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RINGS WITH A FINITELY GENERATED TOTAL QUOTIENT RING

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1. **Introduction and summary.** Let R be a commutative ring with non-zero identity and let K be the total quotient ring of R. We call R a *G-ring* if K is finitely generated as a ring over R. This generalizes Kaplansky's definition of G-domain [5].

Let $Z(R)$ be the set of zero divisors in R. Following [7] elements of $R-Z(R)$ and ideals of R containing at least one such element are called regular. Artin-Tate's characterization of Noetherian G-domains [1, Theorem 4] carries over with a slight adjustment to characterize a Noetherian G-ring as being semi-local in which every regular prime ideal has rank one. We also characterize a Noetherian G-ring R as a ring whose integral closure in K has only finitely many overrings in K. We provide an example to show that the one dimensionality of Noetherian G-domains does not generalize. Transference of the G-property between a ring and some of its extensions by way of direct products, polynomials, power series, and completions is investigated. Finally we note that a Noetherian Hilbert G-ring must be a total quotient ring in agreement with the case for domains.

The results in this paper form part of the author's doctoral dissertation completed in December 1970 at the University of Colorado. Subsequent to this, the paper of Davison [3], also containing a generalization of Artin-Tate's result, was brought to our attention. The equivalence of (i) and (ii) in our Proposition 2.1 follows from [3, Proposition 2, p. 427].

2. G-**property characterization.** It is clear that K is finite over R if and only if $K=R[1/u]$ for some regular element u in R. Another necessary and sufficient condition is that u lie in the intersection of all regular prime ideals of R.

PROPOSITION 2.1. *Let R be a Noetherian ring. Then the following statements are equivalent.*

(i) *R is a G-ring.*

(ii) *R is semi-local and every regular prime ideal of R has rank one.*

(iii) *The integral closure of R in K has only finitely many overrings in K.*

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Proof. (i) implies (ii): With u in R as above, let $P_1, \ldots, P_n$ be the prime ideals in R minimal over Ru. We are done if each P_i is maximal. Take $P_{n+1}, \ldots, P_m$ to be those prime ideals mamxial in $Z(R)$ which are not contained in the union of $P_1, \ldots, P_n$. Given a $b \in R - P_1$ define b' to be the sum of b and all a_i such that $b \in P_i$ where each $a_i \in \bigcap_j P_j - P_i$. It follows that $b - b' \in P_1$ and $b' \notin P_i$ for $i = 1, \ldots, m$. If $Rb' \neq R$ then any prime ideal minimal over this regular ideal must also be minimal over Ru. Therefore b' is a unit and $(P_1, b) = R$.

(ii) implies (i): This is immediate.

(i) implies (iii): Let $\bar{R}$ be the integral closure of R in K. If R is a G-ring, then $\bar{R}$ is a G-ring and satisfies (ii). The results in [2, Corollary 1, p. 207] and [2, Theorem 4, p. 209] show that every overring of $\bar{R}$ is an intersection of overrings of the form $\bar{R}_{S(P)}$ where P is a regular prime ideal of $\bar{R}$ and $S(P)$ is the complement of the union of P and $Z(\bar{R})$ in $\bar{R}$.

(iii) implies (i): If $P \neq Q$ are regular prime ideals in $\bar{R}$, then $\bar{R}_{S(P)} \neq \bar{R}_{S(Q)}$. So (iii) implies $\bar{R}$ has a finite number of regular prime ideals $P_1, \ldots, P_n$. But each regular prime ideal of R is of the form $P_i \cap R$.

Compare the proof of "(i) implies (ii)" with the proof in [1, Theorem 4].

See [3, Theorem 1, p. 428] for a characterization of the statement "R is a semi-local ring with Krull dimension one." By 2.1 this statement is sufficient for R to be a G-ring. We provide an example to show it is not necessary.

Let k be a field and let $T = k[X_0, \ldots, X_N]$. Let I be an ideal of T belonging to $P_i = (X_0, \ldots, X_i)$ for $i = 1, \ldots, N$ [7, p. 230]. Take S to be the complement of the union of P_N and $Q = (X_0, X_1 - 1)$ in T. Then $R = T_S/I_S$ is a Noetherian G-ring, $R \neq K$, and R has Krull dimension N.

3. **Extensions of the G-property.** Let $\pi_i R_i$ denote the direct product of a collection of rings $\{R_i\}$. If $R = \pi_i R_i$ we have:

PROPOSITION 3.1. *R is a G-ring if and only if each R_i is a G-ring and $R_i = K_i$ for all but finitely many indices i.*

Proof. If $K = R[1/u]$ it is clear that $K_i = R_i[1/u(i)]$. Supose $u(i)$ is not a unit in R_i for a countably infinite number of indices $n = 1, 2, \ldots$. Let v be any regular element of R such that $v(n) = u(n)^n$. Then $1/v \notin R[1/u]$.

The converse is straightforward.

In [5, p. 14] it is noted that a polynomial extension of a domain cannot be a G-domain. This can be extended with a different but very simple argument.

PROPOSITION 3.2. *For any ring, R, $R[X]$ is not a G-ring.*

Proof. Suppose f is a regular element of $R[X]$ lying in all regular prime ideals. Let $g = Xf + 1$. Since f is not nilpotent, g cannot be a unit. Enlarging Rg to a prime ideal P, we see that $g - xf = 1$ lies in P, a contradiction.

PROPOSITION 3.3. *Let R be a Noetherian ring. Then $R[[X]]$ is a G-ring if and only if R has Krull dimension zero.*

Proof. Suppose R has a chain of two distinct prime ideals $P \subset Q$. Let $P'=(P, X)$, $Q'=(Q, X)$ in $R[[X]]$. The regularity of P' implies Q' has rank greater than one.

Conversely, suppose R has Krull dimension zero. If P' is a maximal regular ideal in $R[[X]]$, then $P'=(P, X)$ for some maximal P in R. Krull's Principal Ideal Theorem now shows P' has rank one. Since R is semi-local, we are done.

We now briefly examine how the G-property transfers under completions. We freely utilize the elementary results in the theory of completions given in [6, pp. 53–58] and [8, pp. 251–261].

PROPOSITION 3.4. *Let J be the Jacobson radical of R. If R is a Noetherian G-ring then the J-adic completion of R is a G-ring.*

Proof. If R is semi-local with maximal ideals $P_1, \ldots, P_n$, then the J-adic completion R' is semi-local with maximal ideals $P'_1, \ldots, P_n$ where each $P'_i=P_i \cdot R'$. Furthermore, rank (P_i)=rank (P'_i). This and the fact that $Z(R')$ is closed in the J-adic topology imply the regular prime ideals of R and R' are in one-one correspondence.

We can provide an example of a non-G-ring whose completion is a G-ring. For let $J=(X)$ in $R[X]$. The J-adic completion of this ring is $R[[X]]$. Now utilize Propositions 3.2 and 3.3.

4. Hilbert G-rings.

DEFINITION 4.1. I is a G-ideal of R if T/I is a G-ring.

DEFINITION 4.2. I is a quasi-maximal ideal of R if R/I is a total quotient ring.

One characterization of a Hilbert ring is that every prime G-ideal is maximal [5, p. 18]. We made the definitions above to extend this slightly.

PROPOSITION 4.3. *Let R be Noetherian. Then R is a Hilbert ring if and only if every G-ideal of R is quasi-maximal.*

Proof. Since quasi-maximal prime ideals must be maximal, half of 4.3 is clear.

For the converse, suppose R is a Hilbert G-ring. If $R \neq K$ we can find a non-maximal prime ideal in $Z(R)$ contained in only finitely many primes of R. But this must be a G-ideal, a contradiction.

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