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# A COMMUTATIVITY THEOREM FOR RINGS WITH INVOLUTION

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A ring with involution $R$ is an associative ring endowed with an antiautomorphism $*$ of period 2. One of the first commutativity results for rings with $*$ is a theorem of S. Montgomery asserting that if $R$ is a prime ring, in which every symmetric element $s = s^*$ is of the form $s = s^{n(s)}$ ($n(s) \geqq 2$), then either $R$ is commutative or $R$ is the $2 \times 2$ matrices over a field, which is a nice generalization of a well-known theorem of N. Jacobson on rings all of whose elements $x = x^{n(x)}$. Another classical commutativity theorem, due to I. N. Herstein, asserts that any ring $R$ with centre $Z$ such that every element $x$ satisfies $x - x^2 \cdot p_x(x) \in Z$, where $p_x$ is a polynomial having integral coefficients, is in fact a commutative ring. This theorem was extended to prime rings $R$ with $*$ in the following way: If for every symmetric $s$, $s - s^2 \cdot p_s(s) \in Z$, either $S \subseteq Z$ or $S$ is as in Montgomery's theorem. On the other hand Herstein's theorem was extended to the context of rings without involution in the following way: If $R$ is a semiprime ring and $c$ is a fixed element of $R$ such that $c$ commutes with $x - x^2 \cdot p(x)$ ($p$, depending on $c$ and $x$) then $c$ is a central element. In this paper, we offer an extension to rings with $*$ of the later commutativity theorem. We show the following.

THEOREM 5. *Let $R$ be any prime ring with $*$ having characteristic 0 or greater than 5. Suppose that a fixed element $c$ is such that for each symmetric $s = s^*$ there is $p$, a polynomial having integral coefficients, so that $c$ and $s - s^2 \cdot p(s)$ commute. If, further, $R$ is not the $2 \times 2$ matrices over a field then $c$ is in fact in the centre $Z$ of $R$.*

At the end of the paper we comment on the restriction about the characteristic of $R$ and the nature of the polynomial $p$ intervening in Theorem 5. Essential to this paper will be a result of ours concerning subalgebras preserved by the group of unitaries in matrix algebras with $*$ over division rings containing more than 5 elements.

*Definitions, Notations, and Conventions.* Throughout the paper all rings have characteristic 0 or greater than 5. Except in one case, all homomorphisms preserve the involution and the characteristic assumption. All polynomials $p$ have integral coefficients and all subrings $A$ are $*$-closed ($A = A*$). For $a \in R$, we let $C(A) = C_R(a) = \{x \in R | xa = ax\}$ (*centralizer* of $a$ in $R$). For

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$a, b \in R$, $[a, b] = ab - ba$ (*commutator*). $S$, $K$, $Z$ stand respectively for the symmetrics, the skews, and the central elements of $R$. For $A$ a subring of $R$, $Z(A)$ or $Z_A$ will denote the centre of $A$ viewed as a ring, $S(A)$ or $S_A$, the symmetrics of the ring $A$, and $K(A)$ or $K_A$, the skews of the ring $A$. Finally, $X^+$ (resp. $X^-$) will denote the subset of symmetrics (resp. the skews) in the subset $X$ of $R$.

*Definitions* 1.a) A *co-integral expression* in $x \in A$ is a polynomial expression of the form

$$x^k - x^{k+1} \cdot p(x);$$

$p$ a polynomial having integral coefficients. The integer $k$ is called the *index*.

b) When for every $x \in R$ there is some co-integral expression belonging to the fixed subring $A$ of $R$, we shall say that $R$ is *co-integral* over $A$. If, moreover, the expressions can be taken with fixed index $r$, we use the term "*co-integral of index r*".

c) The ring $R$ is said to be $*$-*co-integral* (resp. $*$-*co-integral of index r*) if for each symmetric $x \in R$, there is some co-integral expression in $x$ (resp. co-integral expression in $x$ of index $r$) belonging to $A$.

*Definitions* 2 (Main definitions). Let $R$ be any ring. Set:

a) $T = T_R = \{a \in R | \forall x \in R \; \exists p; [a, x - x^2 \cdot p(x)] = 0\}$
$= \{a \in R | R, \text{ co-integral of index 1 over } C_R(a)\}$

b) $H = H_{(R,*)} = \{a \in R | \forall x \in S \; \exists p; [a, x - x^2 p(x)] = 0\}$
$= \{a \in R | R, *\text{-co-integral of index 1 over } C_R(a)\}$

The subsets $T$ and $H$ are called respectively *co-hypercenter* and $*$-*co-hypercenter* of $R$.

**1. Basic facts.** In this section we assemble some basic properties of the $*$-co-hypercentre true for arbitrary rings or on the other extreme for simple artinian rings. We begin with formal facts using closure of the co-integral expressions of index 1 under composition of polynomials and standard properties of commutators.

*Remarks* 1.

a) $\forall a \in H, \forall x \in S, \forall n \geqq 1, \exists p;$

(i) $[a, x - x^{2n} \cdot p(x)] = 0.$

In particular if $s$ is a symmetric nilpotent ($s^n = 0$), then $[a, s] = 0$.

b) $\forall a_1, \ldots, a_n \in H, \forall x = x*, \exists p$

(ii) $[a_i, x - x^2 \cdot p(x)] = 0, \quad \forall i = 1, \ldots, n.$

c) $\forall a \in H, \forall x_1, \ldots, x_n \in S, \exists p$

(iii) $[a, x_i - x_i^2 \cdot p(x_i)] = 0, \quad \forall i = 1, \ldots, n.$

Remark 1-c) shows that $H$ is a subring of $A$, containing evidently the co-hypercenter $T = T_R$, and hence, containing the centre $Z$ of $R$. We record these facts as follows.

*Remark* 2. For any ring $R$, the $*$-co-hypercenter $H$ is a subring containing the co-hypercenter, and contained in the centralizer $C(N^+)$, of the symmetric nilpotents $N^+$ of $R$.

Remark 1-b) yields another important property of the $*$-co-hypercenter $H$; namely, $H$ viewed as a ring, will satisfy a polynomial identity of fairly low degree, that it is now convenient to make explicit. Let $H_0$ be any finitely generated $*$-closed subring of $H$ generated by $a_1, \ldots, a_n$. Given $x = x^* \in H_0 \subseteq R$, there is $p(t)$ with

$$[a_i, x - x^2 \cdot p(x)] = 0, \quad \text{for all } i = 1, \ldots, n.$$

Since the $a_i$'s generate $H_0$, $x - x^2 \cdot p(x) \in Z_0 = Z(H_0)$ follows. By the results in [**4**, p. 1125], $H_0$ satisfies the polynomial identity

$$[s_1, s_2, s_3, s_4]^2 \in Z_0 \cap N^+(H_0), \quad \text{for all } s_i = s_i^* \in S(H_0),$$

where $[s_1, s_2, s_3, s_4]$ is the value of the standard polynomial in four non-commuting variables for the specialization $s_1, s_2, s_3, s_4$ in $H_0$. Since $N^+(H_0) \subseteq N^+(R)$, and since $N^+(R)$ centralizes $H$, we get the following.

*Remark* 3. $H$, viewed as a ring with $*$, satisfies the polynomial identity:

$$\forall s_1, s_2, s_3, s_4 \in S(H), [s_1, s_2, s_3, s_4]^2 \in Z \cap N^+ \subseteq Z^+(H).$$

Two more general facts are in order.

*Remarks* 4a) For every subring $R_0 = R_0^*$ of $R$, $H \cap R_0 \subseteq H_{(R_0,*)}$.
b) If $e = e^*$ is a symmetric idempotent, then $eHe \cap H_{(eRe,*)}$.

We digress for a while on *quasi-unitaries*. Recall that if $R$ is a ring with 1, the element $x$ is called *unitary*, if $x$ is an invertible element such that $xx* = 1$. It is natural in the absence of 1, to call $a$ a *quasi-unitary* element, if $a + a^* + aa^* = a + a^* + a^*a = 0$. Such an element induces the quasi-inner automorphism

(1) $x \rightarrow (1 + a)x(1 + a)^{-1} = x + ax + xa* + axa*,$

coinciding with the inner automorphism induced by the unitary $1 + a$ if $R$ happens to possess a unity 1. Generally the automorphism in (1) preserves $S$, $K$, it leaves the elements of $Z$ invariant, and commutes with the integral polynomial expressions. It follows that this automorphism preserves $H$, for all quasi-unitaries. In accordance with [**2**], we shall call $H$ an *invariant* subring, if it is preserved by the quasi-inner automorphisms induced by all quasi-unitary elements of $R$. We have shown:

*Remark* 5. $H$ is an invariant subring.

The invariant property of $H$ will be exploited in what follows for $R$, a simple artinian ring, viewed as the $n \times n$ matrices over a division ring $D$. The involution $*$ induces an involution on $D$. Since $R$ is by our convention of characteristic greater than 5, it follows that $D$ contains more than 5 elements and is 2-torsion free. Thus [**2**] applies and yields the following.

*Remarks* 6([**2**]). Let $W$ be any invariant subalgebra with centralizer $V$ of $R = (D_n, *)$.

1) For $n > 2$, either $W \subseteq Z$, or $V = Z$.

2) For $n = 2$, either $W = O$, $Z$, or $V = Z$, or else the ground involution is the identity mapping, and

$$W = Z + \left\{\begin{bmatrix} 0 & x \\ -qx & 0 \end{bmatrix}\right\}_{x \in D} = W^*$$

contains no symmetric matrix but the scalars.

3) If $W$ satisfies any polynomial identity, then $W = Z$ or $R$, or else $W$ is as in 2)-i).

To be able to apply Remarks 6, we must handle the case $n = 1$. This is done in our first proposition.

PROPOSITION 1. *If $R$ is a division ring either $S \subseteq Z$ (so $R = H$) or $H = Z$.*

*Proof.* Suppose that $S \not\subseteq Z$, but $H \neq Z$. There must be $a \in H$, with $A = C_R(a) \neq R$. We claim that every symmetric $s = s*$ in $R$ has some power $s^{n(s)}$ in $A$. Clearly we may assume $s \notin A$. If $F$ is the subfield generated by $s$ over the subfield $Z^+$ of central symmetrics, then $F$ contains strictly $F \cap A$, which is a subfield. Now $R$ is $*$-co-integral of index 1 over $A$ since, in fact, $a \in H$. Consequently $F$ is co-integral of index 1 over the subfield $F_0 = F \cap A$ (that is, for every $x \in F$, there is a co-integral expression of index 1 in $x$ belonging to $F_0$). By a general result of fields [**8**], $F$ is algebraic over a finite field. Thus $s$ is a root of unity, so certainly $s^{n(s)} \in A$, some $n(s) \geqq 1$. Since $A \neq R$, by a theorem of Herstein and ours [**3**], all norms and traces of $R$ would be central, and consequently in view of the 2-torsion freeness, $S \subseteq Z$, which it is not. This shows that $H = Z$ necessarily as wished.

PROPOSITION 2. *If $R$ is simple artinian and if $R = H$, then either $S \subseteq Z$, or $R$ is the $2 \times 2$ matrices over an algebraic field extension of a finite field, with $*$ a canonical transpose admitting no symmetric nilpotents.*

*Proof.* If $R = H$, then by Remark 3, $R$ is PI, so, by a well-known result of I. Kaplansky, $R$ is finite dimensional over the centre, whence finitely generated over the centre. By the argument used in the proof of Remark 3, $s - s^2 \cdot p_s(s) \in Z$ follows, all $s = s*$. We then quote [**4**, Theorem 3].

We can now describe fully the simple artinian case.

THEOREM 1. *If $R$ is a non-commutative simple artinian ring, either $H = Z$ or $H = R$. In the latter case, $R$ must be of one of the following types:*

(1) *$R$ is a division ring whose symmetrics coincide with the centre, so $R$ is a 4-dimensional division ring.*

(2) *$R$ is the $2 \times 2$ matrices over a field, which is an algebraic extension of a Galois field, with $*$ a canonical transpose admitting no symmetric nilpotents.*

(3) *$R$ is the $2 \times 2$ matrices over a field with $*$ the symplectic involution so that the symmetrics coincide with the centre.*

*Proof.* By Proposition 1, we may assume that $R$ has rank $n$ greater than 1. If $n > 2$, by Remarks 6, $H = Z$ or $R$. The latter case being ruled out by Proposition 2, we get $H = Z$ necessarily.

If $n = 2$. Either $*$ is canonical transpose or symplectic. In the latter case, $S = Z$ necessarily, so evidently $R = H$ is of type (3). In the first case, if $H \neq Z$, necessarily $H = R$ or

$$\text{(i) } H = Z + \left\{\begin{bmatrix} 0 & x \\ -qx & 0 \end{bmatrix}\right\}_{x \in D},$$

where $D$ is a field, and $* = *(q_1, q_2)$ is defined by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} a & cq_1q_2^{-1} \\ bq_2q_1^{-1} & d \end{bmatrix}.$$

If $H = R$ we use, again, Proposition 2 to get that $R$ is of type (2). We are left with the case (i), that we shall now rule out.

For let $0 \neq \begin{bmatrix} 0 & x \\ -qx & 0 \end{bmatrix} \in H$. Given $a \in D$, a field, $\underline{s} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ is a symmetric matrix. By the assumption, for some polynomial $p(t)$ with integral coefficients, $0 \neq \begin{bmatrix} 0 & x \\ -qx & 0 \end{bmatrix}$ commutes with

$$\underline{s} - \underline{s}^2 \cdot p(\underline{s}) = \begin{bmatrix} a - a^2 \cdot p(a) & 0 \\ 0 & 0 \end{bmatrix}.$$

This is possible only if $a = a^2 \cdot p(a)$. Thus $D$ is co-integral over the zero subring. It follows that $D$ is algebraic over a finite field.

If $R$ contained some symmetric nilpotent matrix, the subalgebra $W$ generated by all these would be a non-zero invariant subalgebra obviously not of the form (i), so necessarily would coincide with $R$. Since $H$ centralizes $W$, this contradicts the relation $H \nsubseteq Z$. This shows that $R$ contains no symmetric nilpotents. Because $D$ is algebraic over a finite field so will be $R$, and in the absence of symmetric nilpotents, every symmetric in $R$ becomes co-integral of index 1 over the zero subring (in fact, of the form $s = s^{n(s)}$, $n(s) \geqq 2$). But, in the latter case, $H = R$, which is ruled out. With this the theorem is proved.

We inspect the nature of the simple artinian ring $R$ in the special case $H^+ (= H \cap S) \nsubseteq Z$. To begin with, $R$ can not be of type (1) in Theorem 1, or type (3). By Theorem 1, $R$ is necessarily of type (2). Something more can be said about type (2). Since $R$ contains no symmetric nilpotents, $R$ contains no skew nilpotents either. For otherwise, the involution $*$ would induce a non-trivial involution on the ground field, forcing $*$ to be of the second kind. On the other hand, we claim that every commutative subring $V$ of $R$ consisting entirely of symmetrics must be central. For by Remarks 6, adjoining the center $Z$ to $V$, we get the subalgebra

$$W = V + Z \subseteq Z + \left\{\begin{bmatrix} 0 & x \\ -qx & 0 \end{bmatrix}\right\}_{x \in D},$$

and consequently $V^+ \subseteq W^+ \subseteq Z$. We record these facts in the following corollary.

COROLLARY 1. *Any simple right artinian ring $R$ such that $H^+ \not\subseteq Z$, is necessarily of type (2) as in Theorem 1. It follows that $R$ contains no skew or symmetric nilpotents. Moreover, every invariant commutative subring of symmetrics must be central.*

**2. Nil radical of $H$.** At the outset (Theorem 5) $R$ is taken to be a prime ring. However, at later stages of the paper it will be necessary for us to deal with certain subrings of $R$ that can be of arbitrary prime radical. For this reason we shall relax throughout the prime condition by $*$-prime (e.g. non-zero $*$-closed ideals in $R$). We wish to show that $H$, viewed as a ring, contains no non-zero nil ideals. This is carried out by looking first at the $*$-prime, not prime, case. As one would expect, the prime case is more complex, and will be studied alone.

2.1 *$*$-prime case.* Suppose that $R$ contains a non-zero ideal $I$ of the type $I \cap I^* = 0$. Denote by $\bar{R}$ the factor ring $R/I$ (the involution $*$ is disregarded in $\bar{R}$), $\bar{H}$, the image of $H$ in $\bar{R}$, and by $J$, the image of $I^*$ in $\bar{R}$.

PROPOSITION 3. *For every $\bar{a} \in \bar{H}$, and every $\bar{x} \in J$, a non-zero ideal of $\bar{R}$,* $[\bar{a}, \bar{x} - \bar{x}^2 p(\bar{x})] = 0$.

*Proof* (sketched). Pick any $x \in I^*$, and apply the basic property of $a \in H$ via the symmetric $x \oplus x^* \in I \oplus I^*$. Then pass to $R/I$.

In [**1**] we have shown that if $R$ is any semiprime ring then $T = Z$. This property is used freely throughout. Proposition 3 suggests the following.

*Question.* If $R$ is a prime ring and $a$ is a fixed element of $R$ such that for some non-zero ideal $J$ of $R$, $J$ is co-integral of index 1 over $J \cap C_R(a)$, does it follow that $a \in Z$?

All our concern in this section is the study of the nilpotents and for these special elements we get indeed that they commute with such elements $a$. This is the content of the following result.

PROPOSITION 4. *Let $R$ be a $*$-prime, not prime, ring. Then the $*$-co-hypercentre has the following properties.*

1) *$H$ centralizes all symmetric nilpotents of $R$.*

2) *$H$ contains no symmetric nilpotents* (*other than* 0).

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*Proof.* It suffices to prove this for the image $\bar{H}$ of $R$ in the factor ring (deprived of involution) $\bar{R} = R/I$, with $I \neq 0$ an ideal verifying $I \cap I^* = 0$.

1) By Proposition 3, $\bar{H}$ centralizes all nilpotents in $J = I^*/I$. Then let $e = e^2 \in \bar{H}$. If $y = ex - exe$, $x \in J$, then $y$ is a square-zero element in $J$. Then $ya = ay$, $\bar{a} \in \bar{H}$. Thus $(ex - exe)\bar{a}e = \bar{a}(ex - exe)e = 0$, for all $x \in J$. Consequently $eJ(1 - e)\bar{a}e = 0$. Since $\bar{R}$ is prime, if then $e \neq 0$, $(1 - e)\bar{a}e = 0$ follows, that is, $\bar{a}e = e\bar{a}e$. By symmetry, $e\bar{a} = e\bar{a}e = \bar{a}e$, for all $e = e^2$, and $\bar{a} \in \bar{H}$.

2) Suppose that $\bar{a}^2 = 0$, $\bar{a} \in \bar{H}$. By an argument similar to [**1**], it can be shown that $\bar{a} \cdot J$ is co-integral of index 2 over the zero subring. This forces $R$ to be primitive with a socle containing $\bar{a} \cdot J$. It follows that $J$ is primitive with socle. If $J$ has a unity, by the primeness of $\bar{R}$, $\bar{R} = J$, placing $\bar{a}$ in $T(\bar{R}) = Z(\bar{R})$, so $\bar{a} = 0$. If, on the other hand, $J$ has no unity, the socle $J_0$ of $J$ must be generated by nilpotents centralized by $\bar{a}$. Thus $\bar{a}$ centralizes the ideal $J_0JJ_0$ of $R$, giving $\bar{a} \in Z$, whence $\bar{a} = 0$.

2.2 *Prime case.* We take $R$ to be prime, and let $P = P^*$ be a nil ideal of $H$ viewed as a ring. Concerning the center $Z_H$ of $H$, or the $*$-center $Z_H{}^+$ of the ring $H$, it is convenient to notice that $Z_H$ (as well as $H$) contains $P^+$, and contains along with $2x$, the element $x$ (by 2-torsion freeness). Also, since the quasi-unitaries induce automorphisms on $H$, then $Z_H$, $Z_H{}^+$ are invariant subrings. In this connection we recall a remark due to Herstein [**7**, Theorem 6.1.1].

*Remark* 7. If $W$ is any invariant subring of $R$ such that $2x \in W$ implies $x \in W$, then for every quasi-unitary skew $k$ of $R$, and every $a \in W$,

$$(1 - k)^{-1}[a, k](1 + k)^{-1} \in W.$$

We proceed to a very special case that will be used partly in this section, and fully at later parts of the paper.

PROPOSITION 5. *If $R$ is a prime PI ring such that $H^+ \not\subseteq Z$, then necessarily $R$ is as in Theorem* 1, *type* (2). *Consequently $R$ contains no symmetric nilpotents.*

*Proof.* We claim that $R$ cannot be a domain. If not, take any $a \in H$, $a \notin Z$. For every $s = s^* \in R$, $Z^+[s]$ is a commutative domain, which is co-integral of index 1 over $Z^+[s] \cap C_R(a)$. By [**4**, Lemma 5], the field of quotients of $Z^+[s]$ is radical over the subfield of quotients of $Z^+[s] \cap C_R(a)$. Thus for some integer $n$, and some $u, v \neq 0 \in C_R(a)$, $us^{n(s)} = v \in C_R(a)$. Consequently

$$0 = [a, v] = [a, us^n] = u[a, s^n].$$

It follows that $[a, s^{n(s)}] = 0$, that is, $s^{n(s)} \in C_R(a)$, all $s = s^* \in R$. If $\bar{R} = R(Z^+)^{-1}$ is the ring of fractions of $R$, we get a division ring, for $R$ satisfies a polynomial identity. By the above, for every symmetric $\bar{s}$ in $\bar{R}$, $\bar{s}^{n(s)} \in C_{\bar{R}}(a) = C_R(a)(Z^+)^{-1}$. Since $a \notin Z$, $C_{\bar{R}}(a) \neq \bar{R}$. By [**3**, Theorem 1], all symmetrics in $\bar{R}$ are central, contradicting the assumption on $R$. This shows that $R$ cannot be a domain. Equivalently $\bar{R}$ is a simple finite dimensional algebra having rank greater than 1.

Let $W$ be the subalgebra generated by the symmetric idempotents. Clearly $W$ is an invariant subalgebra. Now the centralizer $V$ of $W$ is necessarily $Z(\bar{R})$. This is certainly true if $\bar{R}$ has rank $\geqq 3$. For $\bar{R}$ of rank 2, the case where $*$ is symplectic in $\bar{R}$ must be ruled out as $S(R) \not\subseteq Z(R)$. Thus by Remarks 6, if $V \neq Z$ necessarily $W$ has all its diagonal matrices with equal diagonal coefficients, which is evidently false as $*$ is canonical transpose.

Now let $s \in H$ ($s$ can be any element in $H$) and let $e = e* = e^2 \in \bar{R}$, with $[s, e] \neq 0$. Write $e = f \cdot z_0^{-1}$, $f = f* \in S(R)$, $z_0 \in Z^+(R)$. Given $z \in Z^+$, it is clear that $f \cdot z \in S(R)$. By the basic property of $s$, we have $[s, f \cdot z] = [s, (fa)^2 p(fz)]$, for some $p(t)$. Now

$$(fz)^2 = f^2z^2 = z^2(e \cdot z_0)^2 = ez_0^2z, \ldots, (fz)^n = e(z_0z)^n.$$

Thus

$$[s, ez_0z] = [s, e(z_0z)^2p(z_0z)];$$
$$(z_0z - (z_0z)^2p(z_0z))[s, e] = 0;$$
$$z_0z = (z_0z)^2p(z_0z);$$
$$z = z_0z^2p(z_0z);$$
$$z = z^2z_1, \quad \text{for some } z_1 \in Z^+.$$

Thus $Z^+$ is a field, so $Z$ is a field, giving $\bar{R} = RZ^{-1} = R$. We then quote Theorem 1.

If $R$ is a $PI$ $*$-prime ring with $H \not\subseteq Z$, what can be said about $R$? To begin with, if $S \subseteq Z$, this forces $R$ to be a prime ring. For if in the contrary case, we get trivially that $R = Z$, contrary to the assumption $H \not\subseteq Z$. Since $R$ is a prime non-commutative ring verifying $S \subseteq Z$, it follows that $R$ must be an order in the $2 \times 2$ matrices with the symplectic involution. Next suppose that $S \not\subseteq Z$. The first argument in the proof of Proposition 5 shows that $R$ cannot be a domain. Thus $R$ must be simple artinian verifying $S \not\subseteq Z$ and $H \not\subseteq Z$. By Theorem 1 from Section 1, necessarily $R$ must be of type (2) of that theorem. We have shown the following.

COROLLARY. *If $R$ is a $PI$ $*$-prime ring such that $H \not\subseteq Z$, then necessarily $R$ is a prime ring, which is either an order in the $2 \times 2$ matrices with symplectic involution, or simple artinian of type* (2) *in Theorem* 1.

PROPOSITION 6. *Let $R$ be a prime ring with a square-zero symmetric $a$ such that $aka = 0$. Then $R$ contains a $*$-closed prime subring $R_0$ containing $a$, which is an order in the $2 \times 2$ matrices over a field.*

*Proof.* This proposition is essentially a special case of a theorem of S. Montgomery [**7**, Theorem 2.5.1]. For the convenience of the reader we give a self-contained proof. By an observation due to Herstein and Montgomery,

$R$ satisfies the generalized polynomial identity $[ax, ay]^2 = 0$, all $x, y \in R$. By a theorem of Martindale [**10**], the central closure $Q = R \cdot C$ of $R$ is a primitive ring with socle, whose underlying division ring $D$ must be a field, and $a$ is of rank $= 1$. In fact, $aQ$ satisfies the polynomial identity $[x', y']^2 = 0$, all $x', y' \in aQ$. If then $aQ = eQ$, $e = e^2 \in$ Socle $(Q)$, then $eQe$ is primitive with polynomial identity $[x, y]^2 = 0$, giving that $eQe = D$ is a field.

Write $e = ay$, $y \in Q$. We have $e* = y*a$, and $e*e = y*a^2y = 0$ follows. If $f = e + e* - ee* = (e - \frac{1}{2}ee*) + (e - \frac{1}{2}ee*)*$, a routine computation shows that: $e_1 = e_1^2 = e - \frac{1}{2}ee*$; $e_1e_1^* = e_1^*e_1 = 0$; $e_1Q = eQ$. Consequently $fQf = e_1Qe_1 \oplus e_1^*Qe_1^* \approx D_2$. Also, $a \in fQf$. For the equality $aQ = eQ = e_1Q$ gives $fa = e_1a + e_1^*a = q + e_1^*a = a + (e*a - \frac{1}{2}ee*a) = a$, since $e*a = (y*a)a = 0$, and similarly $af = a$.

Since $Q$ is a subring of the ring of quotients of $R$, for every $x \in Q$, there is an ideal $0 \neq I$ of $R$ such that $xI \subseteq R$. In particular there must be $J \neq 0$ with

$$fJ \subseteq R \quad \text{and} \quad J*f \subseteq R = R.$$

Then $fJJ*f \subseteq R$, where $JJ* = I \neq 0$ is an ideal of $R$. Let $R_0 = R \cap fQf$. Clearly $R_0$ is a subring containing $a$, satisfying the standard identity in 4 variables. If $uR_0v = 0$; $u, v \in R_0$, then $u(fJJ*f)v = 0$. Since $u, v \in R_0 \subseteq fQf$, $uf = u$ and $fv = v$, so $u(JJ*)v = uIv = 0$. Since $I$ is an ideal of the prime ring $R$, either $u = 0$ or $v = 0$. This shows that $R_0 = R_0^*$ is a prime ring, which by the above satisfies the standard identity in 4 variables. Now $R_0$ contains the square-zero element $a$. Consequently $R_0$ is an order in the $2 \times 2$ matrices over a field.

COROLLARY. *If $R$ is prime with $a = a^*$ a square-zero element in $H$ such that $aKa = 0$, then $a = 0$ necessarily.*

*Proof.* If $a$ were $\neq 0$, by Proposition 6, there is a prime PI subring $R_0 = R_0^*$ containing $a$. Clearly $a = a* \in H(R_0)$, with $a^2 = 0$, so $H^+(R_0) \not\subseteq Z(R_0)$. In view of Proposition 5, $R_0$ contains no symmetric nilpotents, a contradiction. We have to agree that $a = 0$ necessarily.

PROPOSITION 7. *If $R$ is prime, then $H$ contains no non-zero symmetric nilpotents.*

*Proof.* The proof breaks in several steps.

*Step* 1. *If $R$ contains an idempotent $e$ with $e \oplus e* = 1$, then $H$ contains no symmetric nilpotents.*

Let $T_{eRe}$ be the co-hypercenter of $eRe$, and let $Z_{eRe}$ be the center of $eRe$. We have $T_{eRe} = Z_{eRe}$. Given $a \in H$, and $x \in eRe$, we have

$$\begin{aligned} 0 &= [a, (x + x*) - (x + x*)^2p(x + x*)] \\ &= [a, x - x^2p(x)] + [a, x* - x*^2p(x*)]. \end{aligned}$$

Then $[eae, x - x^2p(x)] = 0$ necessarily, placing $eae$ in $T_{eRe} = Z_{eRe}$. Now let

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$a \in Z_H^+$ ($= *$-center of $H$) and let $k \in K$. The element $k_1 = eke*$ is a square-zero skew. Since $k_1$ is quasi-unitary, $(1 + k_1)a(1 - k_1) \in Z_H$ follows, that is, $k_1a - ak_1 - k_1ak_1 \in Z_H$. Changing $k_1$ to $2k_1$ gives $[k_1, a] \in Z_H$. Thus $[a, [a, k_1]] = 0$. On the other hand,

$$[a, eke + e*ke*] = [eae + e*ae* + e*ae + e*ae, eke + e*ke*]$$
$$= [e*ae + eae*, eke + e*ke*],$$

for $[eke, eae] = [e*ke*, e*ae*] = 0$. Thus

$$[a, eke + e*ke*] = [eae* + e*ae, eke + e*ke*]$$
$$= eae*ke* + e*aeke - ekeae* - e*ke*ae$$
$$= (eae*ke* - ekeae*) + (e*aeke - e*ke*ae).$$

Now

$$s_1 = eae*ke* - ekeae* = eae*ke + (eae*ke*)*$$

is a square-zero symmetric. Thus $[a, s_1] = 0$, and similarly for $s_2 = e*aeke - e*ke*ae$. From this $[a, [a, eke + e*ke*]] = 0$. Since we had $[a, [a, k_1]] = 0$, we get $[a, [a, k]] = 0$, for all $k \in K$.

If then $a = a^*$ is a square-zero element in $H$, $a \in Z_H^+$ follows giving $[a, [a, k]] = -2aka = 0$, so $aka = 0$, for all $k \in K$. In view of Proposition 5, $a = 0$ necessarily.

*Step* 2. *If $e = e^2$ is an idempotent of $R$ such that $ee* = 0$, and if $a$ is a square-zero symmetric in $H$, then $eae* = e*ae = 0$.*

For let $e_1 = e - \frac{1}{2}e*e$, $e_1* = e* - \frac{1}{2}e*e$. It was already observed that $e_1 \oplus e_1* = f$ is a symmetric idempotent. If $R_1 = fRf$, it is clear that $R_1$ contains in its $*$-co-hypercenter $H_1 = fHf$.

Since $a \in Z_H^+$, $(1 - 2f)a(1 - 2f) \in Z_H$ follows, giving $b = af + fa - 2faf \in Z_H^+$. Consequently $[a, b] = 0$. Since $a^2 = 0$, we get $afa - 2afaf = afa - 2fafa$; $(af)^2 = (fa)^2$. Thus $a_1 = faf$ is a symmetric cube-zero in $H_1$. Consequently $a_1 \in Z_{H_1}$, the center of $H_1$, By Step 1, $a_1 = faf = 0$ necessarily.

Now $f = e_1 + e_1* = e + e* - e*e$, where $e*e$ is a symmetric nilpotent commuting with $a \in H$. Thus

$$0 = faf = (e + e* - e*e)a(e + e* - e*e)$$
$$= (eae + eae* - ee*ea) + (e*ae + e*ae* - e*e*ea)$$
$$- (e*eea + e*ee*a + e*ee*ea)$$
$$= eae + eae* + e*ae + e*ae* - 2ae*e.$$

Right multiplication by $e*$ combined with the relation $ee* = 0$ gives

$$eae* + e*ae* = 0;$$
$$eae* = -e*ae* = e*(eae*) = (e*e)ae* = ae*ee* = 0;$$
$$e*ae* = 0; \quad eae = 0;$$
$$0 = eae + eae* + e*ae + e*ae* - 2e*ea; = e*ae - 2e*ea;$$
$$e*ae = 2e*ea = (2e*ea)e = 2e*(eae) = 0.$$

*Step* 3. *If* $a^2 = 0$ *with* $a = a* \in H$, *then* $aKa = 0$.

Let $v = v_1 + v_2$ with $v_i \in R$, $v_1 \cdot v_2 = 0$. For every $n \geqq 1$, we have $v^n = v_1^n + v_2^n + v_2^{n-1} \cdot v_1$. Setting $v = [k, a]$, we get for

$$v_1 = ka, v_2 = -ak = v_1^*, v_1v_2 = -ka^2k = 0;$$
$$v^n = (ka)^n + (-1)^n(ak)^n + (n-1)(-1)^{n-1}(ak^{n-1}(ka)).$$

Now

$$[a, v] = 2aka; [a, v^2] = [a, v^4] = \ldots = [a, v^{2n}] = 0;$$
$$[a, v^{2k+1}] = 2a(ka)^{2m+1}.$$

Since $v = v*$, we get by the basic definition that

$$2aka = [a, v] = [a, v^2p(v)] = 2\{\alpha_1a(ka)^3 + \alpha_2a(ka)^5 + \ldots\};$$
$$aka = \alpha_1a(ka)^3 + \alpha_2a(ka)^5 + \ldots;$$
$$(ak)^2 = \alpha_1(ak)^4 + \alpha_2(ak)^6 + \ldots = (ak)^2p((ak)^2)(ak)^2.$$

Let $e = e^2 = (ak)^2p((ak)^2)$. We have $e*e = (ka)^2p((ka)^2) \cdot e = 0$. By Step 2, $eae* = 0$. Explicitly we get

$$\begin{aligned} 0 = y = eae* &= (ak)^2p((ak)^2)(ak)^2a(ka)^2p((ka)^2)(ka)^2 \\ &= \alpha_1^2(ak)^2a(ka)^2 + (\alpha_1\alpha_2(ak)^2a(ka)^4 + \alpha_1\alpha_2(ak)^4a(ka)^2) + \ldots \\ &= (\alpha_1^2(ak)^4 + 2\alpha_1\alpha_2(ak)^6 + \ldots) \cdot a \\ &= (\alpha_1(ak)^2 + \alpha_2(ak)^4 + \ldots)^2 \cdot a = p^2((ak)^2) \cdot a, \end{aligned}$$

so,

$$\begin{aligned} e = (ak)^2 \cdot p((ak)^2) = (ak)^4 \cdot p^2(ak)^2 &= p^2(ak)^2 \cdot (ak)^4 \\ &= p^2((ak)^2) \cdot a(ka)^3k = 0; \end{aligned}$$
$$(ak)^2 = e(ak)^2 = 0; \quad (ka)^3 = k(ak)^2a = 0;$$
$$aka = \alpha_1a(ka)^3 + \alpha_2a(ka)^5 + \ldots = 0.$$

Having shown that $aka = 0$, we then quote the corollary to Proposition 6, which completes the proof.

2.3 *Skew nilpotents in* $H$. One difference from the symmetric case is that $H$ could very well contain non-zero skew nilpotents. Take for example $R$ to be the $2 \times 2$ matrices occurring in Theorem 1, type (3). Here $H = R$ certainly has skew nilpotents. An other obstruction is that an arbitrary nil ideal $P$ of $H$ is not *a priori* invariant. We circumvent the latter obstruction by choosing $P$ to be the *prime* radical of $H$. Once we can show that $P = 0$ necessarily, using the fact that $H$ contains no symmetric nilpotents $\neq 0$, clearly we get that $H$ contains no nil ideals $\neq 0$. To circumvent the former obstruction, let us show the following.

PROPOSITION 8. *For every* $a \in P$ (= *prime radical of* $H$) *and every square-zero skew* $k$, *in* $R$, $ak$ *is nilpotent.*

*Proof*. Since $k$ is quasi-unitary with quasi-inverse $-k$, for every $a \in P$, $(1 + k)a(1 - k) \in P$ follows. Thus $ka - ak - kak \in P$. Changing $k$ to $-k$ gives $kak \in P$. Thus $akak \in P$, whence $ak$ is nilpotent.

PROPOSITION 9. *Let $R$ be a prime PI ring, and let $a \in H$ be a square-zero skew such that $ak$ is nilpotent for any square-zero skew $k$. Then $a = 0$.*

*Proof*. By the corollary to Proposition 6 (Section 2.2), and the corollary to Theorem 1 (Section 1), we may take $R$ to be an order in the $2 \times 2$ matrices $\bar{R}$ over a field with symplectic involution. Moreover, since $\bar{R}$ is obtained by localizing $\text{re}Z^+(R)$, the property of $a$ remains true under the square-zero skews in $\bar{R}$. Now the square-zero skews in $\bar{R}$ are of one of the following types:

i) $k = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$

ii) $k = \begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}$

iii) $k = \lambda\begin{bmatrix} 1 & x \\ y & -1 \end{bmatrix}, \quad \lambda \neq 0, xy = -1.$

Since $a$ is a square-zero skew of $\bar{R}$, $a$ is of one of the types i)-iii). Assume that $a$ is of type i), $a = \begin{bmatrix} 0 & a_0 \\ 0 & 0 \end{bmatrix}$. Then

$$a\begin{bmatrix} 1 & x \\ y & -1 \end{bmatrix} = \begin{bmatrix} 0 & a_0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & x \\ y & -1 \end{bmatrix} = \begin{bmatrix} a_0 y & -a_0 \\ 0 & 0 \end{bmatrix}$$

is certainly non-nilpotent for $a_0 \neq 0$, that is, $a \neq 0$. Thus $0 \neq a$ cannot be of type i), and, by symmetry, $a$ is not of type ii). On the other hand, if $a$ is of type iii), the argument can be reversed. We have to agree that $a = 0$ necessarily,

PROPOSITION 10. *The prime radical of $H$ is zero.*

*Proof*. By Proposition 7, from Section 2.2, $P$ consists entirely of square-zero skews.

*Step* 1. *If $a \in P$ is such that $aSa = 0$, then $a = 0$.*

Exactly as in the parallel situation treated in Proposition 6, we can find a *PI* prime subring $R_1$ containing in its $*$-co-hypercenter the given element $a = -a*$ in $P$. Because $ak$ is nilpotent for every square-zero skew in $R$, clearly this property holds in $R_1$. By Proposition 9, $a = 0$ necessarily.

*Step* 2. *If $R$ contains some idempotent $e$ with $e \oplus e* = 1$, then $P = 0$.*

Let $a \in P$ and let $s \in S$. We have

$$\begin{aligned}[a, s] &= [eae* + e*ae + eae + e*ae*, ese + ese* + ese + e*se*] \\ &= [eae* + e*ae + eae + e*ae*, ese + e*se*] = [eae* + e*ae, ese + e*se*],\end{aligned}$$

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for $[a, ese* + e*se] = 0$, since $ese*$, $e*se$ are symmetric nilpotents; $aea \in eHe \subseteq T_{eRe} = Z_{eRe}$; $e*ae* \in e*He* \subseteq T_{e*re*} = Z_{e*Re*}$. Now

$$\begin{aligned}[a, s] &= [eae* + e*ae, ese + e*se*] = \\ &= (eae*se* + (eae*se*)*) + (e*aese + (e*aese)*) \\ &= s_1 + s_2; \end{aligned}$$

$$s_1^2 = 0, s_i = s_i^* \quad (a \in P \text{ implies } a = -a*).$$

Thus $[a, [a, s]] = [a, s_1 + s_2] = 0$, so, $asa = 0$, all $s = s*$, that is, $aSa = 0$. By Step 1, $a = 0$ follows.

*Step* 3. *If e is any idempotent of R such that* $ee* = 0$, *then* $eae* = e*ae = 0$.

Let $f = e + e* - e*e = e_1 \oplus e_1^*$. Let $a \in P_H$, and $a_1 = faf$. We have $(1 - 2f)a(1 - 2f) \in P_H$, so, $af + f \cdot a - 2faf \in P_H$. Thus $afa - 2afaf = -afa + 2faf$ (observed that $a$ anti-commutes with $af + fa - 2faf$); $afa = afaf + fafa$

$$\begin{aligned}afa &= (afa)f + f(afa) = (afaf + fafa)f + f(afaf + fafa) \\ &= afaf + fafaf + fafaf + fafa = (afaf + fafa) + 2fafaf \\ &\qquad\qquad\qquad\qquad = afa + 2fafaf; \quad fafaf = 0;\end{aligned}$$

$$a_1^2 = (faf)(faf) = fafaf = 0.$$

Moreover, if $k_1$ is a square-zero skew in $R_1 = fRf$, then $a_1k_1$ is nilpotent ($a_1 \cdot k_1 = fafk_1 = fak_1$, and $a_1k_1a_1k_1 = fafk_1fafk_1 = fak_1ak_1 \ldots$). By Step 2, $a_1 = faf = 0$ necessarily. This gives, as in step 2 of Proposition 7, $eae* = e*ae = 0$ necessarily.

*Step* 4. *Every* $a \in P$ *satisfies* $aSa = 0$, *so* $a = 0$.

Set $v = v_1 + v_2$, $v_1v_2 = 0$, where $v_1 = sa$, $v_2 = v_1^* = -as$, and use an argument similar to Step 3 of Proposition 7, to get $aSa = 0$ as wished.

2.4 *Skew nilpotents in R*. So far, we have shown that $H$ has no non-zero nil ideals where $R$ is any $*$-prime ring. To get that $H^+$ centralizes all skew nilpotents, we shall use a subdirect representation argument. In this connection we observe that any semi-prime ring $R$, whose characteristic is greater than 5, has a subdirect representation into $*$-prime rings inheriting the characteristic assumption.

Then let $a \in H^+$ and let $k$ be a skew nilpotent. Denote by $A$ the subring generated by $a$ and $k$. Factoring out the nil radical $P$, we get a ring $\bar{A}$ whose characteristic is zero or greater than 5, which by the above has a subdirect representation into $*$-prime rings $\Lambda$ with the same characteristic assumption.

In any $*$-prime image $\Lambda$, if $\alpha$, $\sigma$ are the images of $a$ and $k$ respectively, clearly $\alpha = \alpha^* \in H(\Lambda)$, while $\sigma$ is a skew nilpotent. Thus $\sigma^2$ is a symmetric nilpotent and consequently $[\alpha, \sigma^2] = 0$. Because $\sigma^2$ evidently commutes with $\sigma$, $\sigma^2$ is then a central symmetric, so in view of the $*$-primeness, $\sigma^2 = 0$ necessarily.

Thus $\sigma\alpha - \alpha\sigma - \sigma\alpha\sigma \in H(\Lambda)$. Changing $\sigma$ to $2\sigma$ gives $\sigma\alpha - \alpha\sigma \in H(\Lambda)$ and $\sigma\alpha\sigma \in H(\Lambda)$. Since $\sigma\alpha\sigma$ is a symmetric square-zero element in $H(\Lambda)$, and since by Proposition 4 and 7, $H(\Lambda)$ contains no symmetric nilpotents, $\sigma\alpha\sigma = 0$ follows. Then $\tau = \sigma\alpha - \alpha\sigma$ is a symmetric in $H(\Lambda)$, whose square is

$$\tau^2 = \sigma\alpha\sigma\alpha + \alpha\sigma\alpha\sigma - \sigma\alpha^2\sigma - \alpha\sigma^2\alpha = -\sigma\alpha^2\sigma,$$

so $\tau$ is a symmetric nilpotent, whence $\tau^2 = 0$. Thus $\tau = 0$, that is, $[\sigma, \alpha] = 0$.

We return to the subring $A$. We claim that $(1 + k)^{-1} [a, k] (1 - k)^{-1}$ is nilpotent. In fact in every $*$-prime image $\Lambda$ of $A/P$ and hence of $A$, it was seen that $[a, k] = 0$. However by Remark 7 from Section 2.2, $a \in H$ gives $(1 + k)^{-1} [a, k] (1 - k)^{-1} \in H$. Thus $(1 + k)^{-1} [a, k] (1 - k)^{-1}$ is a symmetric nilpotent of $R$, which is $*$-prime. It follows that

$$(1 + k)^{-1} [a, k] (1 - k)^{-1} = 0$$

giving $[a, k] = 0$ as desired, and we have proved the following result.

Proposition 11. *If $R$ is $*$-prime, then $H^+$ centralizes both the symmetric and skew nilpotents.*

Using Propositions 4, 7, 10, and 11 (Sections 2.1, 2.2, 2.3), and using a routine subdirect representation argument, we derive the following interesting theorem.

Theorem 2. *Let $R$ be any semi-prime ring. Then $H$ has the following properties:*
i) *$H$ contains no non-zero symmetric nilpotents.*
ii) *$H$ contains no non-zero nil ideals (in $H$).*
iii) *$H^+$ centralizes both the symmetric and skew nilpotents in $R$.*

**3. Center of $H$.** In this section we will establish an important step towards the main theorem stated at the outset; namely, every symmetric of the ring $H$ belonging to the centre $Z(H)$ of $H$ is in fact in $Z$. We will have to break the given ring $R$ into subrings having two generators.

3.1 *Subrings with two generators.* Start with any ring $R$, and pick $a$ in $H$, and $b$ in $S \cup K$. Denote by $A = A(a, b)$ the subring generated by $a$ and $b$. Of course $a$ will remain in the $*$-co-hypercenter of $A$. Denote by $B$ the centralizer of $b$ in $A$. Clearly $Z(A) = C_A(a) \cap C_A(b)$. We proceed to the following proposition.

Proposition 11. *In the ring $A$, $b$ is co-integral of index 2 over the center, with a centralizer $B$ satisfying a polynomial identity.*

*Proof.* For let $s = s* \in C(B)$. By the basic property of $a \in H(A)$, there is $p$ such that $[s - s^2 \cdot p(s), a] = 0$. Since $s - s^2 \cdot p(s) \in B$, it follows that $s - s^2p(s) \in C_A(a) \cap C_A(b) = Z(A)$. By [**4**], every ring $B$ satisfying $s - s^2 \cdot p(s) \in Z(B)$ must satisfy a polynomial identity. Moreover, since $b^2$ is certainly symmetric, $b^2$ is co-integral of index 1 over the center of $A$, which completes the proof.

By a result of S. Montgomery, as generalized by M. Smith [**15**], if the ring $A$ as in Proposition 11 is a prime ring, then $A$ must satisfy a polynomial identity, which is precisely the information that we are seeking in this subsection. But, if $A$ is only a $*$-prime ring, there is no way to apply directly Montgomery-Smith's result, nor to get directly in the non-prime case, that $H(A) \subseteq Z(A)$. This is circumvented using related results about centralizers.

PROPOSITION 12. *If $A$ is $*$-prime, then $A$ must satisfy a polynomial identity.*

*Proof.*

*Step* 1. *$B$ is semi-prime.*

If $s$ is a symmetric or skew nilpotent in $B$, by Theorem 2, $s$ commutes with $a$. Since $s \in B$, $s \in Z(A)$ follows. In view of the $*$-primeness of $R$, $s = 0$ necessarily.

*Step* 2. *$B$ contains some non-trivial symmetric idempotent.*

Let $e = e* = e^2 \neq 0, 1$ in $B$. Clearly $[a, e] \neq 0$. Now in the course of the proof of Proposition 4 (Section 2.1) it was seen that if $A$ were not prime, necessarily $H(A)$ centralizes all symmetric idempotents. Consequently $A$ is necessarily a prime ring. We can finish up the proof by a localization argument. But there is no need for that. In fact, given $z \in Z^+$, $z \neq 0$, $ze$ is symmetric, so $[a, ze - (ze)^2p(ze)] = 0$ forces $z = z^2p(z)$, $z \in Z^+$. It follows that $B$ is $*$-co-integral of index 1 over the zero subring. Now $B$ cannot be nil (otherwise $b$ is nilpotent, so $[a, b] = 0$, whence $A$ is commutative, which we are ruling out). Thus $R$ has a characteristic $p \neq 0$, and consequently $R$ is an algebra over a field (Galois field). By Montgomery-Smith's result, $A$ must satisfy a polynomial identity.

*Step* 3. *$B$ contains no non-trivial symmetric idempotents.*

We claim that $Z^+ \neq 0$ necessarily. Otherwise, take any $0 \neq s = s* \in B$. From $s - s^2p(s) \in Z$ follows $s = s^2p(s)$, giving the idempotent $e = e* = sp(s)$, which must be then the unity of $R$, an impossibility. Thus $B$ contains no symmetrics $\neq 0$, so $b^2 = 0$, whence $[a, b] = 0$, resulting in $A$, commutative, which is ruled out.

Now every symmetric $s = s*$, being of the form $d = s - s^2p(s) \in Z$, is a non-zero divisor on $R$. For if $d = 0$ the argument above gives that $s$ is indeed invertible, while $d \neq 0$ forces $s$ to be non-zero divisor. Localizing $A$ re $Z^+ \neq 0$, $B$ becomes $\bar{B} = B(Z^+)^{-1}$, a semi-prime ring all of whose symmetrics are invertible. By a result of M. Osborn, $\bar{B}$ must be semi-simple artinian (with the extra property that $\bar{B}$ contains no skew nilpotents). We proceed to show that $b$ has some central power in $R$, hence in $\bar{R} = R(Z^+)^{-1}$. Consider the subring $Z^+[b^2]$ generated by $Z^+$ and $b^2$. This is contained in $B$, so $Z^+[b^2]$ must be co-integral of index 1 over $Z^+$. As the later subring is a commutative domain, we derive that $b^2$ has some power in $Z^+(Z^+)^{-1}$, so $b^{2n} \cdot z_1 = z_2$, for some $z_i \in Z^+, z_2 \neq 0$. It follows that $b^{2n} \in Z^+$, as wished.

Having shown that $b$ has some power in $Z(\bar{R})$, and that the centralizer $\bar{B}$ of $b$ in $\bar{R}$ is semi-simple artinian, we get using [**9**] that $\bar{R}$ itself is semi-simple artinian. A trivial adaptation of Montgomery's result [**12**] shows that $\bar{R}$ is then $PI$, so $R$ must be $PI$, which completes the proof.

What can be said about any ring $A = A(a, b)$ of the considered generators $a, b$? Denote by $G$ the commutator ideal of $A$. (This is the ideal generated by all commutators in $A$.) We can prove the following theorem.

THEOREM 3. *For any $a = a* \in H(R)$, and $b \in S \cup K$, $A = A(a, b)$ satisfies a polynomial identity modulo the prime radical, and the commutator ideal $G = G(A)$ of the ring $A$ is *-co-integral over the zero subring.*

*Proof.* It suffices to prove the theorem for $R = A(a, b)$, a *-prime ring with characteristic zero or greater than 5 (provided we can establish a ploynomial identity of fixed degree, the reduction for the $PI$ conclusion is clear. As for the nature of the commutator ideal $G$, reduce to the *-prime case by considering an $m$-system

$$M = \{2^n \cdot 3^m \cdot 5^{r'} g(s)\}_{n,m,r;g=t^r-t^{r+1}p(t)}$$

and take a *-prime ideal maximal re the exclusion of $M$, where $s = s*$ is a fixed symmetric in $G$). By Proposition 12, $R$ must satisfy a polynomial identity. If $H^+(R) \subseteq Z$, clearly $a \in H^+(R)$ commutes with $b$, so $R$ is commutative, whence $G = 0$. If, on the other hand, $H^+(R) \not\subseteq Z$, Proposition 5, applies and yields $R$ to be as in Theorem 1, type (2). It follows that $R$ satisfies the standard identity in 4 variables, and that $G$ is clearly *-co-integral over the zero subring. The theorem is proved.

3.3. *Symmetric idempotents.* We take $R$ to be a *-prime ring, and let $a = a* \in Z_H$, the centre of $H$. We wish to show that for every symmetric idempotent $e = e*$ of $R$, $[a, e] = 0$ necessarily. As observed earlier this property is certainly true when $R$ is not prime.

PROPOSITION 13. 1) *If $[a, e] \neq 0$, then $R$ must have finite characteristic.*
2) *If $b = ae + ea - 2eae$, then $b = b* \in Z_H$, $[b, e] \neq 0$, and the subring $A(b, e)$ generated by $b$ and $e$ is finite.*

*Proof.* 1) Suppose, by way of contradiction, that $R$ has characteristic 0. Given any $c = c* \in H(R)$ and any $x \in S \cup K(R)$, we know by Theorem 3, Section 2.4, that the corresponding subring $A = A(c, x)$ has a commutator ideal $G$, which is co-integral over the zero subring. Now $G$ is a subring of $R$, which must be of characteristic 0, since $R$ is *-prime. Consequently $G$ must be nil, giving in particular that $[c, x]$ is nilpotent. Since the later element is again in $S \cup K$, by Theorem 2 Section 2.4, $[c, [c, x]] = 0$ follows. Thus $[c, [c, x]] = 0$ for all $x \in R$. By Herstein's Sublemma, $c \in Z$ follows, all $c = c* \in H$, contra-

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dicting the assumption $[a, e] \neq 0$, for the considered elements $a \in H^+$, and $e \in R$. We have to agree that $R$ has non-zero characteristic, so must be an algebra over a Galois field.

2) Since $e = e^*$ is an idempotent, and since $Z_H$ is invariant (for $H$ is invariant) containing $b$, it follows that $(1 - 2e)a(1 - 2e) = a - (2ea + 2ae) + 4eae \in Z_H$, resulting in $b = ea + ae - 2eae \in Z_H$. Observe that $b = be + eb$. If then $b$ commutes with $e$, we get $eb = ebe + eb, be = be + ebe$, so $eb = be = 0$, whence $b = eb + be = 0$, that is, $ea + ae - 2eae = 0$. From this $ea + eae - 2eae = 0$ and $eae + ae - 2eae = 0$, giving $ea = eae = ae$, which is ruled out. Thus $[b, e] \neq 0$ necessarily.

Consider $E = \{e^n \cdot b^m\}_{n=0,1;m \leqq m_0}$, where $m_0$ is the algebraic degree of $b$ over the underlying Galois field. (In fact, $b = ea + ae - 2eae = [ae, e] + [e, ea]$ is in the commutator ideal of the subring $A(e, a)$, which, by Theorem 3 Section 2.4, is co-integral over the zero subring.) By inspection, $E$ has as its span over the Galois field precisely $A(e, b)$, so $A(e, b)$ is finite.

PROPOSITION 14. *If $R$ is $*$-prime, then every symmetric element in the centre of $H$ centralizes every symmetric idempotent in $R$.*

*Proof.* Let $A = A(b, e)$. By Proposition 13, Section 3.3, $A$ is a finite subring of $R$. Let $W = A \cap Z_H^+$. This is a commutative invariant subring of symmetrics containing $b$ (invariant re the ring $A$). If $P$ is the prime radical of $A$, then the factor ring $A/P = \bar{A}$ is certainly finite, and $W$ maps onto a commutative subring of symmetrics $\bar{W}$ containing the image $\bar{b}$ of $a$, which is "almost invariant" in the sense that $\bar{W}$ is preserved under the quasi-unitaries $2\bar{f}$, $\bar{f}$ any symmetric idempotent, or $2\bar{k}(1 - \bar{k})^{-1}$. The later types of quasi-unitaries are in fact liftable re nil ideals.

Now let $\Lambda$ be a $*$-simple component of $\bar{A}$. Clearly $\bar{W}$ maps onto a commutative subring of symmetrics containing the image $\beta$ of $\bar{b}$, which is almost invariant. In the presence of the finiteness of $\Lambda$ (or just the fact that the ground division ring in $\Lambda$ is not 4-dimensional), Remarks 6 extend to the almost invariant subalgebras. But we must first ensure that $\Lambda$ is simple artinian. If not, taking into account that $e$ maps onto an idempotent $\epsilon = \epsilon^*$ of $\Lambda$, and that $b$ maps onto the element $\beta \in H^+(\Lambda)$, we get immediately $[\beta, \epsilon] = 0$ necessarily. This allows us to take $\Lambda$ to be simple. Clearly we may suppose that $H^+(\Lambda) \nsubseteq Z(\Lambda)$. By Corollary to Theorem 1, Section 1, $\Lambda$ enjoys the property that every commutative subring of symmetrics, which is almost invariant, must be central. Then $[\beta, \epsilon] = 0$ necessarily.

All in all, we have shown that $[b, e] = 0$ in every $*$-prime image of $A$. In view of the construction of $b$, this means that $b = 0$ in every $*$-prime image of $A$, resulting in $b$, a symmetric nilpotent of $A$. Since $v$ was in $Z_H^+ \subseteq H$, by Theorem 2, Section 2.4, $b = 0$ follows. Thus $[b, e] = 0$, whence $[a, e] = 0$, proving the proposition.

3.4 *Structure of the $*$-center of $H$.* In this closing subsection, we let $R$ be any $*$-prime ring and wish to establish that every central symmetric $c$ of $H$, is a central element of $R$. As already observed, we may take $R$ to be with finite characteristic (Proposition 13, part 1) Section 3.3). Thus every co-integral element $x \in R$ over the zero subring is of the form $x^{n(x)} = e = e^2$. If, moreover, $x$ is in $S \cup K$, $x^{n(x)}$ is a symmetric idempotent of $R$. By Proposition 14, Section 3.3, $[c, x^{n(x)}] = 0$ follows. Let then $b$ be a fixed element of $S \cup K(R)$, and let $A(c, b)$ be the subring generated by $c$ and $b$. By Theorem 3, Section 2.4, for every $x = x*$ in the commutator ideal $G = G(A)$ of $A$, $x$ is co-integral over the zero subring, and consequently $[c, x^{n(x)}] = 0$.

Let $\Lambda$ be a $*$-prime image of the ring $A$. By Theorem 3, $A$ is $PI$. We claim that $\Lambda$ is actually commutative. For in the contrary case, $[\alpha, \beta] \neq 0$, where $a$ and $b$ map respectively an $\alpha$ and $\beta$. Since $a = a*$ was in $H(R) \cap A \subseteq H(A)$, it follows that $\alpha = \alpha* \in H^+(\Lambda)$. Thus $H^+(\Lambda) \not\subseteq Z(\Lambda)$. In view of Proposition 5, $\Lambda$ is necessarily of type (2) in Theorem 1, Section 1. In particular $\Lambda$ is simple and non-commutative. Thus the commutator ideal $G(A)$ of $A$ maps onto a non-zero ideal necessarily equal to $\Lambda$. Thus $\alpha$ has the property $[\alpha, x^{n(x)}] = 0$, for all $x \in \Lambda$. Consequently $\alpha$ centralizes all symmetric idempotents in $\Lambda$. However the subalgebra generated by these being invariant must be all of $\Lambda$ forcing $\alpha \in Z(\Lambda)$. We conclude that $\Lambda$ was commutative.

Since $[a, b]$ is zero in every $*$-prime image of $A(a, b)$, it follows that $[a, b]$ is nilpotent. Because $[a, b] \in S \cup K$ and $a = a* \in H$, by Theorem 2, $[a, [a, b]] = 0$ follows. Consequently $[a, [a, x]] = 0$ for all $x \in R$. By Herstein's Sublemma, $a \in Z$ follows. We have proved the following result.

THEOREM 4. *If $R$ is $*$-prime, then every symmetric element in the centre of $H$ is in fact a central element of $R$.*

**4. Structure of $H$.** In this section we complete the proof of Theorem 5, as stated at the outset. We are given any $*$-prime ring $R$ with characteristic 0 or greater than 5. We now examine the case where $H^+ \not\subseteq Z$.

PROPOSITION 15. *If $H^+ \not\subseteq Z$, then $R$ must be of type (2) in Theorem 1, Section 1.*

*Proof.* By Theorem 2, $H$ is a semi-prime ring. By Remark 3, $H$ satisfies a polynomial identity. If $J = J^*$ is a non-zero ideal of the ring $H$, then by a result of L. Rowen [**16**], $J$ contains a central element $c$ of $H$. If both $c + c*$ and $cc*$ were equal to zero, $c$ would be a central square-zero element of $H$, contrary to the semi-primeness (and the fact that $H \neq 0$ necessarily, since $H^+ \not\subseteq Z$). This shows that either $c + c* \neq 0$ or $cc* \neq 0$. If $cc* \neq 0$, $J$ contains the central symmetric element $z = cc*$ in $H$. If, on the other hand, $c + c* \neq 0$, then $z_1 = c + c*$ is a central symmetric in $J$. This shows that $J$ must contain an element $z \neq 0$ in $Z^+(H)$. By Theorem 4, $z \in Z(R)$ follows. Thus $J$ contains a non-zero divisor on $R$. Consequently $H$ must be a $*$-prime ring.

We claim that the ring $H$ must be of type (2), Theorem 1. To see this observe that since $H^+ \not\subseteq Z$ there must be $a = a* \in H$, $a \notin Z$. By the contra-positive of Theorem 4, $a \notin Z(H)$. In view of the $*$-primeness of $H$ and the presence of a polynomial identity in the ring $H$, we can then apply Proposition 5, Section 2.2, and get the desired information on $H$.

Since $H$ is isomorphic to the $2 \times 2$ matrices over a field with a canonical transpose involution, it follows that $H$ contains a unity $f$. Now $f$ is a central element $H$, so must be central in $R$. Because $f = f* = f^2$, by the $*$-primeness of $R$, $f = 1$ necessarily, the unity of $R$. Also $H$ contains a symmetric idempotent $e = e*$ and some skew $k_0$, such that $[e, k_0] = c \neq 0$. Now $c = c*$ is a square-central symmetric in $H$, which can of course be taken such that $c^2 \neq 0$. It follows that $c^2 \neq 0$ is a central element of $R$ (Theorem 4, Section 3.4), and consequently $c$ is a non-zero divisor on $R$.

Now let $s = s* \in C_R(e) = B$. Since both $s$ and $se$ are symmetrics we can find a polynomial $p(t)$ so that $[k_0, s - s^2 \cdot p(s)] = [k_0, (se) - (se)^2p(se)] = 0$. Then

$$0 = [k_0, (s - s^2p(s)e] = (s - s^2p(s))[k_0, e] = (s^2p(s) - s) \cdot c.$$

Since $c$ is a non-zero divisor on $R$, $s = s^2p(s)$ follows for all symmetrics $s = s*$ in $B = C_R(e)$.

However, $eRe$ and $(1 - e)R(1 - e)$ are $*$-prime rings contained in $B = C_R(e)$, thus inheriting the co-integral assumption $s = s^2 \cdot p(s)$. By Montgomery's result, $eRe$ and $(1 - e)R(1 - e)$ are certainly right artinian and $PI$. It follows that $R$ must be right artinian. Consequently $R$ is semi-simple artinian. Since $B = C_R(e) = C_R(1 - 2e)$, with $(1 - 2e)^2 = 1$, by a result of Montgomery, $R$ satisfies a polynomial identity, which completes the proof (Proposition 5, Section 2.1).

PROPOSITION 16. *Let $R$ be any $*$-prime ring, and suppose that $H^+ \subseteq Z$. Either $S \subseteq Z$ or $H \subseteq Z$, or else $H$ must be a domain.*

*Proof.* If $Z^+ = 0$, we claim that $H = 0$ necessarily, so $H \subseteq Z$ would follow. In fact, since $H^+ \subseteq Z$, we get $H^+ = 0$. Given $k \in H$, $k$ is then a skew, so $k^2 = 0$. Thus every element of $H$ is square-zero, giving that $H$ is nil. By Theorem 2, Section 2.4, $H = 0$ follows as wished. This shows that we may assume $Z^+ \neq 0$.

Let $\bar{R}$ be the partial ring of fractions re $Z^+$, and let $\bar{H}$ be the expansion of $H$. Clearly every symmetric in $\bar{H}$ must be a central element of $\bar{R}$, hence an invertible element. Also, since $H$ is semi-prime (Theorem 2), $\bar{H}$ must be also. It follows that either $\bar{H}$ is a division ring, or $\bar{H}$ is a direct product of division rings, or else $\bar{H}$ is the $2 \times 2$ matrices over a field with symplectic involution.

Assume that $H$ is not a domain. This forces $\bar{H}$ to be a non-division ring. By the above, $\bar{H}$ contains an idempotent $e$ with $e \oplus e* = 1_{\bar{H}} = 1_{\bar{R}}$. We shall now prove that if $S \not\subseteq Z$, necessarily $H \subseteq Z$, which will show the proposition.

Write $e = e_1 \cdot z^{-1}$, $z \in Z^+$. Clearly $e\bar{R}e$ is the localization of the subring $e_1Re_1$. Since $e\bar{R}e$ is certainly semi-prime, $R_1 = e_1Re_1$. must be also. We claim that for every $x \in H$, $x_1 = e_1xe_1$ is in the co-hypercenter of $R_1$. For let $y \in$

$e_1Re_1$. Now $y + y*$ is symmetric in $R$. By the basic property of $x$,

(1) $[x, (y + y*) - (y + y*)^2p(y + y*)] = 0.$

However $yz^{-1} = e_1t_0e_1z^{-1} = e_1t_0e = et_0e_1$, and $y*z^{-1} = e_1{}^*t_0{}^*e_1{}^*z^{-1} = e_1{}^*t_0{}^*e^* = e*t_0{}^*e_1$. Thus $yz^{-1} \cdot y*z^{-1} = e_1t_0e \cdot e*t_0{}^*e_1 = 0 = y*z^{-1} \cdot yz^{-1}$, giving $yy* = y*y = 0$. Thus (1) becomes

$$0 = [x, y - y^2p(y)] + [x, y* - (y*)^2p(y*)].$$

Then

$$x(y - y^2p(y)) - (y - y^2p(y))x = [y* - (y*)^2p(y*), x]$$

Now $y - y^2p(y) \in e_1Re_1$, so $(y - y^2p(y))e = e(y - y^2p(y)) = (y - y^2p(y))$. Thus

(2) $xe(y - y^2p(y)) - (y - y^2p(y))ex = [y* - (y*)^2p(y*), x]$

Multiply (2) on the left by $e$ and on the right by $e$, to get

$$[exe, y - y^2p(y)] = 0;$$
$$0 = [e_1xe_1 \cdot z^{-2}, y - y^2p(y)] = z^{-2}[e_1xe_1, y - y^2p(y)];$$
$$[e_1xe_1, y - y^2p(y)] = 0,$$

placing $x_1 = e_1xe_1$ in the co-hypercenter of the ring $R_1 = e_1Re_1$. Consequently $e_1xe_1$ is a central element of $e_1Re_1$. By symmetry, for $x$ as before in $H$, $e_1{}^*xe_1{}^*$ is a central element of $R_1{}^{*} = e_1{}^*Re_1{}^*$.

Consider an arbitrary skew $k$ in $H$, and an arbitrary symmetric $s = s*$ in $\bar{R}$. At this point let us observe that since $H$ centralizes all symmetric nilpotents in $R$, so will $\bar{H}$ in $\bar{R}$, and by the above, that $eke$, $e*ke*$ are respectively central elements in the corner subrings $e\bar{R}e$ and $e*\bar{R}e*$. Write

$$[k, s] = [k, ese + e*se* + e*se + ese*].$$

Since $ese*$ and $e*se$ are symmetric nilpotents, we get

$$[k, s] = [k, ese + e*se*].$$

Now $[k, s] = [eke + eke* + e*ke + e*ke*, ese + e*se*]$. Since $[eke, ese] = [eke, e*se*] = 0 = [e*ke*, e*se*] = [e*ke*, ese]$, we obtain

$$[k, s] = [eke* + e*ke, ese + e*se*] = s_1 + s_2,$$

where $s_i$ are again, symmetric nilpotents. Thus

(3) $[k, [k, s]] = [k, s_1 + s_2] = 0.$

Since $H$ is semi-prime, with $H^+ \subseteq Z$, if then $H$ were not contained in $Z$, in particular $H^- \neq 0$. If now $H^-$ is nil, necessarily $k^2 = 0$ for all $k = -k*$ in $H$, giving by a straightforward linearization $kk' = 0$, all $k, k' \in H^-$. Consequently $H$ would have the nil radical $H^-$, which is ruled out by Theorem 2, Section 2.4. This shows that some $k \in H^-$ is a non-square zero. Because $k^2 = z \in Z$,

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$k$ is a non-zero divisor on $R$. However, by (3),

$$0 = [k, [k, s]] = k^2 s - 2ksk + sk^2$$

Since $k^2 \in Z$, we get $2k^2 s = ksk$, which on cancellation by $k$ gives $ks = sk$ for all $s = s* \in \bar{R}$, forcing $k \in Z$, for we had $S \not\subseteq Z(R)$, by a well-known result of Herstein. Knowing that $H$ contains a central skew, we can now derive trivially the conclusion $H \subseteq Z$. For if $k_0$ is any skew in $H$, $k_0 \neq 0$, then $k_0 k$ is a non-zero symmetric in $H$, so $k_0 k \in Z$ with $k \in Z$ whence $k_0 \in Z$, all $k_0 \in H^-$, $k_0 \neq 0$, so $H = H^+ \subseteq Z$, which completes the proof.

We have all the pieces to prove Theorem 5. We slightly re-phrase the statement.

THEOREM 5. *Let $R$ be any $*$-prime ring having characteristic* 0 *or greater than* 5. *Suppose that the fixed element $c$ of $R$ is such that for every symmetric $s = s*$ of $R$, there is a polynomial $p(t)$ depending on $c$ and $s$ such that $c$ commutes with $s - s^2 \cdot p(s)$. Then $c$ is in fact a central element, except when $R$ is of one of the following types*:

1) *$R$ is an order in the $2 \times 2$ matrices over a field with symplectic involution (so, all symmetrics are central).*

2) *$R$ is the $2 \times 2$ matrices over an algebraic field extension of a Galois field with a canonical transpose involution admitting no symmetric (or skew) nilpotents (so, every symmetric satisfies $s = s^{n(s)}$, $n(s) \geqq 2$).*

*Proof*. Suppose that $R$ is not of type (2) and that $H \not\subseteq Z$. By the contrapositive of Proposition 15, $H^+ \subseteq Z$ follows. By Proposition 16, either $S \subseteq Z$ or $H \subseteq Z$, or else $H$ must be a domain. Since we had $H \not\subseteq Z$, it must be that $S \subseteq Z$ or that $H$ is a domain. Now the case $S \subseteq Z$ gives that $R$ is necessarily prime (for $R$ is non-commutative, whence $R$ must be of type (1).

We are left with the following possibility: $H^+ \subseteq Z$, $H^- \not\subseteq Z$, $S \not\subseteq Z$, and $H$ a domain, that we must now rule out.

*Step* 1. *Let $A(k, s)$ be the subring generated by a fixed skew $k$ in $H$, and a fixed symmetric $s = s*$ in $R$. Then $A$ is* PI *modulo the prime radical, and the commutator ideal of $A$ is co-integral over the zero subring.*

It suffices to show this assertion for $A$ a $*$-prime non-commutative ring. We may of course assume that $S(A) \not\subseteq Z(A)$, and by Propositions 15, 16, that $H^-(A)$ consists entirely of non-nilpotent square-central skews. Observe that $k \in H(A)$ is one such element. Let $B = C_A(k)$. Given $\sigma = -\sigma* \in B$, we claim that $\sigma$ is non-nilpotent (for $\sigma \neq 0$). Suppose the contrary. Then $\sigma^2$ is a symmetric nilpotent. By the basic property of $k$, $\sigma^2$ commutes with $k$. Since $\sigma^2 \in B = C_A(s)$, $\sigma^2 \in Z(A)$ follows, giving $\sigma^2 = 0$. Because $H(A)$ is invariant, we get $(1 - \sigma)k(1 + \sigma) \in H(A)$. Changing $\sigma$ to $2\sigma$ give $\sigma k\sigma$ and $\sigma k - k\sigma \in H(A)$. Because $\sigma k \sigma$ is square-zero, $\sigma k \sigma = 0$. It follows that

$$(\sigma k - k\sigma)^2 = -\sigma k^2 \sigma = -\sigma^2 k^2 = 0,$$

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so, by the same token, $\sigma k = k\sigma$. Consequently $\sigma \in Z$, whence $\sigma = 0$ necessarily. Clearly $B$ contains no symmetric nilpotents neither, since in fact, $B$ is $*$-co-integral of index 1 over $Z$. A trivial adaptation of the proof of Proposition 12, gives that $A$ is PI. By Corollary to Proposition 5, $A$ is either an order in the $2 \times 2$ matrices with symplectic involution, but then $A = A(k, s(=s*))$ would be commutative, or, the $2 \times 2$ matrices over a field, which is algebraic over a Galois field. Thus the later case must occur, giving immediately the conclusions in the assertion.

*Step* 2. *Let* $e = e*$ *be any symmetric idempotent of* $R$. *Then* $[k, e] = 0$.

Let $y = ek + ke - 2eke$. We have $y = -y* \in H$ (using as in a previous case the invariance of $H$ via the quasi-unitary $-2e$). Suppose that $y \neq 0$. By an argument (in the fourth paragraph of the proof) of Proposition 15, for every $b = b* \in C_R(e)$ there is a polynomial $p(t)$ such that

$$[y, e](b - b^2 \cdot b(b)) = 0.$$

Now

$$[y, e] = ye - ey = ye - (y - ye) = 2ye - y = y(2e - 1),$$

so

$$y(2e - 1)(b - b^2 p_b(b)) = 0.$$

On cancellation by $y = -y* \in H$, and by the formal unit $2e - 1$, we get $b = b^2 \cdot p_b(b)$, all $b = b* \in C_R(e)$. As in the proof of Proposition 15, this would give that $R$ must be simple artinian, and Theorem 1 would apply, yielding the theorem. This shows that we may assume $y = 0$, so that $[k, e] = 0$ as desired.

*Step* 3. *For every* $x = x*$ *in the commutator ideal* $G$ *of* $A(k, s)$, $[k, x^{n(x)}] = 0$.

If $[k, s] = 0$ there is nothing to prove. If not, we claim that $[k, s]$ is non-nilpotent. Otherwise, $[k, s]$ would be a symmetric nilpotent. Since $k \in H$, $0 = [k, [k, s]] = k^2 s - 2ksk + sk^2$ follows. Because $0 \neq k^2 \in Z$, we would get $ks = sk$, which is false. Thus $G$ is non-nil. By 1, $G$ was co-integral over the zero subring. Consequently, $R$ must be of finite characteristic, and every $x = x* \in G$ is of the form $x^{n(x)} = e = e*$. By 2, $[k, x^{n(x)}] = 0$ follows.

We can now easily reach a contradiction to the assumption $[k, s] \neq 0$. For if $\Lambda$ is a $*$-prime image of $A(k, s)$, this is a PI ring. If $\Lambda$ were non-commutative, by the corollary to Proposition 5 (noting that $H(\Lambda) \not\subseteq Z(\Lambda)$ and that $S(\Lambda) \not\subseteq Z(\Lambda)$), $\Lambda$ should be of type (2) in Theorem 1, Section 1, which would yield as in a previous situation that the image $\sigma$ of $k$ is such that $[\sigma, x^{n(x)}] = 0$, for all $x = x* \in \Lambda$, $n(x) \geqq 2$, forcing $\sigma \in Z(\Lambda)$ necessarily. We conclude that $[k, s]$ is zero in every $*$-prime image of $A$, giving that $[k, s]$ is a symmetric nilpotent in $A \subseteq R$, so $[k, [k, s]] = 0$ whence as in the above $[k, s] = 0$, all $s = s* \in R$, a contradiction to the assumption $k \notin Z$ and $S \not\subseteq Z$. The theorem is proved.

We conclude with some observations and questions. All the results in this paper carry over to the rings $R$ with characteristic possibly 3 or 5, provided $R$ is an algebra over a field containing more than 5 elements. Actually the results remain true for rings $R$ with characteristic 5. This, however, requires rather heavy computations arising in the simple artinian case as our result on invariant subalgebras was assuming a ground division ring containing at least 7 elements. Concerning algebras over commutative rings $\Phi$, the whole paper will extend to this context under a suitable assumption on $\Phi$ extending the integers; namely, if $A$ is a commutative integral domain, which is co-integral over the subalgebra $B$, then $A$ must be radical over the subfield of quotients of $B$.

*Question* 1. Does Theorem 5 carry over to rings with any characteristic?

*Question* 2. If $R$ is semi-prime, in which, given $a = a*$, $b = b*$, $[a - a^2p_1(a), b - b^2 \cdot p_2(b)] = 0$, must $R$ satisfy the standard identity in 4 variables?

## References


1. M. Chacron, *A commutativity theorem for rings*, Proc. Amer. Math. Soc. *59* (1976), 211–216.
2. ——— *Unitaries in matrix algebras with involution*, Can. J. Math. (Submitted for publication).
3. M. Chacron and I. N. Herstein, *Powers of skews and symmetric elements in division rings*, Houston J. Math. *1* (1975), 15–27.
4. M. Chacron, I. N. Herstein, and S. Montgomery, *Structure of a certain class of rings with involution*, Can. J. Math. *27* (1975), 1114–1126.
5. C. Faith, *Radical extensions of rings*, Proc. Amer. Math. Soc. *12* (1961), 274–283.
6. I. N. Herstein, *Topics in ring theory*, Mathematical Lecture Notes, U. of Chicago, Chicago, Illinois.
7. ——— *Lectures on rings with involution*, Chicago Lectures in Mathematics (U of Chicago Press, Chicago, Illinois).
8. ——— *Structure of a certain class of rings*, J. Amer. Math. Soc. *5* (1954), 620.
9. I. N. Herstein and L. Neuman, *Centralizers in rings*, Annali di Mat. (1975), 37–44.
10. W. S. Martindale III, *Prime rings with involution and generalized polynomial identities*, J. Alg. *22* (1972), 502–516.
11. S. Montgomery, *A generalization of a theorem of Jacobson, II*, Pacific J. Math. *44* (1973), 233–240.
12. ——— *Centralizers satisfying polynomial identities*, Israel J. Math *18* (1974), 207–219.
13. M. Osborn, *Varieties of algebras*, Advances in Math. *8* (1972), 163–369.
14. ——— *Jordan algebras of capacity two*, Proc. Nat. Acad. Sci. U.S.A. (1967), 582–588.
15. M. Smith, *Rings with an integral element whose centralizer satisfies a polynomial identity*, Duke Math. J. *42* (1975), 137–149.
16. L. Rowen, *Some results on the center of a ring with polynomial identity*, Bull. Amer. Math. Soc. *79* (1973), 219–223.


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