TREES AND TREE-EQUIVALENT GRAPHS

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1. Introduction. As is well known in the theory of graphs a tree is a connected graph without cycles. Many characterizing properties of trees are known **(1)**, for example the cyclomatic number is equal to zero, which is also equal to $p - 1$, where p is the number of connected components of the graph. The graphs with cyclomatic number equal to $p - 1$ are defined here as *tree-equivalent graphs*. A tree is always a tree-equivalent graph but not conversely. The properties of tree-equivalent graphs are studied here. It is shown that by an operation on tree-equivalent graphs one can obtain a tree without disturbing the set of local degrees. The existence of trees with given local degrees follows as a corollary of the existence of tree-equivalent graphs.

2. Definitions.

2.1. An unoriented *graph* G is defined whenever we have a set X of abstract elements and a set U of edges which are undirected curves joining some pairs of distinct elements of X. It is denoted by (X, U).

2.2. If the maximum number of edges appearing in a graph which join the same two vertices is S, then the graph is called an *S-graph.*

2.3. For any vertex a, the subgraph of all vertices and edges that can be arrived at by travelling along the edges of the graph, including a, is called a *connected component* of G.

2.4. The number d_i of edges incident to a vertex x_i of G is called the *local degree* of G at x_i.

2.5. A *cycle* is a sequence of edges $(u_1, \ldots, u_q)$ such that

(1) u_k is attached to u_{k-1} by one of its extremities and to u_{k+1} by the other for $1 < k < q$,

(2) the initial extremity of u_1 coincides with the terminal extremity of u_q,

(3) no edge appears twice.

2.6. The *cyclomatic number* $K(G)$ of an S-graph G is defined to be $m - n + p$ where m is the number of edges, n is the number of vertices, and p is the number of connected components. This number also has the property that $K(G)$ linearly independent cycles exist in the graph.

2.7. Let $\{C_i\}$, $i = 1, 2, \ldots, r$, and $\{T_j\}$, $j = 1, 2, \ldots, s$, be the p connected

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components of an S-graph G without isolated vertices and let $s \geqslant 1$. If each C_i has at least one cycle, if each T_j is a tree, and if

$$K(G) = p - 1\ (= r + s - 1),$$

then G is said to be a *tree-equivalent graph.*

3. Properties of tree-equivalent graphs.

3.1. *A tree-equivalent graph does not have isolated vertices* (by definition).

3.2. *The local degrees* $(d_1, d_2, \ldots, d_n)$ *of a tree-equivalent graph satisfy the following:*

(1) $1 \leqslant d_i \leqslant n - 1$,
(2) $\sum_i d_i = 2(n - 1)$.

Proof. Since G is tree-equivalent, we have

$$m = k(G) + n - p = (p - 1) + n - p = n - 1;$$

but $\sum_i d_i = 2m$, since each edge contributes one count each to two of the d_i. Hence $\sum_i d_i = 2(n - 1)$.

By 3.1, each $d_i \geqslant 1$. There are exactly $n - 1$ edges in G, since $m = n - 1$ and almost $n - 1$ edges can be incident with any vertex. Therefore $d_i \leqslant n - 1$.

3.3. *A graph for which the above property* 3.2 *holds is a tree-equivalent graph.*

We need only prove that $K(G) = m - n + p = p - 1$, which is obviously true, and that there is at least one connected component which is a tree. But if each connected component contained a cycle, then each component would contain at least as many edges as vertices, whence it would follow that $m \geqslant n$, contradicting the hypothesis that $m = \frac{1}{2} \sum d_i = n - 1$. Therefore G has at least one component which is a tree, i.e. $s \geqslant 1$.

4. Existence of tree-equivalent graphs.

THEOREM 4.1. *A graph with n vertices with prescribed non-zero local degrees which add up to* $2(n - 1)$ *always exists.*

By 3.3, the existence of such a graph is equivalent to the existence of a tree-equivalent graph. For every tree-equivalent graph let us construct an incidence matrix $A = (a_{ij})$ of $n - 1$ rows and n columns such that

$$a_{ij} = \begin{cases} 1 & \text{if the edge } i \text{ is incident to the vertex } x_j, \\ 0 & \text{otherwise;} \end{cases}$$

here $i = 1, 2, \ldots, n - 1$ are the numbered edges of the graph and $\{x_j\}$, $j = 1, 2, \ldots, n$, are the vertices of the graph. The existence of a tree-equivalent graph is equivalent to the existence of an incidence matrix A with each row sum equal to two and the jth column sum equal to d_j. Thus we have to show the existence of a matrix of 0's and 1's with prescribed row and column sums. This is a particular case of a problem solved by Ryser for which we refer to **(2)**. From the majorization conditions given therein, it follows in our case that our matrix always exists, which in turn implies the existence of a tree-equivalent graph with prescribed local degrees.

THEOREM 4.2. *If $p > 1$, a tree-equivalent graph $p - 1$ components on n vertices can be obtained from a tree-equivalent graph of p components and n vertices without altering the local degrees of the graph.*

Proof. Let G be a tree-equivalent graph with p (>1) components. Then G has a component which contains a cycle. Let u be an edge of this cycle and v be a pendant edge of a component T_i which is a tree. If u joins the vertices a and b, and v joins c and d, then the removal of u and v and the insertion of an edge (a, c) and an edge (b, d) converts G into a new graph with the same local degrees and $p - 1$ components. This graph still remains a tree-equivalent graph by 3.3.

COROLLARY 4.1. *A tree with prescribed non-zero local degrees $(d_1, d_2, \ldots, d_n)$ exists if*

$$\sum_i d_i = 2(n - 1).$$

Proof. Given that d_i are non-zero and that $\sum d_i = 2(n - 1)$, a graph with local degrees d_i exists by Theorem 4.1 and it is tree-equivalent by 3.3. Therefore by Theorem 4.2, one can reduce the number of connected components step by step until one obtains a tree-equivalent graph with one component and local degrees d_i, that is, a tree with these local degrees.

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REFERENCES

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2. H. J. Ryser, *Combinatorial properties of matrices of zeros and ones*, Can. J. Math., *9* (1957), 371–377.

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