ON n-DIMENSIONAL STIRLING NUMBERS

by SELMO TAUBER

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1. Introduction

Stirling numbers of the first and second kind play an important part in many branches of mathematics, in particular in combinatorial analysis and statistics. For their definition and properties we refer to (5) where a whole chapter is devoted to their study. Stirling numbers have been generalized in many ways. One generalization is given in (1). In this paper we generalize the results of (1) to n dimensions. In order to simplify the notation we use methods of linear algebra.

Let $x_1, x_2, \ldots, x_n$ be n independent complex variables and X a vector in the n-dimensional vector space V, i.e.

$$X = [x_1, x_2, \ldots, x_n], \quad X \in V. \tag{1}$$

Let I be the set of positive integers and zero and let $p_k \in I$, $k = 1, 2, \ldots, n$. We define $P \in V$ by

$$P = [p_1, p_2, \ldots, p_n] \tag{2}$$

and the norm of P to be

$$p = \| P \| = p_1 + p_2 + \ldots + p_n. \tag{3}$$

We next introduce several summation symbols. First

$$\sum_{p=\alpha}^{\beta}, \tag{4}$$

means a summation where $p = \| P \|$, $\alpha \leqq p \leqq \beta$, $\alpha, \beta \in I$, α and β being fixed positive integers or zero. It should be noted that in this case it may happen that $p_k < \alpha$ for some k, provided that $p \geqq \alpha$. In particular, if $\alpha = 0$, (4) becomes

$$\sum_{p=0}^{\beta}, \tag{5}$$

where all $p_k \geqq 0$, and $p = \| P \| \leqq \beta$. Also if $\alpha = \beta$, (4) becomes

$$\sum_{p=\alpha}^{\alpha} = \sum_{p=\alpha}, \tag{6}$$

which means that $p = \| P \| = \alpha = \text{constant}$.

Another kind of summation is defined by the symbol

$$\sum_{P=A}^{B}, \tag{7}$$

where $A = [a_1, a_2, \ldots, a_n]$, $B = [b_1, b_2, \ldots, b_n]$, and $a_k \leqq p_k \leqq b_k$, for $k = 1$, 2, ..., n, and $a_k, b_k, p_k \in I$. The summation (7) can also be written in one of the following forms:

$$\sum_{p_1=a_1}^{b_1} \sum_{p_2=a_2}^{b_2} \cdots \sum_{p_n=a_n}^{b_n} = (\Sigma^{(n)} p_1[a_1, b_1], p_2[a_2, b_2], \ldots, p_n[a_n, b_n]). \tag{8}$$

Finally, let $m \in I$; we consider the functions $M(m)$, $N_k(m)$, $k = 1, 2, \ldots, n$, defined for $m \in I$ and the vector

$$N(m) = [N_1(m), N_2(m), \ldots, N_n(m)], \tag{9}$$

and the scalar product

$$N(m) . X = \sum_{k=1}^{n} N_k(m) x_k. \tag{10}$$

2. Q-polynomials and A-numbers

Using the notation of Section **1** we define the Q-polynomial

$$Q(X, r) = Q(r) = \prod_{m=1}^{r} [nM(m) + N(m) . X] \tag{11}$$
$$= \sum_{p=0}^{r} A(r, P) \prod_{k=1}^{n} x_k^{p_k},$$

with $A(r, P) = 0$, for $p_k > r$, $r < 0$, or $p_k < 0$, $k = 1, 2, \ldots, n$.

The coefficients $A(r, P)$ are called (generalized) n-dimensional Stirling numbers of the first kind, the n-dimensions corresponding to the polynomial Q which is in n variables.

We first find a recurrence relation for the A-numbers. We clearly have

$$Q(r+1) = \prod_{m=1}^{r+1} [nM(m) + N(m) . X] = Q(r)[nM(r+1) + N(r+1) . X]$$
$$= nM(r+1) \sum_{p=0}^{r} A(r, P) \prod_{k=1}^{n} x_k^{p_k} + N(r+1) . X \sum_{p=0}^{n} A(r, P) \prod_{k=1}^{n} x_k^{p_k}$$
$$= \sum_{p=0}^{r+1} A(r+1, P) \prod_{k=1}^{n} x_k^{p_k}.$$

By equating the coefficients of equal monomials in $x_1, x_2, \ldots, x_n$, we obtain

$$A(r+1, P) = nM(r+1)A(r, P) + N(r+1) . A(r, P_s), \tag{12}$$

where

$$P_s = [p_1, p_2, \ldots, p_{s-1}, p_{s+1}, \ldots, p_n] \tag{12a}$$

and

$$A(r, P_s) = [A(r, P_1), A(r, P_2), \ldots, A(r, P_n)]; \tag{12b}$$

thus

$$A(r, P) = nM(r)A(r-1, P) + N(r) . A(r-1, P_s), \tag{13}$$

which is the recurrence relation for the A-numbers.

3. Examples and special cases

(i) For $n = 1$ we have

$$A(r, p) = M(r)A(r-1, p) + N(r)A(r-1, p-1)$$

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i.e.

$$A_r^p = M(r)A_{r-1}^p + N(r)A_{r-1}^{p-1},$$

which is the same as (8) of (1). In particular, for $M(m) = -m+1$, $N(m) = 1$,

$$Q(r) = (x)_r = x(x-1)(x-2)\dots(x-r+1),$$

$$A_r^p = St_r^p = (-r+1)St_{r-1}^p + St_{r-1}^{p-1},$$

i.e. the Stirling numbers of the first kind.

(ii) For $M(m) = 0$, $N_k(m) = 1$, $k = 1, 2, \dots, n$,

$$A(r, P) = \binom{r}{p_1, p_2, \dots, p_n}, \quad p = \| P \| = r,$$

i.e. the multinomial coefficients as studied in (2) and (3).

4. Inversion problem

Using the results of (1) we can write, using the notation $B(p_k, m)$ for $B_{p_k}^m$ corresponding to $N_k(m)$,

$$x_k^{p_k} = \sum_{m=0}^{p_k} B(p_k, m)Q_k(x_k, m), \tag{14}$$

where

$$Q_k(x_k, m) = \prod_{s=1}^{m} [M(s) + x_k N_k(s)] = \sum_{t=0}^{m} A_k(m, t)x_k^t,$$

and

$$B_k(p_k, m) = 0 \text{ for } p_k < 0,\ m < 0, \text{ and } p_k < m.$$

It follows that

$$\begin{aligned} \prod_{k=1}^{n} x_k^{p_k} &= \prod_{k=1}^{n} \left[\sum_{m_k=0}^{p_k} B_k(p_k, m_k)Q_k(x_k, m_k) \right] \\ &= (\Sigma^{(n)} m_1[0, p_1], m_2[0, p_2], \dots, m_n[0, p_n]) \prod_{k=1}^{n} B_k(p_k, m_k)Q_k(x_k, m_k) \\ &= \sum_{M=0}^{P} B(P, M) \prod_{k=1}^{n} Q_k(x_k, m_k), \end{aligned} \tag{15}$$

where $\prod_{k=1}^{n} B_k(p_k, m_k) = B(P, M)$, $P = [p_1, p_2, \dots, p_n]$, $M = [m_1, m_2, \dots, m_n]$.

(15) gives the inversion of (11), i.e. the expression of an arbitrary monomial in $x_1, x_2, \dots, x_n$ in terms of one-dimensional Q-polynomials. The coefficients $B(P, M)$ will be called (generalized) n-dimensional Stirling numbers of the fourth kind. The name will be justified later.

5. Generalized quasi-orthogonality

For the definition of orthogonality and quasi-orthogonality we refer to (4). It is necessary to generalize these definitions for n dimensions.

We consider the two sets of numbers $A(P, Q)$ and $B(Q, S)$, where

$$P = [p_1, p_2, \ldots, p_n], \quad Q = [q_1, q_2, \ldots, q_n], \quad S = [s_1, s_2, \ldots, s_n],$$
$$\| P \| = p, \quad \| Q \| = q, \quad \| S \| = s,$$

and define quasi-orthogonality between the two sets of numbers by the relation

$$\sum_{Q=S}^{P} B(P, Q)A(Q, S) = \delta_P^S = \prod_{k=1}^{n} \delta_{p_k}^{s_k}, \tag{16}$$

where the summation is to be understood as in (7) and where $A(Q, S) = 0$ for $q_k < s_k$, $q_k < 0$, $s_k < 0$, and $B(P, Q) = 0$ for $p_k < q_k$, $p_k < 0$, $q_k < 0$, $k = 1, 2, \ldots, n$.

We next consider the two sets of numbers $A(p, Q)$ and $B(Q, s)$. We define their quasi-orthogonality by the relation

$$\sum_{q=s}^{p} A(p, Q)B(Q, s) = \delta_p^s, \tag{17}$$

where the summation is to be interpreted as in (4) and the property is commutative.

The definition given by (10) in (2) does not fit either definition given here since the meanings are different.

We shall prove the following theorem that will be important in what follows.

Theorem. *If the numbers $B(P, Q)$ are quasi-orthogonal to the numbers $A(Q, S)$ and to the numbers $C(Q, S)$ then $A(Q, S) = C(Q, S)$.*

Proof. According to the definition we have, with $s_k \leqq q_k \leqq p_k$,

$$\sum_{Q=S}^{P} B(P, Q)A(Q, S) = \delta_P^S,$$
$$\sum_{Q=S}^{P} B(P, Q)C(Q, S) = \delta_P^S.$$

By subtracting these two relations from each other we obtain

$$\sum_{Q=S}^{P} B(P, Q)[A(Q, S) - C(Q, S)] = 0, \tag{18}$$

where $B(P, Q) \neq 0$. Thus for $P = Q = S$, $B(P, P)[A(P, P) - C(P, P)] = 0$, so that $A(P, P) = C(P, P)$. We now generalize the notation of (12*a*) by writing

$$P_{at,\,bu,\,cv} = [p_1, p_2, \ldots, p_{t-1}, p_t - a, p_{t+1}, \ldots, p_{u-1}, p_u - b, p_{u+1}, \ldots,$$
$$p_{v-1}, p_v - c, p_{v+1}, \ldots, p_n],$$

where $a, b, c \in I$ and $a \leqq p_t$, $b \leqq p_u$, $c \leqq p_v$. It is clear that we can change a, b, c into $-a, -b, -c$, with the obvious interpretation. If then, in (18), we take $S = P_t$ we obtain

$$B(P, P)[A(P, P_t) - C(P, P_t)] + B(P, P_t)[A(P_t, P_t) - C(P_t, P_t)] = 0.$$

Since $A(P_t, P_t) - C(P_t, P_t) = 0$, it follows that $A(P, P_t) = C(P_t, P_t)$, and generalizing by induction with $0 \leqq a \leqq p_t$, we obtain

$$A(P, P_{at}) = C(P, P_{at}).$$

Continuing the proof we can fairly easily establish first that

$$A(P, P_{at,\,v}) = C(P, P_{at,\,v})$$

and from there by induction that $A(P, P_{at,\,bv}) = C(P, P_{at,\,bv})$, where $0 \leqq b \leqq p_v$. It follows that

$$A(P_{-at,\,-bv}, P) = C(P_{-at,\,-bv}, P),$$

where $a, b \in I$ without other limitation. From there we finally obtain

$$A(Q, S) = C(Q, S),$$

which holds for any Q and S provided that $q_k, s_k \in I$, $Q, S \in V$, $(k = 1, 2, \ldots, n)$, since for $q_k < s_k$ both A and C numbers are zero.

This is actually a uniqueness theorem concerning the quasi-orthogonality as defined by (16).

6. Quasi-orthogonality relations

According to (1) and the notation used here, we have

$$\sum_{s=m}^{n} B(n, s)A(s, m) = \delta_n^m; \tag{19}$$

thus

$$\sum_{m_k = s_k}^{p_k} B_k(p_k, m_k)A(m_k, s_k) = \delta_{p_k}^{s_k}, \quad k = 1, 2, \ldots, n$$

and

$$\prod_{k=1}^{n} \sum_{m_k = s_k}^{p_k} B_k(p_k, m_k)A(m_k, s_k) = \prod_{k=1}^{n} \delta_{p_k}^{s_k} = \delta_P^S,$$

so that

$$\sum_{M=S}^{P} \left[\prod_{k=1}^{n} B_k(p_k, m_k)\right]\left[\prod_{k=1}^{n} A_k(m_k, s_k)\right] = \delta_P^S,$$

or

$$\sum_{M=S}^{P} B(P, M)A(M, S) = \delta_P^S, \tag{20}$$

where

$$\prod_{k=1}^{n} A_k(m_k, s_k) = A(M, S). \tag{20a}$$

It can be easily checked by constructing a numerical example that, in general, $A(M, S) \neq A(m, S)$.

It follows according to the uniqueness theorem that $B(P, M)$ and $A(m, S)$ are *not* quasi-orthogonal. On the contrary the numbers $B(P, M)$ and $A(M, S)$ are quasi-orthogonal. We shall call the numbers $A(M, S)$ the (generalized) n-dimensional Stirling numbers of the third kind. It is clear that the Stirling numbers of the third and fourth kind are quasi-orthogonal to each other. According to (19) and (20) the quasi-orthogonality property is commutative for the $B(P, M)$ and $A(M, S)$ numbers.

7. Generalized n-dimensional Stirling numbers of the second kind

Since the numbers $B(P, M)$ are not quasi-orthogonal to the numbers $A(m, S)$ we look for a set of numbers $B(S, p)$ that is quasi-orthogonal to the set $A(m, S)$ in the sense of (17). As before we assume that $B(S, p) = 0$ for $p<0$, and for $\| S \| = s<p$. We thus can write

$$\sum_{s=p}^{m} A(m, S)B(S, p) = \delta_m^p, \quad m, p \in I. \tag{21}$$

In (21) put $p = m = 0$; then $A(0, \phi)B(\phi, 0) = 1$, where ϕ is the zero vector, so that since $A(0, \phi) = 1$, $B(\phi, 0) = 1$. We next put $m = 1$, $p = 0$ in (21); with

$$E_k = [\delta_1^k, \delta_2^k, \ldots, \delta_{k-1}^k, \delta_k^k, \delta_{k+1}^k, \ldots, \delta_n^k], \quad k = 1, 2, \ldots, n,$$

we have

$$\sum_{k=1}^{n} A(1, E_k)B(E_k, 0) + A(1, \phi)B(\phi, 0) = 0.$$

This relation shows that the numbers $B(S, p)$ are not defined unless we make an additional assumption. We shall therefore assume that $B(S, p) = \beta(s, p)$, where $s = \| S \|$. This means, for example, $B(E_k, 0) = \beta(1, 0)$, $k = 1, 2, \ldots, n$. Under these conditions we can find the $B(S, p)$ numbers and their recurrence relation. Since these numbers are quasi-orthogonal to the (generalized) n-dimensional Stirling numbers of the first kind we shall call them (generalized) n-dimensional Stirling numbers of the second kind.

We introduce the notation $\sum_{p=q} A(r, P) = \alpha(r, p)$, where $\| P \| = p$ and q is a given positive constant or zero. Starting from (13), we have

$$A(r, P) = nM(r)A(r-1, P) + N(r) . A(r-1, P_s),$$

$$N(r) . A(r-1, P_s) = \sum_{s=1}^{n} N_s(r)A(r-1, P_s).$$

By summing both sides of this, we obtain

$$\sum_{p=q} A(r, P) = \alpha(r, p) = nM(r) \sum_{p=q} A(r-1, P) + \sum_{p=q} \sum_{s=1}^{n} N_s(r)A(r-1, P_s).$$

Since

$$\sum_{p=q} A(r-1, P) = \alpha(r-1, p)$$

and

$$\sum_{p=q} \sum_{s=1}^{n} N_s(r)A(r-1, P_s) = \sum_{s=1} N_s(r) \sum_{p=q} A(r-1, P_s)$$
$$= \sum_{s=1}^{n} N_s(r)\alpha(r-1, p-1),$$

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where $\| P_s \| = \| P \| - 1 = p-1$, it follows that the α numbers satisfy the recurrence relation

$$\alpha(r, p) = nM(r)\alpha(r-1, p) + \| N(r) \| \alpha(r-1, p-1). \tag{22}$$

On the other hand we see that (21) can be written

$$\sum_{s=p}^{m} A(m, S)B(S, p) = \sum_{q=p}^{m} \left[\sum_{s=q} A(m, S)B(S, p) \right] = \sum_{q=p}^{m} \beta(q, p) \sum_{s=q} A(m, S)$$

$$= \sum_{q=p}^{m} \alpha(m, q)\beta(q, p) = \delta_m^p,$$

so that the numbers $\alpha(m, q)$ and $\beta(q, p)$ are quasi-orthogonal in the one-dimensional sense. It follows therefore from theorem I of (**1**) that the $\beta(r, p)$ numbers satisfy the recurrence relation

$$\beta(r, p) = -[nM(p+1)/\| N(p+1)\|]\beta(r-1, p) + \beta(r-1, p-1)/\| N(p+1)\|. \tag{23}$$

8. Summary of relationships

There are four kinds of generalized n-dimensional Stirling numbers. The first kind defined by (11) is quasi-orthogonal in the sense of (17) to the second kind defined in Section 7. The fourth kind defined in Section **4** is quasi-orthogonal to the third kind defined in Section 6, in the sense of (16).

In addition we have defined in Section 7 the α numbers and have shown that they are quasi-orthogonal in the one-dimensional sense (cf. (**4**)) to the generalized n-dimensional Stirling numbers of the second kind, i.e. the β numbers.

For $n = 1$ the numbers of the first and third kind become the numbers of the first kind of (**1**) and the numbers of the second and fourth kind become the numbers of the second kind of (**1**).

9. Generating functions

Equation (11) defines the generalized n-dimensional Stirling numbers of the first kind. At the same time it defines $Q(X, r)$ as the generating function of these numbers.

(i) The generalized n-dimensional Stirling numbers of the second kind are defined in Section 7 and satisfy the recurrence relation (23). We find their generating function by the method given on p. 175 of (**5**). Let

$$G\beta(r, p) = \phi(p, t) = \sum_{r=p}^{\infty} \beta(r, p)t^r,$$

then

$$G\beta(r, p+1) = \phi(p+1, t), G\beta(r+1, p+1) = \phi(p+1, t)/t,$$

and by substituting into (23) we obtain the difference equation satisfied by ϕ, i.e.

$$\phi(p+1, t)[\| N(p+2) \| + tM(p+2)] - t\phi(p, t) = 0.$$

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The general solution of this equation is given by

$$\phi(p, t) = \omega(t) \prod_{k=0}^{p-1} t/[\|N(k+2)\| + tM(k+2)],$$

where $\omega(t)$ is an arbitrary periodic function of period 1. To find $\omega(t)$ we use the boundary condition

$$\phi(1, t) = \sum_{r=0}^{\infty} \beta(r, 1)t^r = \omega(t)t/[\| N(2)\| + tM(2)].$$

The same procedure can be applied to the generating function of the α numbers.

(ii) Referring to Section **4** we can write

$$\prod_{k=1}^{n} Q_k(t_k, m_k) = \prod_{k=1}^{n} \sum_{s_k=0}^{m_k} A_k(m_k, s_k)t_k^{s_k} = \sum_{S=0}^{M} A(M, S)T^S = \phi(M, T),$$

where, symbolically,

$$T^S = \prod_{k=1}^{n} t_k^{s_k}, \quad S = [s_1, s_2, \ldots, s_n].$$

This relation defines the generating function $\phi(M, T)$ of the numbers $A(M, S)$.

(iii) According to (12) of (**1**) for each k (which we leave out in order to simplify the notation)

$$B(p, m) = -M(m+1)B(p-1, m)/N(m+1) + B(p-1, m-1)/N(m+1). \quad (24)$$

On the other hand

$$GB(p, m) = \phi(m, t) = \sum_{p=m}^{\infty} B(p, m)$$

$$GB(p, m+1) = \phi(m+1, t)$$

$$GB(p+1, m+1) = \phi(m+1, t)/t,$$

since $B(0, m+1) = 0$. Substituting into (24) we obtain

$$\phi(m+1, t)[N(m+2) + tM(m+2)] - t\phi(m, t) = 0,$$

which is readily solved to give

$$\phi(m, t) = \omega(t) \prod_{k=0}^{m-1} t/[N(k+2) + tM(k+2)],$$

where

$$\phi(1, t) = \omega(t)t/[N(2) + tM(2)] = \sum_{p=1}^{\infty} B(p, 1)t^p,$$

which defines $\omega(t)$.

It follows that

$$\prod_{k=1}^{n} \phi_k(m_k, t_k) = \prod_{k=1}^{n} \omega_k(t_k) \prod_{s=0}^{m_k-1} t_k/[N_k(s+2) + t_k M_k(s+2)]$$

$$= \prod_{k=1}^{n} \sum_{p_k=m_k}^{\infty} B_k(p_k, m_k)t_k^{p_k} = \phi(M, T),$$

where $\phi(M, T)$ is the generating function for the generalized n-dimensional Stirling numbers of the fourth kind, since

$$\phi(M, T) = \sum_{P=M}^{\infty} B(P, M)T^P,$$

where

$$T^P = \prod_{k=1}^{n} t_k^{p_k}, \quad P = [p_1, p_2, \ldots, p_n].$$

Remark. Special cases of the preceding numbers have been studied. One complete example will be found in (6).

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Portland State University
Portland, Oregon