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GENERALIZED FREDHOLM TRANSFORMATIONS

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Abstract

In an earlier paper we showed that the set $\psi_+(X, Y)$ of super Tauberian transformations between two Banach spaces X and Y forms an open subset of $\mathfrak{B}(X, Y)$ which is closed under perturbation by super weakly compact transformations. In this note we characterize a class dual to $\psi_+(X, Y)$ which we denote by $\psi_-(X, Y)$. We show that

$$T \in \psi_+(X, Y) \quad \text{if and only if} \quad T' \in \psi_-(Y', X')$$

and that

$$T' \in \psi_+(Y', X') \quad \text{if and only if} \quad T \in \psi_-(X, Y)$$

and provide standard and nonstandard characterizations of elements of $\psi_-(X, Y)$. These two classes thus play in some ways analogous roles to the sets of semi-Fredholm transforms $\phi_+(X, Y)$ and $\phi_-(X, Y)$.

Moreover $\psi(X, Y) = \psi_+(X, Y) \cap \psi_-(X, Y)$ then forms an open subset of $\mathfrak{B}(X, Y)$ closed under the taking of adjoints, under the taking of nonstandard hull extensions, and under perturbation by super weakly compact transformations.

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1. Preliminaries

This paper is a continuation of the investigation begun in an earlier paper [8]. We are concerned with transformations between (real infinite dimensional) Banach spaces and with their extensions on the nonstandard hulls of these spaces. Our notation is generally consistent with [8] except for a limited number of instances which we comment on explicitly. As before we are assuming that our objects of study are embedded in some set theoretical structure $\mathfrak{M}$ of which $^*\mathfrak{M}$ is an $\aleph_1$-saturated enlargement. For a Banach space X the *nonstandard hull* $\hat{X}$ (with respect to $^*\mathcal{M}$) is constructed by factoring the infinitesimal elements of $^*X$ from the finite elements of $^*X$. The original space X is embedded in $\hat{X}$ and $\hat{X}$ is a Banach space under the norm $\|\hat{p}\| =$ standard part $^*\|p\|$ where $\hat{p}$ denotes the equivalence class determined by the finite element $p \in {}^*X$. An element $S \in$ finite $^*\mathcal{B}(X, Y)$ defines an element $\hat{S} \in \mathcal{B}(\hat{X}, \hat{Y})$ by the equation $\hat{S}(\hat{p}) = (S(p))\hat{}$ where $p \in$ finite $^*X$.

We remind the reader that the class of *Tauberian transformations* $\mathcal{T}(X, X)$ consists of those transformations T between X and Y for which $T''x'' \in Y$ implies $x'' \in X$. The class of *super Tauberian transformations*, which we *now* denote by $\psi_+(X, Y)$, consists of those transformations which have Tauberian extensions between the nonstandard hulls, that is, $T \in \psi_+(X, Y)$ if $\hat{T} \in \mathcal{T}(\hat{X}, \hat{Y})$. Theorem 3 in [8] provides alternate characterizations of $\psi_+(X, Y)$ the simplest of which is the condition that $\ker \hat{T}$ is reflexive or superreflexive. It seems to be an open question whether or not $T \in \mathcal{T}(X, Y)$ implies $T'' \in \mathcal{T}(X'', Y'')$ (see Kalton and Wilansky [7] and [8, Proposition]). The principal difficulty in establishing a result like this arises since a Tauberian transformation T need not have closed range. Thus one cannot assume that the range of the adjoint is the set of $f \in X'$ for which $Tx = 0$ implies $f(x) = 0$, that is, we cannot assume $\mathcal{R}(T') = (\ker T)^\perp$ (see Dunford and Schwartz [2, page 487]).

Without the conclusion that $T'' \in \mathcal{T}(X'', Y'')$ it is impossible to define a class of transformations which is completely dual to $\mathcal{T}(X, Y)$. Yang [10] counters this problem by calling a transformation T *co-Tauberian* if T has closed range and reflexive cokernel. Then, for transformations with closed range, Tauberian and co-Tauberian transformations are completely dual. It is not true that a super Tauberian transformation T need have closed range (see Section 4) but nevertheless if T is super Tauberian then T'' is super Tauberian.

2. The existence of $\psi_-(X, Y)$

We show in this section that there exists a class of transformations satisfying the duality properties stated in the abstract.

LEMMA 1. *Let X and Y be Banach spaces and suppose T: $X \to Y$. Let $S = \{f_1, f_2, \ldots, f_n\}$ be a finite subset of X' and suppose $\phi \in X''$ is such that $\|T''\phi\| < \varepsilon$ where $\varepsilon > 0$. Then, given $\delta > 0$, there exists a point $x \in X$ such that*

(i) $\|x\| \leq 3\|\phi\| + \delta$;

(ii) $f_i(x) = \phi(f_i)$ *for* $i = 1, 2, \ldots, n$; *and*

(iii) $\|Tx\| < \varepsilon$.

PROOF. By Helly's theorem (Wilansky [9, page 103]) there exists a point $x_0 \in X$ with $\|x_0\| \leqslant \|\phi\| + \delta/2$ such that $(f_i)(x_0) = \phi(f_i)$ for $i = 1, 2, \ldots, n$. Let $S_\perp = \{x \in X\colon f_i(x) = 0$ for $i = 1, 2, \ldots, n\}$ and $S^\perp = \{x'' \in X''\colon x''(f_i) = 0$ for $i = 1, 2, \ldots, n\}$. Then $\phi \in x_0 + S^\perp$ and so we can write $\phi = x_0 + x''$ where $x'' \in S^\perp$ and $\|x''\| \leqslant 2\|\phi\| + \delta/2$. Suppose $A = \{x \in S_\perp : \|x\| \leqslant 2\|\phi\| + \delta/2\}$ and $B = T(x_0) + T(A)$. Then there exists a net of points $\{x_\alpha\} \subset A$ such that $x_\alpha \to x''$ in the weak* topology. Consequently $Tx_\alpha \to T''x''$ in the weak* topology or equivalently $Tx_0 + Tx_\alpha \to T''\phi$ in the weak* topology. But $\|T''\phi\| < \varepsilon$ so for all $g \in Y''$ with $\|g\| \leqslant 1$ there exists a point $b \in B$ such that $|g(b)| < \varepsilon$. Now let us suppose that $d(0, B) \geqslant \varepsilon$, so that $Y_\varepsilon \cap B = \varnothing$. Then B and Y_ε can be separated by a non-zero continuous linear functional (Dunford and Schwartz [2, page 417]). This means there exists an element $g \in Y'$ with $\|g\| = 1$ and a real constant d such that

$$g(B) \geqslant d \quad \text{and} \quad g(Y_\varepsilon) \leqslant d.$$

But $\sup g(Y_\varepsilon) = \varepsilon$ and so $d \geqslant \varepsilon$ forcing the inequality $g(B) \geqslant \varepsilon$. This is a contradiction and so $d(0, B) < \varepsilon$. Thus there is a point $x_1 \in A$ such that $\|T(x_0 + x_1)\| < \varepsilon$ and $x = x_0 + x_1$ then satisfies the three conditions of the lemma.

THEOREM 1. *Let X and Y be Banach spaces and suppose $T \in \psi_+(X, Y)$. Then $T'' \in \psi_+(X'', Y'')$.*

PROOF. Suppose $T'' \notin \psi_+(X'', Y'')$. Then, by [8, Theorem 3], there exists a real number r satisfying $0 < r < 1$ such that for all positive integers n there exist finite sequences of elements $\{\phi_1, \phi_2, \ldots, \phi_n\}$ in X'' and $\{F_1, F_2, \ldots, F_n\}$ in X'' such that $\|\phi_k\|, \|F_k\| < 1$ for $k = 1, 2, \ldots, n$ satisfying

$$F_j(\phi_i) > r \quad \text{for } 1 \leqslant j \leqslant i \leqslant n \quad \text{and}$$
$$F_j(x_i) = 0 \quad \text{for } 1 \leqslant i < j \leqslant n.$$

with $\|x_k\| < 3$ and $\|Tx_k\| < 1/k$ for $k = 1, 2, \ldots, n$. It follows by [8, Theorem 3] that $T \notin \psi_+(X, Y)$.

We now define $\psi_-(X, Y)$ to consist of those transformations $T \in \mathfrak{B}(X, Y)$ for which $T' \in \psi_+(Y', X')$. Since $\psi_+(Y', X')$ is open it follows that $\psi_-(X, Y)$ is an open subset of $\mathfrak{B}(X, Y)$. Further, since the converse of Theorem 1 is also true, we have

$$T \in \psi_+(X, Y) \quad \text{if and only if} \quad T' \in \psi_-(Y', X').$$

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3. Characterizations of the set $\psi_-(X, Y)$

If $T \in \mathfrak{B}(X, Y)$ we let $\overline{\mathfrak{R}}(T)$ denote the closure of the range of T, and we then call the quotient space $Y/\overline{\mathfrak{R}}(T)$ the cokernel of T. We shall show that $T \in \psi_-(X, Y)$ if and only if $\hat{T}$ has reflexive cokernel, or equivalently, a superreflexive cokernel. The proof of this result would be immediate except that, in general, $(\hat{T})' \neq (T')\hat{}$; recall $(\hat{X})' = (X')\hat{}$ if and only if X is superreflexive (see Henson and Moore [3, Theorem 8.5]).

Lemma 2. *Let X and Y be Banach spaces and suppose T: $X \to Y$. If* $\ker(T')\hat{}$ *is reflexive (respectively, superreflexive) then $\hat{T}$ has reflexive cokernel (respectively, superreflexive cokernel).*

Proof. Let W and Z denote $\ker(T')\hat{}$ and $\hat{Y}/\overline{\mathfrak{R}}(\hat{T})$ respectively. We can consider $(X')\hat{}$ to be embedded in $(\hat{X})'$ in which case $(T')\hat{}$ is the restriction of $(\hat{T})'$ to $(X')\hat{}$. Thus we can suppose $W \subset \ker(\hat{T})'$ and thus that W is a subspace of Z' (see, for example, Brown and Page [1, page 196]). Let π: $Z \to W'$ be the canonical map defined by $(\pi(z))w = w(z)$. If π is an isometric embedding then it follows that Z is reflexive (respectively, superreflexive) since it is then a closed subspace of the reflexive space (respectively, superreflexive space) W'. To establish that π is an isometry it suffices to show that if $\|z\| = 1$ then for each $\varepsilon > 0$ there exists an element $w \in W_1$ such that $|w(z)| > 1 - 2\varepsilon$. Suppose to the contrary that $z = \hat{q} + \overline{\mathfrak{R}}(\hat{T})$ is an element of Z for which $w(z) < 1 - 2\varepsilon$ for all $w \in W_1$ where $\varepsilon > 0$ is fixed. This implies $g(q) < 1 - 2\varepsilon$ for all norm 1 elements $g \in {}^*(Y')$ such that $g \simeq 0$ on $T(X_1)$. Now $d(\hat{q}, \overline{\mathfrak{R}}(\hat{T})) = 1$ and so $d(q, T(X_n)) > 1 - \varepsilon$ for $n = 1, 2, 3, \ldots$. Consequently there is an $\omega \in {}^*\mathbf{N}\backslash\mathbf{N}$ such that $d(q, T(X_\omega)) > 1 - \varepsilon$. We now argue in a similar way to the last part of the proof of Lemma 1. Specifically there exists a norm 1 functional $g \in Y'$ with the property that $g(T(X_\omega) - q) \geqslant 1 - \varepsilon$. If $g(T(X_1)) \simeq 0$ then $-g(q) \geqslant 1 - 2\varepsilon$ which contradicts the above assumption on q. If $g \not\simeq 0$ on $T(X_1)$ then $g(T(X_\omega))$ contains infinite values and $g(q)$ must take an infinite value which is impossible. Thus we can conclude that there is no point z with the stated property and it follows that π is an isometry.

Theorem 2. *Let X and Y be Banach spaces and suppose T: $X \to Y$. Then $T \in \psi_-(X, Y)$ if and only if $\hat{T}$ has reflexive cokernel (or, equivalently, superreflexive cokernel).*

Proof. Suppose $\hat{T}$ has reflexive cokernel. Then the conjugate space of $Y/\overline{\mathfrak{R}}(\hat{T})$ is reflexive, that is, $(\overline{\mathfrak{R}}(\hat{T}))^\perp = \ker \hat{T}$, is reflexive. Consequently $\ker(T')\hat{}$ is reflexive whence $T' \in \psi_+(Y', X')$ by the characterization of [8]. The converse implication now follows by this characterization and Lemma 2. The equivalent result in term of superreflexivity follows by the same argument.

We comment that if M is a closed subspace of Y then Y/M is reflexive if and only if $Y'' = Y + M^{\perp\perp}$. This fact shows the connection between what we are now doing and the class in [8] denoted by $\mathfrak{D}\mathfrak{T}(X, Y)$ (see [8, Proposition]).

THEOREM 3. *Let X and Y be Banach spaces and suppose T: $X \to Y$. Then $T \in \psi_-(X, Y)$ if and only if* $\ker(\hat{T})' = \ker(T')\hat{}$.

PROOF. We begin by supposing that $\ker(T')\hat{} \subset \ker(\hat{T})' = (\overline{\mathfrak{R}}(\hat{T}))^{\perp}$. Then there exists a nonzero $\phi \in ((\overline{\mathfrak{R}}(\hat{T}))^{\perp})' = (\hat{Y}/\overline{\mathfrak{R}}(\hat{T}))''$ which vanishes on $\ker(T')\hat{}$. By Theorem 2 $\hat{Y}/\overline{\mathfrak{R}}(\hat{T})$ is reflexive and thus we can suppose $\phi \in \hat{Y}/\overline{\mathfrak{R}}(\hat{T})$, say $\phi = \hat{q} + \overline{\mathfrak{R}}(\hat{T})$. We then have $\hat{g}(q + \overline{\mathbf{R}}(\hat{T})) = 0$ for all $\hat{g} \in \ker(T')\hat{}$. Consequently $g(q) \simeq 0$ whenever $g \simeq 0$ on $T(X_1)$. We then argue as in Lemma 2. Since ϕ is nontrivial $\hat{q} \notin \overline{\mathfrak{R}}(\hat{T})$ and thus there exists a (standard) positive real δ such that $d(q, T(X_n)) > \delta$ for $n = 1, 2, \ldots$. Thus there is an $\omega \in {}^*\mathbf{N}\backslash\mathbf{N}$ such that $d(q, T(X_\omega)) > \delta$ and in turn a norm 1 element $g \in {}^*Y'$ such that $g(T(X_\omega) - q) \geqslant \delta$. If $g \simeq 0$ on $T(X_1)$ then $g(q) < -\delta/2$ which contradicts our above assumption on q. On the other hand if $g \not\simeq 0$ on $T(X_1)$ then $\inf g(T(X_\omega))$ is an infinite negative nonstandard real. This then contradicts the inequality $g(q) \geqslant -\|q\|$. Consequently ϕ does not exist and we have the conclusion $\ker(T')\hat{} = \ker(\hat{T})'$.

The converse argument is essentially that used by Henson and Moore in [3, Theorem 8.5]. Suppose that $\ker(T')\hat{} = \ker(\hat{T})'$, and that $T \notin \psi_-(X, Y)$. Following the notation of Lemma 2 let W and Z denote $\ker(T')\hat{}$ and $\hat{Y}/\overline{\mathfrak{R}}(\hat{T})$ respectively so that $W = Z'$. Since Z is not reflexive by James' characterization of reflexivity, [6, Theorem 3], there exists a real number r satisfying $0 < r < 1$ such that there exist bounded sequences $\{q_n + \overline{\mathbf{R}}(\hat{T})\}$ and $\{\hat{g}_n\}$ in Z and W respectively such that $\hat{g}_i(\hat{q}_j + \overline{\mathfrak{R}}(\hat{T})) > r$ for $i \leqslant j$, and such that $\hat{g}_i(\hat{q}_j + \overline{\mathfrak{R}}(\hat{T})) = 0$ for $j < i$. Since ${}^*\mathfrak{M}$ is assumed to be $\aleph_1$-saturated we can suppose that the sequences $\{q_n$: $n \in \mathbf{N}\}$ and $\{g_n$: $n \in \mathbf{N}\}$ are restrictions of internal sequences $\{q_n$: $n \in {}^*\mathbf{N}\}$ and $\{g_n$: $n \in {}^*\mathbf{N}\}$ respectively. Thus we can assume there is an element $\omega \in {}^*\mathbf{N}\backslash\mathbf{N}$ such that $g_i(q_j) > r$ for $1 \leqslant i \leqslant j \leqslant \omega$, and such that $g_i(q_j) < r/2$ for $1 \leqslant j < i \leqslant \omega$. Now the sequence $\{\hat{g}_n$: $n \in \mathbf{N}\}$ has a $\sigma(W, Z)$-limit point $\hat{g} \in W$. Hence $g(q_j) \geqslant r$ for $j \in {}^*\mathbf{N}\backslash\mathbf{N}$ provided $j \leqslant \omega$, whilst $g(q_j) \leqslant r/2$ for $j \in \mathbf{N}$. This implies $\mathbf{N}$ is internal which is incorrect.

Before setting our final characterizations of $\psi(X, Y)$ we need to introduce two definitions. We say T has *property Q* if for all reals r satisfying $0 < r < 1$ there do not exist sequences of norm 1 elements $\{y_1, y_2, \ldots\}$ in Y and $\{g_1, g_2, \ldots\}$ in Y' such that

(i) $|g_k| < 1/k$ on $T(X_1)$ for all k;

(ii) $g_j(y_i) > r$ for $1 \leqslant i \leqslant j$, and $g_j(y_i) = 0$ for $1 \leqslant j < i$.

We say T has *property* $\hat{Q}$ if for all reals r satisfying $0 < r < 1$ there exists a positive integer n for which there do not exist finite sequences of norm 1 elements $\{y_1, y_2, \ldots, y_n\}$ in Y and $\{g_1, g_2, \ldots, g_n\}$ in Y' satisfying conditions (i) and (ii) above.

THEOREM 4. *Let X and Y be Banach spaces and suppose T: $X \to Y$. Then the following conditions are equivalent*:

(i) *T has property $\hat{Q}$*;

(ii) $T \in \psi_-(X, Y)$;

(iii) *$\hat{T}$ has property $\hat{Q}$*;

(iv) *$\hat{T}$ has property Q*.

PROOF. We show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(i) *implies* (ii). Suppose $T \notin \psi_-(X, Y)$ so that $T' \notin \psi_+(Y', X')$. Then, see [8], there exists a real number r satisfying $0 < r < 1$ such that for all positive integers n there exist finite sequences $\{g_1, \ldots, g_n\} \subset Y'$ and $\{\phi_1, \ldots, \phi_n\} \subset Y''$ such that $\|T'g_k\| < 1/k$ for $k = 1, 2, \ldots, n$; $\phi_i(g_j) > r$ for $1 \leqslant i \leqslant j \leqslant n$ and $\phi_i(g_j) = 0$ for $1 \leqslant j < i \leqslant n$. Then by Helly's theorem we can assume that $\phi_k \in Y$ for $k = 1, 2, \ldots, n$; and it follows that T doesn't possess property $\hat{Q}$.

(ii) *implies* (iii). Let $E = \hat{X}$, $F = \hat{Y}$, $S = \hat{T}$ and suppose these objects are embedded with X, Y, T *etc.* in a structure $\mathfrak{N}$ of which $^*\mathfrak{N}$ is an $\aleph_1$-saturated enlargement. If S doesn't possess property $\hat{Q}$ then for some (standard) r satisfying $0 < r < 1$ there exist, for $\omega \in {}^*\mathbf{N}\backslash\mathbf{N}$, finite sequences of norm one elements $\{q_1, q_2, \ldots, q_{2\omega}\}$ in $^*\mathbf{F}$ and $\{g_1, g_2, \ldots, g_\omega\}$ in F' for which $|g_k| < 1/k$ on $S(E_1)$ for $k = 1, 2, \ldots, 2\omega$; $g_j(q_i) > r$ for $1 \leqslant i \leqslant j \leqslant 2\omega$, and $g_j(q_i) = 0$ for $1 \leqslant j < i \leqslant 2\omega$. For $k = 1, 2, 3, \ldots$ let $\hat{p}_k = \hat{q}_{\omega+k}$ and $\hat{f}_k = \hat{g}_{\omega+k}$. These are elements in the hulls $\hat{F}$ and $(F')\hat{}$ constructed with respect to $^*\mathfrak{N}$. Then $\hat{S}'\hat{f}_k = 0$ for $k = 1, 2, 3, \ldots$; $\hat{f}_j(\hat{p}_i) > r$ for $i \leqslant j$ and $\hat{f}_j(\hat{p}_i) = 0$ for $j > i$. Consequently by the James' characterization of reflexivity $\ker \hat{S}'$ is not reflexive. C. Ward Henson has shown that a Banach space and its hull have isometric hulls when constructed from an $\aleph_1$-saturated enlargement which has the $\aleph_0$-isomorphism property (see [4, Propositions 1 and 2] and [5]). Moreover he has an "isometric nonstandard hulls" theorem for operators in which it is established that the isometries respect the induced action of T (private communication).

Consequently $\ker \hat{T}$ is not reflexive (when constructed with respect to such a $^*\mathfrak{N}$), and therefore $T \notin \psi_-(X, Y)$. Since (iii) trivially implies (iv) we are finished once we show (iv) implies (i).

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(iv) *implies* (i). Suppose T doesn't possess property $\hat{Q}$. Then for some (standard) real r satisfying $0 < r < 1$ and $\omega \in {}^*\mathbf{N}$ there exist finite sequences of norm 1 elements $\{q_1, q_2, \ldots, q_\omega\}$ in ${}^*Y$ and $\{g_1, g_2, \ldots, g_\omega\}$ in ${}^*Y'$ satisfying conditions (i) and (ii) above. But then the sequences $\{\hat{q}_k\colon k \in \mathbf{N}\}$ and $\{\hat{g}_k\colon k \in \mathbf{N}\}$ satisfy $|\hat{g}_k| < 1/k$ on $\hat{T}(\hat{X}_1)$ for all k, $\hat{g}_j(\hat{q}_i) > r$ for $i \leqslant j$, and $\hat{g}_j(\hat{q}_i) = 0$ for $j \leqslant i$. Thus $\hat{T}$ does not possess property Q.

One consequence of the above result is that $T \in \psi_-(X, Y)$ if and only if $\hat{T} \in \psi_-(\hat{X}, \hat{Y})$. Now let $\psi(X, Y) = \psi_+(X, Y) \cap \psi_-(X, Y)$, that is, $T \in \psi(X, Y)$ if and only if $\hat{T}$ has reflexive kernel and cokernel. It is a consequence of results proven here and in [8] that:

(i) $\psi(X, Y)$ is an open subset of $\mathfrak{B}(X, Y)$;

(ii) $T \in \psi(X, Y)$ if and only if $T' \in \psi(Y', X')$;

(iii) $T \in \psi(X, Y)$ if and only if $\hat{T} \in \psi(\hat{X}, \hat{Y})$; and

(iv) T is closed under perturbation by super weakly compact transformations.

4. Transformations with closed range

If T is a transformation with closed range the conditions for membership of $\psi_+(X, Y)$ or $\psi_-(X, Y)$ can be simplified.

LEMMA 3. *Let X and Y be Banach spaces and suppose $T\colon X \to Y$. Then the following properties are equivalent*:

(i) $\mathfrak{R}(T)$ *is closed*;

(ii) $(\mathfrak{R}(T))\hat{} = \mathfrak{R}(\hat{T})$;

(iii) $\mathfrak{R}(\hat{T})$ *is closed*.

PROOF. Suppose $\mathfrak{R}(T)$ is closed and let $(T(p))\hat{} \in (\mathfrak{R}(T))\hat{}$. By the open mapping theorem we can assume p is finite so that $(T(p))\hat{} = \hat{T}(\hat{p})$. This shows $(\mathfrak{R}(T))\hat{} = \mathfrak{R}(\hat{T})$. Since the hull of a normed space constructed with respect to an $\aleph_1$-saturated model is complete it follows that (ii) implies (iii). Finally suppose $\mathfrak{R}(\hat{T})$ is closed. Let $Z = \overline{\mathfrak{R}}(T)$ and suppose $y \in Z_1$. Then $\hat{y} \in \overline{\mathfrak{R}}(\hat{T}) = \mathfrak{R}(\hat{T})$ and so, by the open mapping theorem, there exists a positive constant r independent of y, such that $y = \hat{T}(\hat{p})$ for some point $p \in {}^*X_r$. By transfer it follows that $Z_1 \subseteq (T(X_r))^-$ whence $Z_1 \subseteq T(X_{2r})$ (see Brown and Page [1, Lemma 8.5.2]). This proves that (iii) implies (i).

We then have

THEOREM 5. *Let X and Y be Banach spaces and suppose* $T: X \to Y$ *has closed range. Then*

(i) $T \in \psi_+(X, Y)$ *if and only if* $\ker T$ *is superreflexive*;

(ii) $T \in \psi_-(X, Y)$ *if and only if* $Y/\mathfrak{R}(T)$ *is superreflexive.*

PROOF. We check (ii) first. We have $T \in \psi_-(X, Y)$ if and only if $\hat{Y}/\mathfrak{R}(\hat{T})$ is reflexive, or equivalently if and only if $\hat{Y}/(\mathfrak{R}(T))\hat{}$ is reflexive. But $\hat{Y}/(\mathfrak{R}(T))\hat{}$ is isomorphically isometric to $(Y/\mathfrak{R}(T))\hat{}$ which is reflexive if and only if $Y/\mathfrak{R}(T)$ is superreflexive.

Next $T \in \psi_+(X, Y)$ if and only if $T' \in \psi_-(Y', X')$, that is, if and only if $X'/\mathfrak{R}(T')$ is superreflexive since T' has closed range. But for transformations with closed range $\mathfrak{R}(T') = (\ker T)^\perp$ so that $X'/\mathfrak{R}(T')$ equals $X'/(\ker T)^\perp$ which is isometrically isomorphic to $(\ker T)'$. Thus $T \in \psi_+(X, Y)$ if and only if $(\ker T)'$ is superreflexive, or equivalently if and only if $\ker T$ is superreflexive.

We finish by remarking that elements of $\psi_+(X, Y)$ with closed range do not in general form an open subset in $\mathfrak{B}(X, Y)$. To see this let T be the zero operator on l^2, and let S be any operator on l^2 which doesn't have closed range. Then T has closed range and is a member of $\psi_+(X, Y)$ although $T = \lambda S = \lambda S$ does not have closed range for any value of the scalar λ.

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