PROPAGATION OF THERMAL STRESSES IN AN INFINITE MEDIUM

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1. Introduction

In the full linear theory of thermoelasticity † there is a coupling between the thermal and the purely mechanical effects so that not only does a non-uniform distribution of temperature in the solid produce a state of stress but dynamical body forces or applied surface tractions produce variations in temperature throughout the body. In a recent paper (Eason and Sneddon, (**2**)) an account was given of the calculation of the dynamic stresses produced in elastic bodies, both infinite and semi-infinite, by uneven heating. In this paper we shall consider the propagation of thermal stresses in an infinite medium when, in addition to heat sources, there are present body forces which vary with the time.

In § 2 we derive the general solution of thermoelastic equations, written in terms of cartesian coordinates, by using four-dimensional Fourier transforms, and in § 3 we get the solution for axially symmetrical problems by using a mixed Fourier-Hankel transform. In §§ 5, 6 we consider the effects produced by time-dependent body forces ; the expressions for the temperature and the components of the displacement vector are given in the form of multiple integrals which, in the general case, are difficult to evaluate in closed form.

The solution assumes a simpler form in the quasi-static approximation in which the stress-strain relation and the equation governing the conduction of heat remain unaltered but the equations of motion are replaced by equations of equilibrium. The quasi-static approximation is valid when the heat sources and body forces do not vary too violently with the time, ‡ and it is considered in § 7. One result of physical interest emerges from this analysis. In the quasi-static approximation we find that the temperature field due to a body force $F(x, y, z)f(t)$ in the z-direction can be determined from the classical equation for the conduction of heat (i.e. with no term involving the time rate of change of the dilatation) by replacing the body force by an equivalent heat source which is proportional to

$$f'(t)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(z-z')F(x', y', z')dx'dy'dz'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}$$

2. General Theory : Solution in Rectangular Cartesian Coordinates

Using rectangular cartesian coordinates and the dimensionless forms due

† For references see (2).

‡ The numerical values are considered in (2).

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to Sneddon and Berry (3), the thermoelastic equations can be written as

$$\tau_{ij,\,j}+X_i = a\ddot{u}_i \qquad i, j = 1, 2, 3 \quad \text{(2.1)}$$

$$\tau_{ij} = \{(\beta^2-2)\Delta-b\theta\}\delta_{ij}+2\gamma_{ij} \qquad i, j = 1, 2, 3 \quad \text{(2.2)}$$

$$\Theta+\nabla^2\theta = f\dot{\theta}+g\dot{\Delta} \quad \text{(2.3)}$$

where the rectangular cartesian coordinates are denoted by x_1, x_2, x_3 and the other quantities have a similar convention.

If we eliminate the stresses between (2.1) and (2.2) we get an equation

$$(\beta^2-2)\Delta_{,i}-b\theta_{,i}+2\gamma_{ij,\,j}+X_i = a\ddot{u}_i \quad \text{(2.4)}$$

which can be differentiated with respect to x_i to give

$$\beta^2\Delta_{,ii}-b\theta_{,ii}+X_{i,\,i} = a\ddot{\Delta} \quad \text{(2.5)}$$

We now define the multiple integral transform

$$\bar{f}(\xi_1, \xi_2, \xi_3, \omega) = \frac{1}{4\pi^2}\int_{V_4} f(x_1, x_2, x_3, t)\exp\{i(\xi_i x_i+\omega t)\}dV \quad \text{(2.6)}$$

where $dV = dx_1dx_2dx_3dt$ and where the integration is taken over the entire $x_1x_2x_3t$-space.

Then by multiplying throughout equations (2.3), (2.4) and (2.5) by

$$(4\pi^2)^{-1}\exp\{i(\xi_i x_i+\omega t)\}$$

and integrating over V_4, we get the transformed equations

$$\bar{\Theta}-\xi^2\bar{\theta} = -i\omega f\bar{\theta}-i\omega g\bar{\Delta} \quad \text{(2.7)}$$

$$\bar{X}_i-i\xi_i(\beta^2-2)\bar{\Delta}+ib\xi_i\bar{\theta}-\xi^2\bar{u}_i-i\xi_i\bar{\Delta} = -a\omega^2\bar{u}_i \quad \text{(2.8)}$$

$$-i\xi_i\bar{X}_i-\beta^2\xi^2\bar{\Delta}+b\xi^2\bar{\theta} = -a\omega^2\bar{\Delta} \quad \text{(2.9)}$$

where $\xi^2 = \xi_1^2+\xi_2^2+\xi_3^2$.

The solutions of equations (2.7) and (2.9) are

$$\bar{\theta} = \frac{\omega g\xi_q\bar{X}_q+(\beta^2\xi^2-a\omega^2)\bar{\Theta}}{\mathscr{D}(\omega, \xi^2)} \quad \text{(2.10)}$$

$$\bar{\Delta} = \frac{-i\xi_q\bar{X}_q(\xi^2-i\omega f)+b\xi^2\bar{\Theta}}{\mathscr{D}(\omega, \xi^2)} \quad \text{(2.11)}$$

and we can now use (2.8) to give

$$\bar{u}_i = \frac{\bar{X}_i}{\xi^2-a\omega^2}-\frac{\{(\beta^2-1)(\xi^2-i\omega f)-i\omega bg\}\xi_i\xi_q\bar{X}_q}{(\xi^2-a\omega^2)\mathscr{D}(\omega, \xi^2)}+\frac{ib\xi_i\bar{\Theta}}{\mathscr{D}(\omega, \xi^2)} \quad \text{(2.12)}$$

where we have defined

$$\mathscr{D}(\omega, \xi^2) = (\beta^2\xi^2-a\omega^2)(\xi^2-i\omega f)-i\omega bg\xi^2. \quad \text{(2.13)}$$

Finally, application of the inverse transforms to the expressions (2.10) and (2.12) gives us the displacement and temperature fields

$$u_i = \frac{1}{4\pi^2}\int_{W_4}\left\{\frac{\bar{X}_i}{\xi^2-a\omega^2} - \frac{\{(\beta^2-1)(\xi^2-i\omega f)-i\omega bg\}\xi_i\xi_q\bar{X}_q}{(\xi^2-a\omega^2)\mathscr{D}(\omega,\,\xi^2)} + \frac{ib\xi_i\bar{\Theta}}{\mathscr{D}}\right\} \exp\{-i(\xi_p x_p+\omega t)\}dW \quad \text{...(2.14)}$$

$$\theta = \frac{1}{4\pi^2}\int_{W_4}\left\{\frac{\omega g\xi_q\bar{X}_q+(\beta^2\xi^2-a\omega^2)\bar{\Theta}}{\mathscr{D}(\omega,\,\xi^2)}\right\}\exp\{-i(\xi_p x_p+\omega t)\}dW \quad \text{...(2.15)}$$

where $dW = d\xi_1 d\xi_2 d\xi_3 d\omega$ and the integration is taken over the entire $\xi_1\xi_2\xi_3\omega$-space.

3. General Theory : Problems with Axial Symmetry

It is often interesting to consider problems in which, if we use cylindrical polar coordinates, there is symmetry about the z-axis. We shall therefore consider this type of problem in its most general form.

Assuming symmetry about the z-axis, the thermoelastic field equations can be written in the form

$$\beta^2\left\{\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right\}+\frac{\partial^2 u}{\partial z^2}+(\beta^2-1)\frac{\partial^2 w}{\partial r\partial z}+X_r-b\frac{\partial\theta}{\partial r}=a\frac{\partial^2 u}{\partial t^2} \quad \text{...(3.1)}$$

$$\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\beta^2\frac{\partial^2 w}{\partial z^2}+(\beta^2-1)\frac{\partial}{\partial z}\left\{\frac{\partial u}{\partial r}+\frac{u}{r}\right\}+X_z-b\frac{\partial\theta}{\partial z}=a\frac{\partial^2 w}{\partial t^2} \quad \text{...(3.2)}$$

$$\Theta+\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}+\frac{\partial^2\theta}{\partial z^2}=f\frac{\partial\theta}{\partial t}+g\frac{\partial}{\partial t}\left\{\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}\right\} \quad \text{.........(3.3)}$$

Thus, if we make use of the transforms

$$(\bar{u},\,\bar{X}_r)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(\zeta z+\omega t)}dzdt\int_0^{\infty}rJ_1(\xi r)(u,\,X_r)dr \quad \text{.........(3.4)}$$

$$(\bar{w},\,\bar{\theta},\,\bar{\Theta},\,\bar{X}_z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(\zeta z+\omega t)}dzdt\int_0^{\infty}rJ_0(\xi r)(w,\,\theta,\,\Theta,\,X_z)dr \quad \text{...(3.5)}$$

then equations (3.1), (3.2) and (3.3) transform to

$$(\beta^2\xi^2+\zeta^2-a\omega^2)\bar{u}-i(\beta^2-1)\xi\zeta\bar{w}-\bar{X}_r=b\xi\bar{\theta} \quad \text{..................(3.6)}$$

$$i(\beta^2-1)\xi\zeta\bar{u}+(\xi^2+\beta^2\zeta^2-a\omega^2)\bar{w}-\bar{X}_z=ib\zeta\bar{\theta} \quad \text{..................(3.7)}$$

$$-\bar{\Theta}+(\xi^2+\zeta^2-i\omega f)\bar{\theta}=i\omega g(\xi\bar{u}-i\zeta\bar{w}) \quad \text{..........................(3.8)}$$

The solutions of these equations are

$$\bar{u}=\frac{\{(\xi^2+\beta^2\zeta^2-a\omega^2)(\xi^2+\zeta^2-i\omega f)-i\omega bg\zeta^2\}\bar{X}_r}{(\xi^2-a\omega^2)\mathscr{D}(\omega,\,\xi^2)} + \frac{i\{(\beta^2-1)(\xi^2+\zeta^2-i\omega f)-i\omega bg\}\xi\zeta\bar{X}_z}{(\xi^2-a\omega^2)\mathscr{D}(\omega,\,\xi^2)}+\frac{b\xi\bar{\Theta}}{\mathscr{D}(\omega,\,\xi^2)} \quad \text{......(3.9)}$$

$$\bar{w} = \frac{-i\{(\beta^2-1)(\xi^2+\zeta^2-i\omega f)-i\omega bg\}\xi\zeta\bar{X}_r}{(\xi^2-a\omega^2)\mathscr{D}(\omega,\,\xi^2)} + \frac{\{(\beta^2\xi^2+\zeta^2-a\omega^2)(\xi^2+\zeta^2-i\omega f)-i\omega bg\xi^2\}\bar{X}_z}{(\xi^2-a\omega^2)\mathscr{D}(\omega,\,\xi^2)} + \frac{ib\zeta\bar{\Theta}}{\mathscr{D}(\omega,\,\xi^2)} \quad \ldots(3.10)$$

$$\bar{\theta} = \frac{g\omega(i\xi\bar{X}_r+\zeta\bar{X}_z)+(\beta^2\xi^2-a\omega^2)\bar{\Theta}}{\mathscr{D}(\omega,\,\xi^2)} \quad \ldots\ldots(3.11)$$

where $\xi^2 = \xi^2+\zeta^2$ and $\mathscr{D}$ is defined by (2.13).

The expressions for u, w and θ can now be obtained from (3.9), (3.10) and (3.11) by means of the transforms inverse to (3.4) and (3.5). In some of the most interesting applications the radial component of the body force X_r is zero. The expressions for the components of displacement and for the temperature distribution then become

$$u = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}d\zeta d\omega \int_0^{\infty} \frac{i\{(\beta^2-1)(\xi^2+\zeta^2-i\omega f)-i\omega bg\}\xi\zeta\bar{X}_z+b\xi\bar{\Theta}(\xi^2-a\omega^2)}{(\xi^2-a\omega^2)\mathscr{D}}\,\xi J_1(\xi r)d\xi \quad \ldots(3.12)$$

$$w = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}d\zeta d\omega \int_0^{\infty} \frac{\{(\beta^2\xi^2+\zeta^2-a\omega^2)(\xi^2-i\omega f)-i\omega bg\xi^2\}\bar{X}_z+ib\zeta\bar{\Theta}(\xi^2-a\omega^2)}{(\xi^2-a\omega^2)\mathscr{D}}\,\xi J_0(\xi r)d\xi \quad \ldots(3.13)$$

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}d\zeta d\omega \int_0^{\infty} \frac{g\omega\zeta\bar{X}_z+(\beta^2\xi^2-a\omega^2)\bar{\Theta}}{\mathscr{D}}\,\xi J_0(\xi r)d\xi \quad \ldots\ldots(3.14)$$

4. Effects due to Uneven Heating

Rectangular Cartesian Coordinates. It is easily seen from the foregoing work, that the components of displacement and the temperature distribution produced by a heat source $\Theta(x_1,\,x_2,\,x_3,\,t)$ in the absence of body forces are given by

$$u_i = \frac{ib}{4\pi^2}\int_{W_4} \frac{\xi_i\bar{\Theta}}{\mathscr{D}} \exp\{-i(\xi_p x_p+\omega t)\}dW \quad \ldots\ldots(4.1)$$

$$\theta = \frac{1}{4\pi^2}\int_{W_4} \frac{(\beta^2\xi^2-a\omega^2)\bar{\Theta}}{\mathscr{D}} \exp\{-i(\xi_p x_p+\omega t)\}dW \quad \ldots\ldots(4.2)$$

which are the expressions derived by Eason and Sneddon (**2**). There are two special cases which are of particular interest. These are the solutions to the steady-state and the two-dimensional problems, and they can be found in the reference mentioned above.

Problems with Axial Symmetry. From equations (3.12)-(3.14) it is immediately seen that the components of displacement and the temperature

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distribution due solely to the action of a heat source $\bar{\Theta}$ which is symmetrical about the z-axis are

$$u = \frac{b}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}d\zeta d\omega \int_0^{\infty} \xi^2\mathscr{D}^{-1}\bar{\Theta}J_1(\xi r)d\xi \quad \ldots\ldots\ldots\ldots\ldots\ldots(4.3)$$

$$w = \frac{ib}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \zeta e^{-i(\zeta z+\omega t)}d\zeta d\omega \int_0^{\infty} \xi\bar{\Theta}\mathscr{D}^{-1}J_0(\xi r)d\xi \quad \ldots\ldots\ldots\ldots\ldots\ldots(4.4)$$

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}d\zeta d\omega \int_0^{\infty} (\beta^2\xi^2-a\omega^2)\mathscr{D}^{-1}\bar{\Theta}\xi J_0(\xi r)d\xi \ldots\ldots(4.5)$$

where $\bar{\Theta}$ is defined by (3.5).

5. Effects produced by Time-dependent Body Forces: Rectangular Coordinates

It is readily seen from expressions (2.14) and (2.15) that the components of displacement and the temperature distribution due to body forces **X** are

$$u_i = \frac{1}{4\pi^2}\int_{W_4} \left\{\frac{\bar{X}_i}{\xi^2-a\omega^2} - \frac{\{(\beta^2-1)(\xi^2-i\omega f)-i\omega bg\}\xi_i\xi_q\bar{X}_q}{(\xi^2-a\omega^2)\mathscr{D}}\right\} \exp\{-i(\xi_p x_p+\omega t)\}dW \;\ldots\ldots\ldots(5.1)$$

$$\theta = \frac{g}{4\pi^2}\int_{W_4} \frac{\omega\xi_q\bar{X}_q}{\mathscr{D}} \exp\{-i(\xi_p x_p+\omega t)\}dW \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.2)$$

where $\bar{X}_r$ is defined by (2.6). It may be noted that, since the classical equations may be obtained from the linked equations by putting $g = 0$, the classical solution for the temperature distribution will be $\theta = 0$. This is, of course, obviously true, since in the classical solution the temperature is given by the heat conduction equation quite independently of the other equations, which contain the mechanical effects.

STEADY-STATE PROBLEM. If the body forces do not depend on the time t, so that

$$X_i = F_i(x_1, x_2, x_3)$$

then we find that

$$\bar{X}_i = (2\pi)^{\frac{1}{2}}\bar{F}_i\delta(\omega)$$

where $\bar{F}_i$ is the three-dimensional Fourier transform of F_i.

On substituting this value into (5.1) and (5.2) we see that $\theta = 0$, showing that a steady body force does not produce a thermal effect, and

$$u_i = \frac{1}{(2\pi)^{3/2}\beta^2}\int_{W_3} \frac{\beta^2\xi^2\bar{F}_i-(\beta^2-1)\xi_i\xi_q\bar{F}_q}{(\xi^2)^2} \exp\{-i\xi_p x_p\}d\xi \ldots\ldots\ldots(5.3)$$

where $d\xi = d\xi_1 d\xi_2 d\xi_3$. This latter expression can be converted to the standard expression for the statical solution (see e.g. Eason *et al.* (1), p. 581).

THE TWO-DIMENSIONAL PROBLEM. The solutions for the two-dimensional problem can be obtained by putting $\bar{X}_3 = 0$ and

$$\bar{X}_i = (2\pi)^{\frac{1}{2}}\bar{F}_i(\xi_1, \xi_2, \omega)\delta(\xi_3) \qquad i, j = 1, 2.$$

where $\bar{F}_i$ is the three-dimensional Fourier transform of the components of the body force F_i.

These solutions are

$$u_i = \frac{1}{(2\pi)^{3/2}}\int_{W_3}\left\{\frac{\bar{F}_i}{\gamma^2-a\omega^2} - \frac{\{(\beta^2-1)(\gamma^2-i\omega f)-i\omega bg\}\xi_i\xi_q\bar{F}_q}{(\gamma^2-a\omega^2)\mathscr{D}(\omega,\gamma^2)}\right\} \exp\{-i(\xi_p x_p+\omega t)\}dW_3 \quad \text{......(5.4)}$$

$$\theta = \frac{g}{(2\pi)^{3/2}}\int_{W_3}\frac{\omega\xi_q\bar{F}_q}{\mathscr{D}(\omega,\gamma^2)}\exp\{-i(\xi_p x_p+\omega t)\}dW_3 \quad \text{......(5.5)}$$

where $dW_3 = d\xi_1 d\xi_2 d\omega$ and $\gamma^2 = \xi_1^2+\xi_2^2$.

6. Effects produced by Time-dependent Body Forces: Axial Symmetry

From equations (3.12)-(3.14) we see that the solutions corresponding to the application of body forces X_z are

$$u = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}d\zeta d\omega\int_0^{\infty} \frac{i\{\beta^2-1)(\xi^2+\zeta^2-i\omega f)-i\omega bg\}\xi\zeta\bar{X}_z J_1(\xi r)d\xi}{(\xi^2+\zeta^2-a\omega^2)\mathscr{D}} \quad \text{......(6.1)}$$

$$w = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}d\zeta d\omega\int_0^{\infty} \frac{\{(\beta^2\xi^2+\zeta^2-a\omega^2)(\xi^2-i\omega f)-i\omega bg\xi^2\}\bar{X}_z\xi J_0(\xi r)d\xi}{(\xi^2+\zeta^2-a\omega^2)\mathscr{D}} \quad \text{......(6.2)}$$

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}d\zeta d\omega\int_0^{\infty}\frac{g\omega\zeta\bar{X}_z}{\mathscr{D}}\,\xi J_0(\xi r)d\xi \quad \text{......(6.3)}$$

whilst the complete solution, corresponding to the application of both components X_r and X_z, can be easily obtained by inverting the relevant terms of equations (3.9)-(3.11).

7. Quasi-static Solutions: Equivalent Heat Sources

The constant a is usually very small for problems in which the c.g.s. system of units is the natural system to use (see Eason and Sneddon (**2**)). We would therefore expect to get a good approximation to the exact solution by neglecting the terms in which a occurs. This approximate solution is known as the quasi-static solution. It can be seen from equation (2.1) that this approximation is physically equivalent to neglecting the inertia of the medium.

In the quasi-static approximation

$$\mathscr{D} = \beta^2\xi^2(\xi^2-i\omega f_1)$$

where $f_1 = f(1+\epsilon)$ and $\epsilon = bg/\beta^2 f$.

Using (6.3) the quasi-static approximation to the temperature distribution produced by a body force X_z is

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}d\zeta d\omega\int_0^{\infty}\frac{\omega g\zeta\bar{X}_z}{\beta^2\xi^2(\xi^2-i\omega f_1)}\,\xi J_0(\xi r)d\xi \quad \text{......(7.1)}$$

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Equivalent Heat Source. Let us consider the classical solution of the problem of the temperature distribution produced by a distributed heat source. The solution satisfies the heat conduction equation

$$\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}+\frac{\partial^2\theta}{\partial z^2}+\Theta=f\frac{\partial\theta}{\partial t}$$

which can be transformed, using (3.5), to the form

$$-\xi^2\bar{\theta}-\bar{\Theta}=-i\omega f\bar{\theta}$$

so that

$$\theta=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(\zeta z+\omega t)}d\zeta d\omega\int_0^{\infty}\frac{\bar{\Theta}}{\xi^2-i\omega f}\xi J_0(\xi r)d\xi \qquad (7.2)$$

Thus, by comparing equations (6.3) and (7.2), we can see that the same temperature distribution would be given by the body force X_z as would be given, in the classical theory, by a heat source Θ obeying the equation

$$\bar{\Theta}=\frac{\omega g\zeta(\xi^2-i\omega f)\bar{X}_z}{\mathscr{D}} \qquad (7.3)$$

We call the heat source Θ, given by (7.3), the equivalent heat source for the body force X_z. Having found the equivalent heat source, the problem is identical with a problem in the classical heat conduction theory.

Similarly we can see from (7.1) and (7.2) that the quasi-static solution can be obtained from a solution of the classical heat conduction equation. All we need do, is to replace f in that equation by f_1, and consider a heat source Θ given by

$$\bar{\Theta}=\frac{\omega g\bar{X}_z\zeta}{\beta^2(\xi^2+\zeta^2)} \qquad (7.4)$$

Equivalent Heat Source for a Point Force. As an illustration of the above, we shall calculate the heat source equivalent to the quasi-static treatment of a point force at the origin. Thus

$$X_z=\frac{1}{2\pi r}\delta(r)\delta(z)f(t)$$

and

$$\bar{X}_z=\frac{1}{(2\pi)^{3/2}}G(\omega) \qquad (7.5)$$

where

$$G(\omega)=\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}f(t)e^{i\omega t}dt \qquad (7.6)$$

Inversion of (7.6) and differentiation with respect to t gives the relation

$$f'(t)=\frac{-i}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\omega G(\omega)e^{-i\omega t}d\omega \qquad (7.7)$$

Now, using (7.4) and (7.5), the required heat source is given by

$$\bar{\Theta}=\frac{g\omega\zeta}{\beta^2(\xi^2+\zeta^2)}\frac{G(\omega)}{(2\pi)^{3/2}}$$

so that

$$\Theta = \frac{g}{\beta^2(2\pi)^{5/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\zeta z+\omega t)}d\zeta d\omega \int_0^\infty \xi J_0(\xi r)\frac{\omega\zeta}{\xi^2+\zeta^2}G(\omega)d\xi$$

$$= \frac{igf'(t)}{\beta^2(2\pi)^2}\int_{-\infty}^{\infty}\zeta e^{-i\zeta z}d\zeta\int_0^\infty \xi(\xi^2+\zeta^2)^{-1}J_0(\xi r)d\xi$$

using (7.7). Evaluating the integrals we finally get

$$\Theta = \frac{gf'(t)}{4\pi\beta^2}z(r^2+z^2)^{-3/2} \quad \text{.......(7.8)}$$

This result can be immediately generalised to give, in the quasi-static approximation, the heat source equivalent to the distributed body force

$$X_z = F(x, y, z)f(t)$$

This heat source is given by

$$\Theta = \frac{gf'(t)}{4\pi\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(z-z')F(x', y', z')dx'dy'dz'}{\{(x-x')^2+(y-y')^2+(z-z')^2\}^{3/2}} \quad \text{.......(7.9)}$$

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