true if $\bar{R} = WF$, F a field, $\mathrm{char}(F) \neq 2$, e.g., see [4]. In what follows, this special property is assumed whenever necessary.

Let $\rho: R \to \bar{R}$ be a homomorphism. $\rho^{-1}(\bar{I})$ is an ideal of index 2 in R so $\rho^{-1}(\bar{I}) = I$. In particular, $\rho(I) \subseteq \bar{I}$. For $a \in G$, consider $\rho(a) \in \bar{R}^*$. This decomposes uniquely as $\rho(a) = \bar{\rho}(a)(1 + x(a))$ where $\bar{\rho}(a) \in \bar{G}$ and $x(a) \in \bar{I}^2$. Thus $\bar{\rho}: G \to \bar{G}$ is a group homomorphism. Since $\rho(-1) = -1 \in \bar{G}$, it follows that $\bar{\rho}(-1) = -1$. Now suppose $a, b \in G$ satisfy $1 \in D\langle a, b\rangle$. Then $(1 - \rho(a))(1 - \rho(b)) = 0$. Since $\bar{\rho}(c) - \rho(c) \in \bar{I}^2$ holds for any $c \in G$, this implies that $(1 - \bar{\rho}(a))(1 - \bar{\rho}(b)) \in \bar{I}^3$ and hence, by $(*)$, that $(1 - \bar{\rho}(a))(1 - \bar{\rho}(b)) = 0$. Thus $\bar{\rho}$ induces a scheme homomorphism $\bar{\rho}: R \to \bar{R}$.

$\bar{\rho}$ is characterized as the unique scheme homomorphism satisfying $\bar{\rho}(x) \equiv \rho(x) \pmod{\bar{I}^2}$ for all $x \in R$. Clearly $\bar{\rho} = \rho$ if and only if ρ is a scheme homomorphism. Also $\rho \to \bar{\rho}$ is functorial in the sense that $\overline{\psi \circ \rho} = \bar{\psi} \circ \bar{\rho}$ and $\bar{1} = 1$. In particular, if ρ is bijective, then $\bar{\rho}$ is bijective.

2.3. Note. For $a \in G$, $\rho(a) = \bar{\rho}(a)(1 + x(a))$ with $x(a) \in \bar{I}^2$. $1 + x(a)$ is a unit of order 2 in $\bar{R}$. Thus $x(a)$ is 2-primary torsion so $\rho(a) - \bar{\rho}(a) = \bar{\rho}(a)x(a)$ is 2-primary torsion. Since G generates R additively, this implies that $\rho(x) \equiv \bar{\rho}(x) \pmod{(\bar{I}^2)_{\mathrm{tor}}}$ holds for all $x \in R$. Here, $(\bar{I}^k)_{\mathrm{tor}}$ denotes the torsion part of $\bar{I}^k$. Actually, if we use (1.2), we can conclude that $2x(a) = 0$ for each $a \in G$ so $2\rho = 2\bar{\rho}$.

2.4. Proposition. *Suppose $G \neq \{\pm 1\}$, $\bar{I}^2$ is not torsion free, but $\bar{I}^k$ is torsion free for some $k \geqq 3$. Then, for each scheme homomorphism $\alpha: R \to \bar{R}$, there exists a homomorphism $\rho: R \to \bar{R}$ such that $\bar{\rho} = \alpha$, $\rho \neq \alpha$.*

Proof. We may as well assume $(\bar{I}^{k-1})_{\mathrm{tor}} \neq 0$ so $G/\{\pm 1\}$ and $(\bar{I}^{k-1})_{\mathrm{tor}}$ are non-trivial groups of exponent 2. Thus there exists a non-trivial group homomorphism $x: G \to (\bar{I}^{k-1})_{\mathrm{tor}}$ with $x(-1) = 0$. Pick any such homomorphism and define $\rho: G \to R^*$ by $\rho(a) = \alpha(a)(1 + x(a))$ $\forall a \in G$. Now $x(a)x(b) \in (\bar{I}^k)_{\mathrm{tor}} = 0$ so $(1 + x(a))(1 + x(b)) = 1 + x(a) + x(b) = 1 + x(ab)$. Thus ρ is a group homomorphism. Also $x(-1) = 0$ and $\alpha(-1) = -1$ so $\rho(-1) = -1$. Assume $1 \in D\langle a, b\rangle$, $a, b \in G$. Then

$$(1 - \rho(a))(1 - \rho(b)) = (1 - \alpha(a))(1 - \alpha(b)) + (1 - \alpha(a))\alpha(b)x(b)$$
$$+ (1 - \alpha(b))\alpha(a)x(a) + \alpha(a)x(a)\alpha(b)x(b).$$

The first term here is zero since α is a homomorphism. The last three terms are zero since $(\bar{I}^k)_{\mathrm{tor}} = 0$. Thus ρ induces a homomorphism $\rho: R \to \bar{R}$.

3. **Automorphisms.** Consider a homomorphism $\rho:R \to R$ satisfying $\bar{\rho} = 1$. That is, assume $\rho(x) \equiv x \pmod{I^2}$ holds for all $x \in R$.

3.1. Lemma. *Suppose $k \geqq 2$ and that $\rho(x) \equiv x \pmod{I^k}$ holds for all $x \in R$. Then $\rho(x) \equiv x \pmod{I^{i+k}}$ holds for all $x \in I^{i+1}$, $i \geqq 0$.*

Proof. The result is clear if $i = 0$. If $i \geqq 1$, the result follows by induction using

$$\rho(xy) - xy = \rho(x)(\rho(y) - y) + (\rho(x) - x)y$$

with $x \in I$, $y \in I^i$.

3.2. Lemma. *If $k \geqq 2$ and $\rho(x) \equiv x \pmod{I^k}$ holds for all $x \in R$ then $\rho^2(a) \equiv a \pmod{I^{2k-1}}$ holds for all $a \in R$.*

Proof. Since G generates R we can assume $a \in G$. Thus $\rho(a) = a(1 + x)$ with $x \in I^k$.

$$\begin{aligned} \rho^2(a) &= \rho(\rho(a)) = \rho(a(1 + x)) = \rho(a)(1 + \rho(x)) \\ &= a(1 + x)(1 + \rho(x)) = a + a(x + \rho(x) + x\rho(x)). \end{aligned}$$

Thus we have to show that

$$x + \rho(x) + x\rho(x) = 2x + (\rho(x) - x) + x\rho(x) \in I^{2k-1}.$$

Clearly $x\rho(x) \in I^{2k}$. By (3.1), $\rho(x) - x \in I^{2k-1}$. Also, $1 + x$ has order 2 in R^*, so $2x + x^2 = 0$. Thus $2x = -x^2 \in I^{2k}$. (In fact, by (1.2), $2x = 0$.)

3.3. Proposition. *Suppose I^2 is not torsion free but I^k is torsion free for some $k \geqq 3$. Then there exists an automorphism $\rho:R \to R$ such that $\bar{\rho} = 1$, $\rho \neq 1$.*

Proof. If $G = \{\pm 1\}$ then $R = \mathbf{Z}$, $\mathbf{Z}/(2)$, or $\mathbf{Z}/(4)$ and I^2 is torsion free. Thus $G \neq \{\pm 1\}$. Thus, by (2.4), there is some homomorphism $\rho:R \to R$ such that $\bar{\rho} = 1$, $\rho \neq 1$. Pick any such ρ and pick s so large that $2^s + 1 \geqq k$. Then for any $x \in R$, (3.2) implies that $\rho^{2^s}(x) \equiv x \pmod{I^{2^s+1}}$. By (2.3), $\rho^{2^s}(x) - x$ is torsion so $\rho^{2^s}(x) = x$. Thus, $\rho^{2^s} = 1$. This implies ρ is bijective.

Let $\mathrm{Aut}(R)$ denote the group of automorphisms $\rho:R \to R$. Let $\mathrm{Aut}_{sc}(R) \subseteq \mathrm{Aut}(R)$ be the subgroup consisting of scheme automorphisms. For $j \geqq 1$ let $\mathrm{Aut}_j(R) \subseteq \mathrm{Aut}(R)$ be the subgroup of automorphisms satisfying $\rho(x) \equiv x \pmod{I^{j+1}}$ for all $x \in R$. Harrison's map $\rho \to \bar{\rho}$ is a group homomorphism from $\mathrm{Aut}(R)$ onto $\mathrm{Aut}_{sc}(R)$ with kernel $\mathrm{Aut}_1(R)$. Since $\bar{\rho} = \rho$ for $\rho \in \mathrm{Aut}_{sc}(R)$, $\mathrm{Aut}(R)$ is a semi-direct product of $\mathrm{Aut}_1(R)$ and $\mathrm{Aut}_{sc}(R)$. If I^2 is torsion free, $\mathrm{Aut}_1(R) = 1$ and $\mathrm{Aut}(R) = \mathrm{Aut}_{sc}(R)$. Suppose I^2 is not torsion free but I^{k+1} is torsion free for some $k \geqq 2$. Then $\mathrm{Aut}_1(R) \neq 1$ but $\mathrm{Aut}_k(R) = 1$. Each $\mathrm{Aut}_j(R)$ is normal in $\mathrm{Aut}(R)$. Also, by (3.2), $\rho \in \mathrm{Aut}_j(R) \Rightarrow \rho^2 \in \mathrm{Aut}_{2j}(R)$. Thus, in this case, $\mathrm{Aut}_1(R)$ is solvable and each element of $\mathrm{Aut}_1(R)$ has finite 2-power order.

3.4. PROPOSITION. *If* I^3 *is torsion free then* $\mathrm{Aut}_1(R)$ *is canonically isomorphic to the group* $\mathrm{Hom}_{\mathrm{gr}}(G/\{\pm 1\}, (I^2)_{\mathrm{tor}})$. *(Here, "*$\mathrm{Hom}_{\mathrm{gr}}$*" denotes group homomorphisms.)*

PROOF. If $x:G \to (I^2)_{\mathrm{tor}}$ is any group homomorphism satisfying $x(-1) = 0$ then, by the proof of (2.4), x induces a homomorphism $\rho:R \to R$ given by $\rho(a) = a(1 + x(a))$ for all $a \in G$. As in the proof of (3.3), $\rho^2 = 1$ so ρ is an automorphism and hence $\rho \in \mathrm{Aut}_1(R)$. $x \to \rho$ provides the desired isomorphism.

R is said to be of local type if it is the Witt ring of a local field. R is said to be of elementary type if $|G| < \infty$ and R is built up from $\mathbf{Z}/(2)$, $\mathbf{Z}/(4)$, $\mathbf{Z}$ and local types by forming Witt products and group rings. For elementary types, it is possible to give a precise inductive description of $\mathrm{Aut}_{sc}(R)$. This is an easy consequence of the material on quadratic form schemes developed in [6] and will not be given here.

For local types, $\mathrm{Aut}_1(R)$ and the action of $\mathrm{Aut}_{sc}(R)$ on $\mathrm{Aut}_1(R)$ can be computed explicitly using (3.4). In contrast, the structure of $\mathrm{Aut}_1(R)$ for general elementary types is not at all well understood. This is because $\mathrm{Aut}_1(R)$ is not very well behaved with respect to formation of Witt products and group rings.

Denote by $J_k \subseteq R$ the ideal of elements of (additive) order 2 in I^{k+1}. For elementary types it is known that $J_k \neq 0 \Rightarrow J_k \neq J_{k+1}$. For general Witt rings this appears to be open. Each $\rho \in \mathrm{Aut}_k(R)$ satisfies $\rho(x) \equiv x \pmod{J_k}$ for all $x \in R$. This follows from (1.2) (also see (2.3)). If $k \geqq 1$ is such that $J_k \neq 0$, $J_{k+1} = 0$, then the element $\rho \in \mathrm{Aut}_1(R)$, $\rho \neq 1$, constructed in (3.3), is actually in the group $\mathrm{Aut}_k(R)$. One would hope that if $k \geqq 1$ is arbitrary then $J_k \neq J_{k+1} \Rightarrow \mathrm{Aut}_k(R) \neq \mathrm{Aut}_{k+1}(R)$. In general it is not known if this is true.

3.5. PROPOSITION. *If R is of elementary type,* $k \geqq 1$, *and* $J_k \neq 0$ *then there exists* $\rho \in \mathrm{Aut}_k(R)$, $\rho \notin \mathrm{Aut}_{k+1}(R)$.

PROOF. The proof is by induction on $|G|$. If R is of local type then $k = 1$ and the result is clear. There are two cases left to consider. *Case 1*: $R = R_1 \times R_2$ (Witt product). Then $J_k = (J_1)_k \times (J_2)_k$ (ordinary product) so $(J_i)_k \neq 0$ for $i = 1$ or 2, say $(J_1)_k \neq 0$. Thus, by induction, there exists $\rho_1 \in \mathrm{Aut}_k(R_1)\backslash\mathrm{Aut}_{k+1}(R_1)$. Take $\rho = \rho_1 \times 1$. *Case 2*: $R = \bar{R}[\Delta]$, $\Delta = \{1, g\}$. Then $J_k = \bar{J}_k \oplus (1 - g)\bar{J}_{k-1}$, so $\bar{J}_{k-1} \neq 0$. Pick $x \in \bar{J}_{k-1}\backslash\bar{J}_k$ and define $\rho:R \to R$ by $\rho(a) = a$ for $a \in \bar{G}$, $\rho(g) = g + (1 - g)x$. Then ρ is a ring automorphism, $\rho \in \mathrm{Aut}_k(R)\backslash\mathrm{Aut}_{k+1}(R)$.

If we drop the assumption that I^{k+1} is torsion free for some $k \geqq 2$ then it is not known whether I^2 not torsion free $\Rightarrow \mathrm{Aut}_1(R) \neq 1$. In fact very little is known. If R is the Witt ring of a field then $\cap_j I^j = 0$ by [1]. In this case it follows that $\cap_j \mathrm{Aut}_j(R) = 1$. Combining this with (3.2) one can deduce that any element $\rho \in \mathrm{Aut}_1(R)$ which has finite order has 2-power order. See example 4 below for a case where $\mathrm{Aut}_1(R)$ has elements of infinite order.

4. Examples.

(1) Take $R = \mathbf{Z}/(4)[\Delta]$, Δ a group of exponent 2 (so $G = \Delta \times \{\pm 1\}$). Thus, if $|\Delta| = 2^k$, then $J_k \neq 0$, $J_{k+1} = 0$. We show that $\mathrm{Aut}_1(R)$ has exponent 2. $D\langle 1, 1\rangle = \{\pm 1\}$ so $J_0 \subseteq R$ is the ideal generated by 2. Let $\rho \in \mathrm{Aut}_1(R)$ be arbitrary. Then, for $a \in G$, $\rho(a)$ has the form $\rho(a) = a + 2r$, $r \in I$. Let $\rho(r) - r = 2s$, $s \in I$. Then $\rho^2(a) = \rho(a) + 2\rho(r) = a + 2r + 2(r + 2s) = a + 4r + 4s = a$. This shows $\rho^2 = 1$.

(2) $\mathrm{Aut}_l(R)/\mathrm{Aut}_{2l}(R)$ is abelian of exponent 2. To obtain an example where $\mathrm{Aut}_l(R)/\mathrm{Aut}_{2l+1}(R)$ is not abelian one can take $R = \mathbf{Z}/(2)[\Delta]$ where Δ is a group of exponent 2 with $\mathbf{Z}/(2)$ − basis $a_1, \ldots, a_l, b_0, \ldots, b_l$. Set

$$\rho(a_1) = \psi(a_1) = a_1 + \prod_{i=0}^{l} (1 + b_i) \text{ and}$$

$$\rho(b_0) = b_0 + (1 + b_0) \prod_{i=1}^{l} (1 + a_i), \ \psi(b_0) = b_0.$$

For $i \geqq 2$ and $j \geqq 1$ set $\rho(a_i) = \psi(a_i) = a_i$ and $\rho(b_j) = \psi(b_j) = b_j$. Then ρ, $\psi \in \mathrm{Aut}_l(R)$ but, as one can verify by direct computation, $\rho(\psi(a_1)) \not\equiv \psi(\rho(a_1))$ (mod J_{2l+1}).

(3) If $J_{2^m} = 0$ then each $\rho \in \mathrm{Aut}_1(R)$ has order at most 2^m. To obtain an example where this bound is attained take $m \geqq 1$ and $R = \mathbf{Z}/(2)[\Delta]$ where Δ has exponent 2 and $\mathbf{Z}/(2)$-dimension 2^m. Fix a $\mathbf{Z}/(2)$-basis for Δ of the form $\{a_i, b_i | i \in \mathbf{Z}/(2^{m-1})\}$. Define $\rho \in \mathrm{Aut}_1(R)$ by

$$\rho(a_i) = a_i + (1 + a_{i+1})(1 + b_i),$$
$$\rho(b_i) = b_i.$$

A careful inductive argument shows that

$$\rho^{2^s}(a_i) = a_i + (1 + a_{i+2^s}) \prod_{j=1}^{2^s} (1 + b_{i+j-1})$$

for $s = 0, \ldots, m - 1$. Taking $s = m - 1$ in this formula, it follows that $\rho^{2^{m-1}} \neq 1$.

(4) It is possible to show (for example by patching together automorphisms constructed in the above example) that the Witt ring $R = \mathbf{Z}/(2)[\Delta]$, Δ countably infinite, has elements $\rho \in \mathrm{Aut}_1(R)$ of infinite order.

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