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ON ISOCOMPACTNESS OF FUNCTION SPACES

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Let $C_p(X)$ be the space of all continuous real-valued functions on a Tychonoff space X with the pointwise topology. In this note, we show that if X is a $\mathcal{G}$-space, then $C_p(X)$ is isocompact. This gives an answer to a recent question of Arkhangel'skii in the class of $\mathcal{G}$-spaces.

1. INTRODUCTION

In studying the compactness of countably compact spaces, Bacon [**3**] introduced the notion of an isocompact space. Recall that a topological space X is *isocompact*, if every closed countably compact subspace of X is compact. Obviously, any topological property, which makes a countably compact space compact, implies isocompactness. Among the classes of spaces which are isocompact are the θ-refinable spaces [**10**], the spaces having a G_δ-diagonal [**5**], and the symmetrisable spaces [**9**], to name a few. The main purpose of this note is to study the isocompactness of function spaces and to answer a recent question of Arkhangel'skii in the class of $\mathcal{G}$-spaces defined by a two-person game.

For a Tychonoff space X, let

$$C(X) = \{f : X \to \mathbf{R} \text{ is continuous } \}$$

that is, $C(X)$ is the family of all continuous real-valued functions defined on X. We shall denote by $C_p(X)$ the space of $C(X)$, endowed with the topology of pointwise convergence on X. Obviously, a basic neighbourhood of a function $f \in C_p(X)$ is

$$W(x_0, x_1, \ldots, x_n; \varepsilon)(f) = \left\{g \in C(X) : \left|f(x_i) - g(x_i)\right| < \varepsilon,\ 0 \leqslant i \leqslant n\right\}$$

where $\varepsilon > 0$ and $x_0, x_1, \ldots, x_n \in X$.

No separation axioms are assumed on topological spaces if it is not stated explicitly, and more information on C_p-theory can be found in [**1**].

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2. Main theorem

Let X be a topological space, and let $x \in X$ be a point. The family of all neighbourhoods of x is denoted by $\mathcal{U}(x)$. We shall consider the following $\mathcal{G}(x)$-*game* [4] played in X between two players (α) and (β). Player (α) goes first and chooses a point $x_1 \in X$. Player (β) then responds by choosing $U_1 \in \mathcal{U}(x)$. Following this, player (α) must select another (possibly the same) point $x_2 \in U_1$ and in turn player (β) must again respond to this by choosing (possibly the same) $U_2 \in \mathcal{U}(x)$. The players repeat this procedure infinitely many times. We shall say that the player (β) *wins* the $\mathcal{G}(x)$-game if the sequence $\langle x_n : n \in \mathbf{N}\rangle$ has a cluster point in X. Otherwise, the player (α) is said to *have won* the game. By a *strategy* σ for the player (β), we mean a 'rule' that specifies each move of player (β) in every possible situation. More precisely, a strategy $\sigma = \langle \sigma_n : n \in \mathbf{N}\rangle$ for (β) is a sequence of $\mathcal{U}(x)$-valued functions. We shall call a finite sequence $\langle x_1, x_2, \ldots, x_n\rangle$ or an infinite sequence $\langle x_1, x_2, \ldots\rangle$ a σ-*sequence* if $x_{i+1} \in \sigma_i(\langle x_1, x_2, \ldots, x_i\rangle)$ for each i such that $1 \leqslant i < n$ or $x_{n+1} \in \sigma_n(\langle x_1, x_2, \ldots, x_n\rangle)$ for each $n \in \mathbf{N}$. A strategy $\sigma = \langle \sigma_n : n \in \mathbf{N}\rangle$ for player (β) is called *a winning strategy* if each infinite σ-sequence has a cluster point in X. Finally, we call x a $\mathcal{G}$-point if the player (β) has a winning strategy for the $\mathcal{G}(x)$-game. In addition, if every point of X is a $\mathcal{G}$-point, then X is called a $\mathcal{G}$-*space*.

The class of $\mathcal{G}$-spaces is quite large. In fact, it contains all q-spaces [8] and all W-spaces [7], thus contains all first countable spaces and all locally compact spaces. In [2], the following question was asked.

Question 2.1. [2] *For which spaces X is the space $C_p(X)$ isocompact?*

Now we answer Question 2.1 in the class of $\mathcal{G}$-spaces.

Theorem 2.2. *Let X be a Tychonoff $\mathcal{G}$-space. Then $C_p(X)$ is isocompact.*

Proof: Let $Y \subseteq C_p(X)$ be a closed countably compact subspace. By countable compactness, for every point x in X there is an $M_x > 0$ such that $|f(x)| \leqslant M_x$ for all $f \in Y$. It follows that

$$Y \subseteq \prod_{x \in X} [-M_x, M_x].$$

Therefore, $\overline{Y}$ is a compact subset of $\mathbf{R}^X$, where the closure is taken in $\mathbf{R}^X$. We first show that $\overline{Y} \subseteq C_p(X)$. To this end, assume that there exists some $g \in \overline{Y} \setminus C_p(X)$. Since g is not continuous, there must be some point x_0 and $\varepsilon > 0$ such that for each $U \in \mathcal{U}(x_0)$ we can choose at least one point $x_U \in U$ satisfying

$$|g(x_U) - g(x_0)| \geqslant \varepsilon.$$

Let σ be a winning strategy for player (β) in the $\mathcal{G}(x_0)$-game. Without loss of generality, let x_0 be the first move of player (α). Then player (β) responds by $\sigma(\langle x_0\rangle)$ and $f_1 \in W(x_0; 1)(g) \cap Y$. To respond to this, player (α) chooses a point

$$x_1 \in \sigma(\langle x_0\rangle) \cap \{x \in X : |f_1(x) - f_1(x_0)| < 1\}$$

such that

$$|g(x_1) - g(x_0)| \geqslant \varepsilon.$$

Inductively, players (α) and (β) produce a σ-sequence $\langle x_n : n \in \mathbf{N}\rangle$ in X and a sequence $\langle f_n : n \in \mathbf{N}\rangle$ in Y such that

(1) $x_n \in \sigma\big(\langle x_0, x_1, \dots, x_{n-1}\rangle\big)$ for each $n \in \mathbf{N}$;

(2) $x_n \in \bigcap_{i=1}^{n} \{x \in X : |f_i(x) - f_i(x_0)| < 1/n\}$ for each $n \in \mathbf{N}$;

(3) $|g(x_n) - g(x_0)| \geqslant \varepsilon$ for each $n \in \mathbf{N}$; and

(4) $f_n \in W(x_0, x_1, \dots, x_{n-1}; 1/n)(g)$ for each $n \in \mathbf{N}$.

Since Y is closed countably compact, $\langle f_n : n \in \mathbf{N}\rangle$ has a cluster point, say f in Y. Let x_∞ be a cluster point of the σ-sequence $\langle x_n : n \in \mathbf{N}\rangle$ in X. First of all, f and g coincide on $\{x_n : n \in \omega\}$. To see this, for each fixed $m \in \omega$ and an arbitrary $\delta > 0$ we have the following

(5) $|f(x_m) - g(x_m)| \leqslant |f(x_m) - f_n(x_m)| + |f_n(x_m) - g(x_m)| < \delta.$

whenever $n \in \omega$ is large enough. In particular, $f(x_0) = g(x_0)$. On the other hand, (2) implies that

$$\Big|f_i(x_\infty) - f_i(x_0)\Big| < 1/n$$

for all $i < n$. Let $n \to \infty$ and $i \to \infty$, we have $f(x_\infty) = f(x_0) = g(x_0)$. This contradicts (3). Therefore g must be continuous. We have shown that $\overline{Y} \subseteq C_p(X)$.

Therefore, $\overline{Y} = Y$. We have already observed that $\overline{Y}$ is compact, so that Y is compact. The proof is completed. ∎

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