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Approximation and the Topology of Rationally Convex Sets

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Abstract. Considering a mapping g holomorphic on a neighbourhood of a rationally convex set $K \subset \mathbb{C}^n$, and range into the complex projective space $\mathbb{CP}^m$, the main objective of this paper is to show that we can uniformly approximate g on K by rational mappings defined from $\mathbb{C}^n$ into $\mathbb{CP}^m$. We only need to ask that the second Čech cohomology group $\check{H}^2(K,\mathbb{Z})$ vanishes.

1 Introduction

Let $\mathbb{CP}^m$ be the m-complex projective space, composed of all the complex lines in $\mathbb{C}^{m+1}$ which pass through the origin. It is well known that $\mathbb{CP}^m$ is an m-complex manifold, and that there exists a natural holomorphic projection ρ_m defined from $\mathbb{C}^{m+1} \setminus \{0\}$ onto $\mathbb{CP}^m$, which sends any point $(z_0,\dots,z_m) \neq 0$ to the complex line

$$\rho_m(z_0,\dots,z_m) = [z_0,\dots,z_m] := \{(z_0 t,\dots,z_m t) : t \in \mathbb{C}\}.$$

In particular, we have that the one-dimensional complex projective space $\mathbb{CP}^1$ is the Riemann sphere $\mathbb{S}^2$, and the natural holomorphic projection ρ_1 is given by $\rho_1(w,z) = [\frac{w}{z},1]$ or $[1,\frac{z}{w}]$. Thus, any rational mapping p/q defined on $\mathbb{C}^n$ may be seen as the composition $\rho_1(p,q)$, where (p,q) is a holomorphic polynomial mapping from $\mathbb{C}^n$ into $\mathbb{C}^2$. The critical set E of p/q is the inverse image $(p,q)^{-1}(0)$, and so p/q is a holomorphic mapping defined from $\mathbb{C}^n \setminus E$ into $\mathbb{S}^2$. Previous interpretation allows us to extend the notion of rational mapping to consider the natural projections ρ_m for $m \geq 1$.

Definition 1 A rational mapping based on $\mathbb{C}^n$, and image in $\mathbb{CP}^m$, is defined as the composition $\rho_m(P)$ for a given holomorphic polynomial mapping $P\colon \mathbb{C}^n \to \mathbb{C}^{m+1}$. The critical set E of $\rho_m(P)$ is then defined as the inverse image $P^{-1}(0)$, and so $\rho_m(P)$ is a holomorphic mapping defined from $\mathbb{C}^n \setminus E$ into $\mathbb{CP}^m$.

Recall the fundamentals of rational approximation theory. A compact set K in $\mathbb{C}^n$ is rationally convex if for every point $y \in \mathbb{C}^n \setminus K$ there exists a holomorphic polynomial p such that $p(y) = 0$ and p does not vanish on K. Besides, it is well known that each function $h\colon U \to \mathbb{C}$ holomorphic on a neighbourhood U of K can be approximated on K by rational functions, whenever K is rationally convex; see for

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example [2, 7]. That is, for each $\widehat{\delta} > 0$ there exists a holomorphic rational function p/q such that K does not meet the zero locus of q and $\left|\frac{p(z)}{q(z)} - h(z)\right|$ is strictly less than $\widehat{\delta}$ on K. This result automatically drives us to consider whether the concept of rationally convex sets is strong enough to imply approximation by the kind of rational mappings that we have just introduced in Definition 1. Amazingly, we can get a positive answer by adding a simple cohomological condition.

Theorem 2 (Main theorem) *Let K be a rationally convex set in $\mathbb{C}^n$, and* Dist *a metric on $\mathbb{CP}^m$ which induces the topology, with $m, n \geq 1$. If the second Čech cohomology group $\check{H}^2(K, \mathbb{Z})$ vanishes, then for each $\widehat{\delta} > 0$ and any mapping $g\colon U \to \mathbb{CP}^m$ holomorphic on a neighbourhood U of K, there exists a rational mapping $\rho_m(P)$ defined on $\mathbb{C}^n$ whose critical set does not meet K, and such that* $\mathrm{Dist}\left[\rho_m(P(z)), g(z)\right]$ *is less than $\widehat{\delta}$ on K.*

This result was mainly inspired by the work of Grauert, Kerner and Oka [9, 10, 17]. In an early paper [8] we analysed the approximation by the rational mappings described in Definition 1; we deduced a result similar to Theorem 2 by using the extra topological condition of being null-homotopic. We give a complete reference to this early result in Section 4.

We shall prove Theorem 2 in the third section of this paper. Moreover, we devote the second section to introducing the results on cohomology theory which we need for the proof of the Theorem 2. Finally, examples and corollaries are introduced in Section 4.

2 Cohomology

We strongly recommend [1, 3, 11] for references on homotopy theory and [14, 18, 19], for references on cohomology theory.

We consider two main classes of cohomology groups: Čech and singular. These cohomology groups are both isomorphic on smooth manifolds, and open subsets of $\mathbb{C}^n$, for these spaces are all locally contractible; see for example [18, p. 166] or [19, pp. 334, 341]. However, there is a very nice example in [19, pp. 77, 317] of a compact set $K \subset \mathbb{C}$ whose Čech cohomology group $\check{H}^1(K, \mathbb{Z}) = \mathbb{Z}$, but its singular cohomology group $H^1_s(K, \mathbb{Z})$ vanishes. Now, given a closed set $E \subset \mathbb{C}^n$, we need Čech cohomology groups, because $\check{H}^*(E, \mathbb{Z})$ can be calculated as the direct limit of the sequence $\{\check{H}^*(U, \mathbb{Z})\}$, where U runs over a system (directed by inclusions) of open neighbourhoods of E in $\mathbb{C}^n$; see for example [2, Ch. 15], [3, p. 348], [18, p. 145] or [19, p. 327]. So the Čech cohomology group $\check{H}^*(E, \mathbb{Z})$ vanishes if and only if for each element $\xi \in \check{H}^*(U, \mathbb{Z})$ defined on an open neighbourhood U of E, there exists a second open set W such that $E \subset W \subset U$ and the restriction $\xi|_W$ is equal to zero in $\check{H}^*(W, \mathbb{Z})$.

On the other hand, we need singular cohomology groups because of the following universal result. Let $U \subset \mathbb{C}^n$ be an open subset which has the homotopy type of a CW-complex. The singular cohomology group $H^k_s(U, \mathbb{Z})$ is then isomorphic to the group of homotopy classes $[U, Y]$, where $k \geq 1$ and Y is an Eilenberg–MacLane space of type $(\mathbb{Z}, k)$; see for example [1, p. 183], [3, pp. 488–492], [4, p. 274] or [19, p. 428]. Recall that Y is an Eilenberg–MacLane space of type $(\mathbb{Z}, k)$ if every homotopy group

$\pi_*(Y)$ vanishes, with the only exception of $\pi_k(Y)$, which is equal to $\mathbb{Z}$. Recall that $[U, Y]$ is the group composed by all the homotopy classes of continuous mappings $f: U \to Y$.

Combining the ideas presented in previous paragraphs, we may deduce the following result. Let $E \subset \mathbb{C}^n$ be a closed set whose Čech cohomology group $\check{H}^k(E, \mathbb{Z})$ vanishes, $k \geq 1$. Suppose from now on that E has a system (directed by inclusions) of open neighbourhoods $\{U_\beta\}$ in $\mathbb{C}^n$, where each U_β has the homotopy type of a CW-complex and $E = \bigcap_\beta U_\beta$. Given an Eilenberg–MacLane space Y of type $(\mathbb{Z}, k)$, and since Čech and singular cohomology groups are isomorphic on each U_β, we have that for every continuous mapping $f: U_\beta \to Y$ there exists a second neighbourhood U_θ such that $E \subset U_\theta \subset U_\beta$ and the restriction $f|_{U_\theta}: U_\theta \to Y$ is null-homotopic. The main idea behind this result is to see the homotopy class of the mapping f as an element of $H^k_s(U_\beta, \mathbb{Z}) \cong \check{H}^k(U_\beta, \mathbb{Z})$.

Let us illustrate the previous result with a known example. Recall that $\mathbb{C} \setminus \{0\}$ has the homotopy type of the 1-dimensional sphere $\mathbb{S}^1$ and that $\mathbb{S}^1$ is an Eilenberg–MacLane space of type $(\mathbb{Z}, 1)$ because $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ and $\pi_k(\mathbb{S}^1) = 0$ for every $k \neq 1$. We may deduce that for each non-vanishing continuous function $f: U_\beta \to \mathbb{C} \setminus \{0\}$ and whenever $\check{H}^1(E, \mathbb{Z}) = 0$, there exists a second neighbourhood U_θ such that the restriction $f|_{U_\theta}: U_\theta \to \mathbb{C} \setminus \{0\}$ has a well defined continuous logarithm ($f|_{U_\theta}$ is null-homotopic) [2, Ch. 15], [7, §7, Ch. III].

Coming back to the main theorem of this paper, in the hypotheses we suppose that the second Čech cohomology group $\check{H}^2(K, \mathbb{Z})$ vanishes, so we need an example of an Eilenberg–MacLane space of type $(\mathbb{Z}, 2)$. This example is given by the infinite dimensional complex projective space $\mathbb{CP}^\infty$; see [1, p. 360], [11, p. 157] or [19, p. 425]. Repeating all the calculations done in previous paragraphs, we may prove the following lemma.

Lemma 3 *Let E be a closed subset of $\mathbb{C}^n$ whose second Čech cohomology group $\check{H}^2(E, \mathbb{Z})$ vanishes. Suppose that E has a system (directed by inclusions) of open neighbourhoods $\{U_\beta\}$ in $\mathbb{C}^n$, where each U_β has the homotopy type of a CW-complex and $E = \bigcap_\beta U_\beta$. We have that for every continuous mapping $f: U_\beta \to \mathbb{CP}^\infty$ there exists a second neighbourhood U_θ such that $E \subset U_\theta \subset U_\beta$ and the restriction $f|_{U_\theta}: U_\theta \to \mathbb{CP}^\infty$ is null-homotopic.*

3 Proof of the Main Theorem

We need to recall some properties about the infinite dimensional complex projective space $\mathbb{CP}^\infty$. Consider the infinite dimensional space $\mathbb{C}^\infty$ composed of all the complex sequences $(z_0, z_1, \dots)$, where only a finite number of entries z_k are different from zero. This space $\mathbb{C}^\infty$ is naturally endowed with the standard norm $\sqrt{\sum_k |z_k^2|}$. The complex projective space $\mathbb{CP}^\infty$ is then composed of all the complex lines in $\mathbb{C}^\infty$ which pass through the origin. Besides, there exists a natural projection ρ_∞ defined from $\mathbb{C}^\infty \setminus \{0\}$ onto $\mathbb{CP}^\infty$ which sends any point $(z_0, z_1, \dots) \neq 0$ to the complex line

$$\rho_\infty(z_0, z_1, \dots) = [z_0, z_1, \dots] := \{(z_0 t, z_1 t, \dots) : t \in \mathbb{C}\}.$$

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Finally, it is easy to calculate that ρ_∞ induces a locally trivial fibre bundle in $\mathbb{C}^\infty \setminus \{0\}$, with base on $\mathbb{CP}^\infty$ and fibre $\mathbb{C} \setminus \{0\}$; see [1, p. 360]. We may cover $\mathbb{CP}^\infty$ with open sets W_k composed of all points in $\mathbb{CP}^\infty$ whose k-entry is equal to one. The open set W_0 is equal to $\{[1, y] : y \in \mathbb{C}^\infty\}$, for example. It is now easy to calculate that ρ_∞ induces a trivial fibre bundle on each $\rho_\infty^{-1}(W_k)$, with base on W_k and fibre $\mathbb{C} \setminus \{0\}$. Actually, for every $m \geq 1$, we have that the projection ρ_m induces a locally trivial fibre bundle in $\mathbb{C}^{m+1} \setminus \{0\}$, with base on $\mathbb{CP}^m$ and fibre $\mathbb{C} \setminus \{0\}$, as well.

Proof of Theorem 2 Define the open rational polyhedra V_β in $\mathbb{C}^n$ by the formula,

$$V_\beta := \{z \in \mathbb{C}^n : |p_j(z)| < 1, \forall j \text{ and } |q_k(z)| > 1, \forall k\}, \tag{1}$$

for some given finite collections of holomorphic polynomials $\{p_j\}$ and $\{q_k\}$ in $\mathbb{C}^n$. We have that each rational polyhedron V_β is an open Stein subset of $\mathbb{C}^n$, so V_β has the homotopy type of a CW-complex [15, p. 39]. Moreover, given a rationally convex set $K \subset \mathbb{C}^n$, it is easy to see that the family of all open rational polyhedra V_β which contain K form a system of neighbourhoods in $\mathbb{C}^n$, and that K is equal to the intersection $\bigcap_{K \subset V_\beta} V_\beta$.

On the other hand, let $g \colon U \to \mathbb{CP}^m$ be any mapping holomorphic on a neighbourhood U of K. We can obviously extend this mapping to a second one with range on $\mathbb{CP}^\infty$, we only need to set the first $m + 1$ entries equal to the entries of $g = [g_0, \dots, g_m]$ and the rest of them equal to zero. That is, define $\check{g} \colon U \to \mathbb{CP}^\infty$ by

$$\check{g}(z) := [g_0(z), \dots, g_m(z), 0, \dots, 0, \dots]. \tag{2}$$

We may find an open rational polyhedron V_β such that $K \subset V_\beta \subset U$. Besides, recalling that the Čech cohomology group $\check{H}^2(K, \mathbb{Z})$ vanishes because of the given hypotheses, and considering Lemma 3, we may even find a rational polyhedron V_β such that the restriction $\check{g}|_{V_\beta} \colon V_\beta \to \mathbb{CP}^\infty$ is null-homotopic. Let $I = [0, 1]$ the unit closed interval in the real line. There exists then a continuous mapping G from $V_\beta \times I$ into $\mathbb{CP}^\infty$ such that $G(z, 1) = \check{g}(z)$ and $G(z, 0) = [1, 0, \dots]$, for every $z \in V_\beta$. We have the following commutative diagram,

$$\begin{array}{ccc} V_\beta & \xrightarrow{\;c\;} & \mathbb{C}^\infty \setminus \{0\} \\ \downarrow{\scriptstyle \jmath} & & \downarrow{\scriptstyle \rho_\infty} \\ V_\beta \times I & \xrightarrow{\;G\;} & \mathbb{CP}^\infty, \end{array}$$

where $c(z) = (1, 0, \dots)$ is a constant mapping and $\jmath(z) = (z, 0)$ is the natural inclusion. We know that the projection ρ_∞ induces a locally trivial fibre bundle on $\mathbb{C}^\infty \setminus \{0\}$, with base $\mathbb{CP}^\infty$ and fibre $\mathbb{C} \setminus \{0\}$. This fibre bundle has the homotopy lifting property; see for example [4, pp. 62, 67], [11, p. 87] or [19, p. 96]. Hence, there exists a continuous mapping F from $V_\beta \times I$ into $\mathbb{C}^\infty \setminus \{0\}$ such that $\rho_\infty(F)$ is identically equal to G on $V_\beta \times I$. Recalling equation (2) where $\breve{g}(z) = G(z,1)$ was defined, we can deduce that $F(z,1)$ has the form,

$$F(z,1) := (F_0(z,1),\dots,F_m(z,1),0,\dots,0,\dots).$$

We may then introduce a new continuous mapping f defined from V_β into $\mathbb{C}^{m+1} \setminus \{0\}$, by removing the last entries of $F(z,1)$ equal to zero; that is,

$$f(z) := (F_0(z,1),\dots,F_m(z,1)) \quad \text{for every } z \in V_\beta.$$

It is easy to see that $\rho_m(f(z)) = g(z)$ for every $z \in V_\beta$. The main objective of previous calculations was the construction of the continuous mapping f described above. Actually, we could have showed the existence of such a mapping f by using the results on obstruction theory described in [3, p. 507] and [19, p. 447]. However, we think that the procedure followed in previous paragraphs is simpler and more illustrative.

Nevertheless, we look for a holomorphic (not only continuous) mapping h from V_β into $\mathbb{C}^{m+1} \setminus \{0\}$ such that $\rho_m(h(z)) = g(z)$ for every $z \in V_\beta$. We shall construct this holomorphic mapping h by using Oka's results on the second Cousin problem. Define the space,

$$M := \{(z,w) \in V_\beta \times \mathbb{C}^{m+1} : g(z) = \rho_m(w),\ w \neq 0\}.$$

It is easy to deduce that M is an analytic space because g and ρ_m are both holomorphic mappings. Moreover, we also have the following commutative diagram,

$$\begin{array}{ccc} M & \xrightarrow{\ \eta_2\ } & \mathbb{C}^{m+1} \setminus \{0\} \\ \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \rho_m} \\ V_\beta & \xrightarrow{\ g\ } & \mathbb{CP}^m, \end{array}$$

where $\eta_1(z,w) = z$ and $\eta_2(z,w) = w$ are the basic projections. It is easy to prove that η_1 induces a locally trivial fibre bundle in M, with Stein base on V_β and Stein fibre $\mathbb{C} \setminus \{0\}$. This fibre bundle $M \xrightarrow{\eta_1} V_\beta$ is the *pullback* of the fibre bundle induced by ρ_m on $\mathbb{C}^{m+1} \setminus \{0\}$. Now recalling the continuous mapping f defined above, we automatically have that $z \mapsto (z, f(z))$ is a continuous section of the fibre bundle $M \xrightarrow{\eta_1} V_\beta$, because $g(z) = \rho_m(f(z))$ for every $z \in V_\beta$. Oka's results on the second Cousin problem imply that $z \mapsto (z, f(z))$ is homotopic to a holomorphic section $z \to (z, h(z))$, because V_β is Stein; see for example [5, 6, 10, 17]. Hence, there exists a holomorphic mapping $h\colon V_\beta \to \mathbb{C}^{m+1} \setminus \{0\}$ such that $g(z)$ is equal to $\rho_m(h(z))$ for every $z \in V_\beta$.

On the other hand, let W be an open subset of $\mathbb{C}^{m+1}$ which contains the compact image $h(K)$. Suppose that the closure $\overline{W}$ is compact and does not contains the origin. Notice that the projection ρ_m from $\mathbb{C}^{m+1} \setminus \{0\}$ into $\mathbb{CP}^m$ is continuous with respect

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to the metric Dist, which induces the topology, so ρ_m is also uniformly continuous on $\overline{W}$. Express $h = (h_0, \dots, h_m)$ as a vector in $\mathbb{C}^{m+1}$. There are $m + 1$ small enough constants $\delta_k > 0$ such that, given $z \in K$ and $w \in \mathbb{C}^{m+1}$,

$$\text{(3)}\quad \operatorname{Dist}\big[\rho_m(h(z)), \rho_m(w)\big] < \widehat{\delta} \text{ and } w \in \overline{W}, \quad \text{if } |h_k(z) - w_k| < \delta_k \text{ for } 0 \le k \le m.$$

Recalling that K is rationally convex, we may find $m+1$ rational functions $w_k = \frac{p_k}{q_k}$ defined on $\mathbb{C}^n$, such that K meets the zero locus of no q_k, for every $0 \le k \le m$, and the absolute value $\big|h_k(z) - \frac{p_k}{q_k}(z)\big|$ is strictly less than δ_k on K. Consider the polynomial mapping $P\colon \mathbb{C}^n \to \mathbb{C}^{m+1}$ given by

$$P := \Big(\frac{p_0}{q_0}\prod_{k=0}^{m} q_k, \dots, \frac{p_m}{q_m}\prod_{k=0}^{m} q_k\Big).$$

It is easy to deduce that neither $\big(\frac{p_0}{q_0}, \dots, \frac{p_m}{q_m}\big)$ nor the product $\prod_{k=0}^{m} q_k$ can vanishes on K, because of $0 \notin \overline{W}$ and equation (3). Thus, the compact set K does not meet the critical set $P^{-1}(0)$ of the rational mapping $\rho_m(P)$. We may also deduce that $\rho_m(P)$ is equal to $\rho_m\big(\frac{p_0}{q_0}, \dots, \frac{p_m}{q_m}\big)$ on K. Therefore, recalling equation (3), and that $\rho_m(h(z))$ is equal to $g(z)$ for every $z \in K$, we get the result that we are looking for: the distance $\operatorname{Dist}\big[g(z), \rho_m(P(z))\big]$ is strictly less than $\widehat{\delta}$ on K. ■

4 Examples and Applications

We shall conclude this paper with the observation that the cohomological condition $\check{H}^2(K, \mathbb{Z}) = 0$ in the hypotheses of the main theorem is not a trivial condition. Consider the standard two-sphere in $\mathbb{R}^3$,

$$\mathbb{S}^2 := \{(a, b, c) \in \mathbb{R}^3 : a^2 + b^2 + c^2 = 1\}.$$

We can obviously analyse $\mathbb{S}^2$ as a subset of $\mathbb{C}^3$, embedding it into the real space; it is easy to see that $\mathbb{S}^2$ is rationally convex in $\mathbb{C}^3$. Besides, the groups $\check{H}^2(\mathbb{S}^2, \mathbb{Z})$ and $H^2_s(\mathbb{S}^2, \mathbb{Z})$ are both isomorphic to $\mathbb{Z}$. Finally, consider the following open neighbourhood U of $\mathbb{S}^2$,

$$U := \{(x, y, z) \in \mathbb{C}^3 : -\pi < \arg(x^2 + y^2 + z^2) < \pi\},$$

and the holomorphic mapping $g\colon U \to \mathbb{CP}^1$, where $\sqrt{1} = 1$,

$$g(x, y, z) = \Big[\frac{x + iy}{\sqrt{(x^2 + y^2 + z^2)} + z}, 1\Big] \quad \text{or} \quad \Big[1, \frac{x - iy}{\sqrt{(x^2 + y^2 + z^2)} - z}\Big].$$

It is easy to see that the restriction $g|_{\mathbb{S}^2}$ is the identity mapping from $\mathbb{S}^2$ onto $\mathbb{CP}^1$, because $g(a, b, c) = \big[\frac{a+ib}{1+c}, 1\big]$ is the stereographic projection, so $g|_{\mathbb{S}^2}$ is not null-homotopic. Moreover, we assert that g cannot be approximated on $\mathbb{S}^2$ by rational mappings $\rho_1(P)$. That is, there exists a fixed constant $\beta_g > 0$, such that

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for every rational mapping $\rho_1(P)$ holomorphic on $\mathbb{S}^2$ there is a point $\check{w} \in \mathbb{S}^2$ with $\operatorname{Dist}\big[\rho_1(P(\check{w})), g(\check{w})\big]$ greater than β_g.

Let $P\colon \mathbb{C}^3 \to \mathbb{C}^2$ be any polynomial mapping whose fibre $P^{-1}(0)$ does not meet $\mathbb{S}^2$. We can deduce that the restriction $P|_{\mathbb{S}^2}$ defined from $\mathbb{S}^2$ into $\mathbb{C}^2 \setminus \{0\}$ is null-homotopic, because $\mathbb{C}^2 \setminus \{0\}$ has the homotopy type of the three-sphere $\mathbb{S}^3$ and the second homotopy group $\pi_2(\mathbb{S}^3)$ vanishes. Therefore, we also have that the restriction $\rho_1(P)|_{\mathbb{S}^2}$ is null-homotopic for every rational mapping $\rho_1(P)$ defined on $\mathbb{C}^3$, and whose critical set $P^{-1}(0)$ does not meet $\mathbb{S}^2$. Finally, since $g|_{\mathbb{S}^2}$ is not null-homotopic, we can conclude that there exists a fixed constant $\beta_g > 0$, such that for every rational mapping $\rho_1(P)$ holomorphic on $\mathbb{S}^2$ there is a point $\check{w} \in \mathbb{S}^2$ with $\operatorname{Dist}\big[\rho_1(P(\check{w})), g(\check{w})\big]$ greater than β_g. We only need to recall that $\mathbb{CP}^1$ is an absolute neighbourhood retract [13, pp. 332, 339], and to apply the following lemma, which was originally presented in [8].

Lemma 4 *Let X and (Y, d) be two metric spaces, such that X is compact and Y is an absolute neighbourhood retract. Then, given a fixed continuous mapping $g\colon X \to Y$, there exists a constant $\beta_g > 0$ such that every continuous mapping $f\colon X \to Y$ is homotopic to g, whenever $\mathrm{d}[g(x), f(x)]$ is less than β_g for every $x \in X$.*

Proof Let Y^X be the topological space composed of all the continuous mappings $f\colon X \to Y$, and endowed with the compact-open topology. Since X is compact and (Y, d) is metric, the compact-open topology of Y^X is induced by the metric

$$\mathrm{D}[f_1, f_2] := \sup\big\{ \mathrm{d}[f_1(x), f_2(x)] : x \in X \big\},$$

for any two mappings f_1 and f_2 in Y^X; see for example [13, p. 89]. The space Y^X is locally arcwise connected and an absolute neighbourhood retract, because Y is an absolute neighbourhood retract; see [13, pp. 339–340]. There exists a fixed constant $\beta_g > 0$ such that the open ball in Y^X with centre in g and radius β_g is contained in an arcwise connected neighbourhood of g. That is, for every continuous mapping $f\colon X \to Y$ with $\mathrm{D}[f, g] < \beta_g$, there exists an arc in Y^X whose end points are f and g, and so the mappings g and f are homotopic. ■

Finally, the ideas introduced in the proof of the main theorem may be used to show several versions of this theorem. Consider for example the following definition.

Definition 5 Given a closed set E in $\mathbb{C}^n$, we say that the continuous function $f\colon E \to \mathbb{C}$ can be tangentially approximated by meromorphic mappings, if for every strictly positive continuous function $\hat{\epsilon}\colon E \to \mathbb{R}$, there exists a pair of holomorphic functions ϕ and ψ defined from $\mathbb{C}^n$ into $\mathbb{C}$, such that E does not meet the zero locus of ψ and $\big| f(z) - \frac{\phi(z)}{\psi(z)} \big|$ is less than $\hat{\epsilon}(z)$ for every $z \in E$.

We may now show the following result.

Theorem 6 *Let E be a closed set in $\mathbb{C}^n$ whose second Čech cohomology group $\check{H}^2(E, \mathbb{Z})$ vanishes, and such that every continuous function $f\colon E \to \mathbb{C}$ can be tangentially approximated by meromorphic mappings. Suppose that* Dist *is a metric on $\mathbb{CP}^m$ which induces*

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the topology, and that E has a system of open neighbourhoods $\{V_\beta\}$ in $\mathbb{C}^n$, where each V_β has the homotopy type of a CW-complex and $E = \bigcap_\beta V_\beta$.

For every pair of continuous mappings $\xi\colon E \to \mathbb{CP}^m$ and $\hat{\epsilon}\colon E \to \mathbb{R}$, with $\hat{\epsilon}(z) > 0$, there exists a holomorphic mapping H defined from $\mathbb{C}^n$ into $\mathbb{C}^{m+1}$, such that E does not meet the zero locus of H and $\operatorname{Dist}\big[\rho_m(H(z)), \xi(z)\big]$ *is less than $\hat{\epsilon}(z)$ for every $z \in E$.*

Proof We only give a sketch of this proof, for it is essentially the same one presented in Section 3. First, we have that $\mathbb{CP}^m$ is an absolute neighbourhood retract, for it is homeomorphic to a compact polyhedron; see [13, pp. 332, 339]. Therefore, there exists a continuous mapping $g\colon U \to \mathbb{CP}^m$ defined on an open neighbourhood U of E and such that $g(z) = \xi(z)$ for every $z \in E$. Besides, following equation (2), we can extend g to a continuous mapping $\breve{g}$ defined from U into $\mathbb{CP}^\infty$.

Considering Lemma 3, there exists an open neighbourhood V_β such that $E \subset V_\beta \subset U$ and the restriction $\breve{g}|_{V_\beta}\colon V_\beta \to \mathbb{CP}^\infty$ is null-homotopic. Following the ideas presented in Section 3 of this paper, we can build a second continuous (not necessarily holomorphic) mapping h from V_β into $\mathbb{C}^{m+1} \setminus \{0\}$ such that $\rho_m(h(z))$ is equal to $g(z) = \xi(z)$ for every $z \in E$. Notice that ρ_m is continuous with respect to the metric Dist, which induces the topology. Express $h = (h_0, \dots, h_m)$ as a vector in $\mathbb{C}^{m+1} \setminus \{0\}$. There are $m + 1$ strictly positive continuous functions $\delta_k\colon E \to \mathbb{R}$ such that, given $z \in E$ and $w \in \mathbb{C}^{m+1}$,

$$(4) \qquad \operatorname{Dist}\big[\rho_m(h(z)), \rho_m(w)\big] < \hat{\epsilon}(z), \qquad \text{if } |h_k(z) - w_k| < \delta_k(z) \text{ for } 0 \le k \le m.$$

Recalling that every continuous function can be tangentially approximated by meromorphic mappings on E, we may find holomorphic functions ϕ_k and ψ_k defined from $\mathbb{C}^n$ into $\mathbb{C}$ such that E meets the zero locus of no ψ_k, for $0 \le k \le m$, and the absolute value $\big|h_k(z) - \frac{\phi_k}{\psi_k}(z)\big|$ is strictly less than $\delta_k(z)$ for every $z \in E$. Moreover, we can even choose the functions δ_k in such a way that each $\delta_k(z)$ is strictly less than $\max_j\big\{\frac{|h_j(z)|}{2}\big\}$ for all $z \in E$. Hence, we have that neither $\big(\frac{\phi_0}{\psi_0}, \dots, \frac{\phi_m}{\psi_m}\big)$ nor the product $\prod_{k=0}^m \psi_k$ can vanish on E, and so the set E does not meet the zero locus of the holomorphic mapping $H\colon \mathbb{C}^n \to \mathbb{C}^{m+1}$ defined by

$$H := \Big(\frac{\phi_0}{\psi_0}\prod_{k=0}^m \psi_k, \dots, \frac{\phi_m}{\psi_m}\prod_{k=0}^m \psi_k\Big).$$

It is easy to deduce that $\rho_m(H)$ is equal to $\rho_m\big(\frac{\phi_0}{\psi_0}, \dots, \frac{\phi_m}{\psi_m}\big)$ on E. Whence, recalling equation (4), and that $\rho_m(h(z))$ is equal to $g(z) = \xi(z)$ for every $z \in E$, we get the result that we look for: $\operatorname{Dist}[\xi(z), \rho_m(H(z))]$ is strictly less than $\hat{\epsilon}(z)$ on E. ■

We have already proved a weaker version of this theorem in [8]. We used there the stronger hypotheses that E is compact and $\xi\colon E \to \mathbb{CP}^m$ is null-homotopic, instead of requiring $\check{H}^2(E, \mathbb{Z}) = 0$.

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