A NON SIMPLICITY CRITERION FOR FINITE GROUPS

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M. Suzuki [3] has proved the following theorem. Let G be a finite group which has an involution t such that $C = C_G(t) \cong SL(2, q)$ and q odd. Then G has an abelian odd order normal subgroup A such that $G = CA$ and $C \cap A = \langle 1 \rangle$.

We can prove the following similar result:

THEOREM. *Let G be a finite group of even order with the following property:*

(a) *G has an involution t such that the centralizer $C = C_G(t)$ is the central product of C_1 and C_2 where $C_i \cong SL(2, q_i)$ and q_i odd, $i = 1, 2$. Then G has an abelian odd order normal subgroup A such that $G = CA$ and $C \cap A = \langle 1 \rangle$.*

At first we give two definitions.

DEFINITION. A group G is called the central product of two subgroups H_1 and H_2 if $G = H_1H_2$ and $[H_1, H_2] = 1$, (i.e. H_1 and H_2 commute elementwise).

DEFINITION. Let α be an automorphism of a group G. Then α is called *fixed-point-free* if and only if α fixes only the unit element of G.

In the proof of the theorem we shall use the following result.

FRATTINI LEMMA [1]. *Let N be a normal subgroup of a finite group G, and let K be a Sylow p-subgroup of N for some prime p. Then $G = NN_G(K)$.*

THEOREM OF GLAUBERMAN [2]. *Let t be an involution contained in a Sylow 2-subgroup S of a finite group G. If t is not conjugate in G to any other involution $t' \neq t$ of S, then $t \in Z(G \bmod O_{2'}(G))$, where $O_{2'}(G)$ is the maximal normal odd order subgroup of G.*

THEOREM OF ZASSENHAUS [4]. *If a finite group G has a fixed-point-free automorphism of order 2, then G is an abelian group of odd order.*

We now prove some preliminary results. The first lemma is well known.

LEMMA 1. *An S_2-subgroup of $SL(2, q)$, q odd, is a generalized quaternion group.*

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Lemma 2. *Let Q_i be an S_2-subgroup of C_i, $i = 1, 2$. Then $Q_1 \cap Q_2 = \langle t \rangle$ and $Q = Q_1 Q_2$ is an S_2 subgroup of C and hence of G.*

Proof. $Q_i = \langle a_i, b_i | a_i^{2^{n_i-2}} = b_i^2 = t, b_i^{-1} a_i b_i = a_i^{-1} \rangle$ is an S_2-subgroup of C_i, $i = 1, 2$. Since $t \in Q_i$, $i = 1, 2$, then $\langle t \rangle \subseteqq Q_1 \cap Q_2$. On the other hand, $Q_1 \cap Q_2 \subseteqq C_1 \cap C_2 = \langle t \rangle$, so $\langle t \rangle = Q_1 \cap Q_2$. By consideration of the orders of C_i, $Q = Q_1 Q_2$ is an S_2-subgroup of C. Let T be an S_2-subgroup of G containing Q. Since Q is an S_2-subgroup of C then $T \cap C = Q$. Now $Z(T) \subseteqq C = C_G(t)$ and so $Z(T) \subseteqq Z(Q) = \langle t \rangle$. Thus $\langle t \rangle = Z(T)$ giving $T \subseteqq C$ and hence $T = Q$.

Lemma 3. *Every involution $\tilde{t} \neq t$ of C is of the form $\tilde{t} = x_1 x_2$ where $x_i \in C_i$, $i = 1, 2$, is an element of order 4.*

Proof. Since $C = C_1 C_2$, every non-central involution $\tilde{t}$ can be written as $t = x_1 x_2$ where $x_i \in C$, $i = 1, 2$. Because $\tilde{t}^2 = 1$, $\tilde{t}^2 = x_1^2 x_2^2 = 1$ and so $x_1^2 = (x_2^2)^{-1} \in C_1 \cap C_2 = t$. Thus either $x_i^2 = 1$ or $x_i^2 = t$. But $x_i^2 \neq 1$ since t is the only involution of C_i. Hence x_i is of order 4.

Lemma 4. *C has two conjugate classes of involutions with representatives t and*

$$t_1 = a_1^{2^{n_1-3}} a_2^{2^{n_2-3}}.$$

Proof. From the assumptions of our theorem $\langle t \rangle = Z(C)$ and so t forms a conjugate class of involutions of C. By Lemma 3, every non-central involution of C has the form $x_1 x_2$ where x_i is an element of order 4 in C_i, $i = 1, 2$. However, all elements of order 4 in C_i (fixed i) are conjugate in C_i since $C_i / \langle t \rangle \simeq PSL(2, q_i)$ which has only one class of involutions. Hence any non-central involution in C is conjugate to

$$t_1 = a_1^{2^{n_1-3}} a_2^{2^{n_2-3}}.$$

Lemma 5. *The whole group G has at most two conjugate classes of involutions.*

Proof. This follows from Lemma 2, Lemma 4 and the theorems of Sylow.

Lemma 6. *We have $Q = C_Q(t_1) = \langle a_1, a_2, t_2 \rangle$ and $(\tilde{Q})' = \langle a_1^2, a_2^2 \rangle$* where *$t_2 = b_1 b_2$. Also $\tilde{Q}$ is an S_2-subgroup of $C_C(t_1)$.*

Proof. By a straight forward computation $C_Q(t_1) = \langle a_1, a_2, t_2 \rangle = \tilde{Q}$, which is a non-abelian group of order

$$2^{n_1+n_2-2}.$$

We may write $\tilde{Q} = \langle t_2 \rangle \langle a_1, a_2 \rangle$ where $\langle a_1, a_2 \rangle = H$ is abelian. Since $[a_1, t_2] = a_1^{-2}$ and $[a_2, t_2] = a_2^{-2}$, we get $K = a_1^2, a_2^2 \subseteqq (\tilde{Q})'$. Now $\trianglelefteq H$, since

H is abelian, and K remains invariant under the action of the involution t_2. Thus $K \trianglelefteq \tilde{Q}$. Consider $\tilde{Q}/K$. Since t_2, a_1, a_2 all commute modulo K, then $K \supseteqq (\tilde{Q})'$. Hence $\langle a_1^2, a_2^2 \rangle = K = (\tilde{Q})'$. Let $\tilde{T}$ be an S_2-subgroup of $C_C(t_1)$. Suppose $\tilde{T} \supset \tilde{Q}$. Then $\tilde{T}$ is conjugate to Q and so $Z(\tilde{T})$ is conjugate to $Z(Q)$, that is, t_1 is conjugate to t, a contradiction. Hence $\tilde{T} = \tilde{Q}$.

Lemma 7. *The subgroup $\langle t \rangle$ is characteristic in $(\tilde{Q})'$ and hence $\langle t \rangle$ is characteristic in $C_Q(t_1)$.*

Proof. We note that K is abelian and consider the series

$$\Omega^1(K) \supseteqq \Omega^2(K) \supseteqq \ldots,$$

where $\Omega^i(K)$ is the subgroup of $K = (\tilde{Q})'$ generated by all the elements $x \in K$ such that $x = y^{2^i}$ for some $y \in K$. Clearly, $\Omega^i(K) \supseteqq \Omega^{i+1}(K)$, $i \geqq 1$. Let α be an automorphism of K and $x \in \Omega^i(K)$. Then

$$x^\alpha = (y^{2^i})^\alpha = (y^\alpha)^{2^i} \in \Omega^i(K)$$

for some $y \in K$ and so $\Omega^i(K)$ is characteristic in K, $i \geqq 1$.

Suppose $n_1 = n_2 = n$, then $|a_1| = |a_2| = 2^{n-1}$. We have $K = \langle a_1^2, a_2^2 \rangle$,

$$\Omega^1(K) = \langle a_1^{2^2}, a_2^{2^2} \rangle, \ldots, \Omega^{n-3}(K) = \langle a_1^{2^{n-2}}, a_2^{2^{n-2}} \rangle = \langle t \rangle.$$

So $\langle t \rangle$ is characteristic in $(\tilde{Q})'$ in this case.

Suppose $n_1 \neq n_2$. Without loss of generality we may take $n_1 > n_2$. Again

$$\Omega^1(K) = \langle a_1^{2^2}, a_2^{2^2} \rangle, \quad \Omega^2(K) = \langle a_1^{2^3}, a_2^{2^3} \rangle, \ldots,$$
$$\Omega^{n_2-2}(K) = \langle a_1^{2^{n_2-1}}, a_2^{2^{n_2-1}} \rangle = \langle a_1^{2^{n_2-1}} \rangle, \ldots.$$

Thus $\Omega^{n_2-2}(K)$ is cyclic, so $\langle t \rangle$ is characteristic in $\Omega^{n_2-2}(K)$ and hence $\langle t \rangle$ is characteristic in K.

Lemma 8. *The group G has precisely two conjugate classes of involutions with the representations t and t_1.*

Proof. Suppose t is conjugate in G to t_1. Then in particular,

$$C_G(t) = C \cong C_G(t_1).$$

We know that $C_Q(t_1) = \langle a_1, a_2, t_2 \rangle$ is an S_2-subgroup of $C_C(t_1)$, and that $\langle t \rangle$ is characteristic in $C_Q(t_1)$. Let $\tilde{T}$ be an S_2-subgroup of $C_G(t_1)$ which contains $C_Q(t_1) = \tilde{Q}$. Then $|T : C_Q(t_1)| = 2$ and so $C_Q(t_1) \trianglelefteq \tilde{T}$. Thus $\langle t \rangle \trianglelefteq \tilde{T}$, so $\tilde{T} \subseteqq C_G(t) = C$, a contradiction. Thus t is not conjugate in G to any involution $t_1 \neq t$ of Q.

Proof of the Theorem. We proceed by induction on the order of the group G. Denote by $O_{2'}(G)$ the maximal normal odd order subgroup of G.

Suppose $O_{2'}(G) \neq 1$. Put $\bar{G} = G/O_{2'}(G)$. Denote by $\bar{S}$ the image in $\bar{G}$ of any subset S of G, i.e.

$$\bar{S} = S0_{2'}(G)/0_{2'}(G).$$

Let $M = 0_{2'}(G)$. Clearly, $C_{\bar{G}}(t) = C^*/M$, for some subgroup C^* of G containing M and t. Write $\langle t\rangle M = N \subseteqq C^*$. Then $N \trianglelefteq C^*$ since

$$N/M = \langle \bar{t}\rangle \trianglelefteq C_{\bar{G}}(\bar{t}) = C^*/M.$$

Clearly, $\langle t\rangle$ is an S_2-subgroup of N. By the Frattini argument, $C^* = N_{C^*}(\langle t\rangle)N$. Since $\langle t\rangle$ is a group of order 2, $N_{C^*}(\langle t\rangle) = C_{C^*}(t)$. Thus

$$C^* = C_{C^*}(t)N = C_{C^*}(t)\langle t\rangle M = C_{C^*}(t)M.$$

Since $C = C_G(t) \subseteqq C^*$, we get $C^* = CM$. From the structure of C we know that $O_{2'}(C) = \langle 1\rangle$ so $C \cap M = \langle 1\rangle$. We conclude that

$$C_{\bar{G}}(\bar{t}) = C^*/M = CM/M \cong C.$$

Thus the group $\bar{G}$ satisfies the condition (a) of our Theorem, and $|\bar{G}| < |G|$, so by induction the theorem is true for $\bar{G}$. But $O_{2'}(\bar{G}) = \langle 1\rangle$ since $\bar{G} = G/O_{2'}(G)$, hence $\bar{G} = C_{\bar{G}}(\bar{t})$ and so $G = CO_{2'}(G)$ and $C \cap O_{2'}(G) = \langle 1\rangle$. Now the involution t acts fixed-point-free on $O_{2'}(G)$, so by the result of Zassenhaus [4], $O_{2'}(G)$ is abelian. Hence our theorem is true if $O_{2'}(G) \neq \langle 1\rangle$.

We may assume now that $O_{2'}(G) = \langle 1\rangle$. But then by the theorem of Glauberman [2], $t \in Z(G)$ and so $G = C_G(t) = C$. The theorem is proved.

Remark. It was kindly pointed out by the referee that this paper in fact proves the following slightly stronger result:

If $C/O_{2'}(C)$ is isomorphic to the central product of C_1 and C_2, then $G = CO_{2'}(G)$ and $C \cap O_{2'}(G) = O_{2'}(C)$.

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