# 1

# The driven cavity

## 1.1 The problem

We start with a simple problem for the Navier–Stokes equations, solved by simple methods. We will find the two-dimensional incompressible flow governed by

$$\nabla \cdot \mathbf{u} = 0,$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + \mu \nabla^2 \mathbf{u},$$

with constant density $\rho$ and viscosity $\mu$. These equations are to be solved inside a square $L \times L$ box with boundary conditions of no slip on the bottom and sides and a prescribed horizontal velocity along the top

$$\mathbf{u} = 0 \quad \text{on } y = 0 \text{ and } 0 < x < L, \text{ and on } x = 0 \text{ or } L \text{ and } 0 < y < L,$$

$$\text{and} \quad \mathbf{u} = (U(x), 0) \quad \text{on } y = L \text{ and } 0 < x < L.$$

This rectangular geometry is good for simple numerical methods.

We will evaluate the viscous force on the top

$$F = \int_0^L \mu \left.\frac{\partial u}{\partial y}\right|_{y=L} dx.$$

## 1.2 Know your physics

Before writing any code, it is worth thinking about the physics of the governing equations at the numerical grid level. The converse is also true that when presented with a new system of governing equations thinking about how to solve them numerically often deepens one's understanding of their physics.



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The Navier–Stokes equations have three different physics activities represented by different combinations of the terms.

First on the left-hand side,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}$$

says that information is propagated with the flow, at velocity $\mathbf{u}$. This means in a short time interval $\Delta t$ information has propagated a distance $u\Delta t$ from one grid point towards another. One might want to limit the size of the time-step so that information is not propagated too far, say more than one space grid block, in one time-step.

Looking just at the far left and far right terms of the Navier–Stokes equations, we have

$$\rho \frac{\partial \mathbf{u}}{\partial t} = \mu \nabla^2 \mathbf{u},$$

i.e. a diffusion equation with a diffusivity of the kinematic viscosity $\nu = \mu/\rho$. Thus in one time-step $\Delta t$ information diffuses a distance $\sqrt{\nu \Delta t}$. Keeping this distance less than one grid block requires very small time-steps. While diffusion is relatively fast on small length scales, it is slow on a large length scales, so one often has to wait rather a long time for information to have diffused over the whole grid.

Finally the terms

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p \quad \text{with} \quad \nabla \cdot \mathbf{u} = 0$$

are capable of propagating information to great distances in zero time, particularly in impulsively started motion of bodies in a fluid, reflecting the infinite speed of sound in our incompressible fluid. This behaviour is awkward for numerical work, and an early warning that treating the pressure will not be easy.

The Reynolds number $Re = UL/\nu$ measures the relative importance of inertial to viscous terms in the Navier–Stokes equations. At low Reynolds numbers, typically $Re < 1$, vorticity diffuses rapidly and this must resolved numerically. On the other hand at high Reynolds numbers, typically $Re > 1{,}000$, there are thin boundary layers and sometimes long wakes which must be resolved numerically. To avoid these difficulties in our first simple problem, we shall set the Reynolds number to

$$Re = 10,$$

which is not too low and not too high. Moreover the analytical theories for low and for high Reynolds numbers will not work well at this intermediate value, making numerical solution the only way to solve the problem.

## 1.3 Know your PDEs

Before attempting to solve any equations, it is necessary to know what boundary and initial conditions must be satisfied in order to make the problem well posed, i.e. has a solution which is unique if appropriate and which is not too sensitive to the input data. Applying the wrong type of boundary conditions to an equation can result in there being no solution, although the computer will often misleadingly deliver an output.

The simplest partial differential equation, which is also present in the Navier–Stokes equations, is the first-order hyperbolic equation in one space and one time dimension

$$\frac{\partial \phi}{\partial t} + u(x,t)\frac{\partial \phi}{\partial x} = f(x,t).$$

To make this well posed one needs initial data $\phi(x,0)$ at $t = 0$ over some space interval $a < x < b$ along with inflow boundary data, say $\phi(a,t)$ at $x = a$ for $t > 0$ if $u(a,t) > 0$.

The next prototype equation is the second-order hyperbolic equation, better known as the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}.$$

Being second order in time, this needs both the initial value and the initial time derivative, $\phi(x,0)$ and $\phi_t(x,0)$ at $t = 0$ over some interval $a < x < b$. As information propagates in both directions, boundary data must be supplied at both ends of the interval, e.g. $\phi(a,t)$ and $\phi(b,t)$ for $t > 0$, although in place of the value the spatial derivative $\phi_x$ or some combination such as $\phi + \phi_x$ can be given. On an infinite domain, the boundary conditions are replaced by radiation conditions, which are often tricky to impose numerically.

Another second-order equation is an elliptic equation, better known as the Poisson or Laplace equation

$$\nabla^2 \phi = \rho.$$

This needs boundary data $\phi$ or $\partial\phi/\partial n$ or some combination $\alpha\partial\phi/\partial n + \beta\phi$ (with restrictions on $\alpha$ and $\beta$) given all around the boundary.

Finally there is the parabolic equation, better known as the heat equation

$$\frac{\partial \phi}{\partial t} = D\frac{\partial^2 \phi}{\partial x^2}.$$

Being first order in time, it needs initial data $\phi(x,0)$ at $t = 0$ over some interval, while being second order in space it needs information ($\phi$ or $\phi_x$ or a combination) at the boundaries at both ends $x = a$ and $x = b$.

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The curious nomenclature comes from classifying the general linear second-order partial differential equation in two dimensions

$$a\frac{\partial^2\phi}{\partial x^2} + b\frac{\partial^2\phi}{\partial x\partial y} + c\frac{\partial^2\phi}{\partial y^2} + d\frac{\partial\phi}{\partial x} + e\frac{\partial\phi}{\partial y} + f = 0,$$

by comparing with the conic sections

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

of hyperbolas, ellipses and parabolas. With the obvious exception of the degenerate case of a parabolic equation, the first- and zero-order derivatives play a minor role in determining the mathematical behaviour, and so can be nonlinear without changing what constitutes a well-posed problem.

Numerically, hyperbolic equations are the most difficult to solve. In that they preserve information which they propagate around, any numerical error will be preserved, until it accumulates to swamp the real solution. Elliptic equations are the most costly to solve numerically, because every point in the domain influences every other point, which produces a very large coupled problem. Parabolic equations are the easiest and cheapest equation to solve on a computer. Practically any method works and works well, because little numerical errors made at one time-step decay very rapidly within a few further steps.

## 1.4 Special physics of the corner

In the computational fluid dynamics (CFD) literature it is very common to take a constant uniform velocity along the top lid of the cavity

$$U(x) = U_0.$$

Unfortunately this has a stress singularity in the corners like $\sigma \propto r^{-1}$ due to the discontinuity of the velocity at the corners. The stress singularity gives an infinite force on the top plate.

A better choice would be a velocity which vanishes linearly into the corners

$$U(x) = U_0 \sin(\pi x/L).$$

The viscous stresses are now regular, but the pressure has a logarithmic singularity. This weak singularity is integrable, but still is difficult to represent numerically.

Hence we shall take a velocity of the lid which vanishes quadratically at the corners

$$U(x) = U_0 \sin^2(\pi x/L).$$

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## 1.5 Nondimensionalisation

Engineers always use dimensional variables in computations: scientists do not.[1] We therefore scale the velocity $\mathbf{u}$ with $U_0$, lengths $x$ and $y$ with $L$, time $t$ with $L/U_0$ and pressure $p$ inertially with $\rho U_0^2$. This introduces a single non-dimensional group, the Reynolds number

$$Re = \frac{\text{inertial terms } \rho U_0^2/L}{\text{viscous terms } \mu U_0/L^2} = \frac{U_0 L}{\nu}.$$

The nondimensionalised problem is then

$$\nabla \cdot \mathbf{u} = 0,$$

$$\left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{u},$$

subject to boundary conditions

$$\mathbf{u} = 0 \quad \text{on } y = 0 \text{ and } 0 < x < 1, \text{ and on } x = 0 \text{ or } 1 \text{ and } 0 < y < 1$$

$$\text{and} \quad \mathbf{u} = (\sin^2(\pi x), 0) \quad \text{on } y = 1 \text{ and } 0 < x < 1.$$

We take a state of rest as the initial condition

$$\mathbf{u}(x, y, 0) = 0 \quad \text{at } t = 0 \text{ for } 0 < x < 1 \text{ and } 0 < y < 1.$$

We seek a solution at $Re = 10$.

Finally the force on the lid is scaled viscously by $\mu U_0$, so that we will evaluate

$$F = \int_0^1 \left. \frac{\partial u}{\partial y} \right|_{y=1} dx.$$

## 1.6 Steady vs transient calculations

While one might be interested only in the final steady state, it is normally easier to compute the time evolution from a simple initial condition to the final steady state. This is because the equations for the steady state are nearly always highly nonlinear, whereas the initial value problem is linear in the highest time derivative, e.g. linear in $\partial \mathbf{u}/\partial t$ in the Navier–Stokes equations. Moreover, there is the possibility that a steady state might not exist, or if it does exist might not

[1] An issue of philosophy. Engineers are interested in one practical realisation with all minor complications included, while scientists are interested in the general behaviour in a highly simplified model stripped of all minor complications. Adding complications increases the number of nondimensional groups faster than the number of dimensional variables.

be stable. The initial value problem will always have a solution, subject to there not being a finite-time blowup, and will show that a steady state is unstable if it is unstable.

Sometimes initial value problems approach the final steady state very slowly, and in such cases ways can be found to accelerate the slowly decaying transients.

Note that if one is interested in a series of steady problems, say the steady-state force on the top lid as a function of Reynolds number, then it is not necessary to begin each calculation from rest. Instead one can start the calculation for the next Reynolds number from the steady solution for the last Reynolds number. That would be a crude form of 'parameter continuation'.

Some relaxation methods for finding directly the steady state can be viewed as pseudotime evolutions.

§15.1 in Part III discusses methods for finding steady states.

## 1.7 Pressure!

The general idea for computing the evolution of the flow will be to be given $\mathbf{u}(x,t)$ at one time $t$, from this to evaluate $\partial\mathbf{u}/\partial t$ then and hence calculate $\mathbf{u}(x,t)$ at the next time level $t+\Delta t$. In this scheme we can easily evaluate the contributions to $\partial\mathbf{u}/\partial t$ from $-\mathbf{u}\cdot\nabla\mathbf{u}$ and from $\frac{1}{Re}\nabla^2\mathbf{u}$. The problem arises of how we are going to find the pressure gradient $-\nabla p$. In analytic calculations, the pressure field just seems to drop out of the calculation, so that it is only when one first tries to find a flow numerically one realises that it is a nontrivial issue to find the pressure.

The pressure field enables one to satisfy the conservation of mass: mathematically speaking, it is the 'Lagrangian multiplier' associated with the solenoidal constraint $\nabla\cdot\mathbf{u}=0$. In compressible fluids, the pressure is determined locally by the local density and temperature from an equation of state. In the incompressible limit, the pressure has to be determined globally by the need to make the velocity field solenoidal globally.

There are two alternative ways of tackling the pressure problem. In the so-called primitive variable formulation, we shall find the pressure gradient which makes the velocity solenoidal. Before tackling the problem head on, we will sidestep the pressure problem with the so-called streamfunction-vorticity formulation. This formulation is restricted to two-dimensional problems. The two formulations are taken up in the next two chapters.