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THE EXISTENCE OF LOCALLY FINE SIMPLICIAL SUBDIVISIONS

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Abstract

Given a simplex S and a positive function δ on S, we show that there is a *simplicial* subdivision of S such that the diameter of each subdividing simplex is smaller that δ evaluated at some of its vertices.

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1. Introduction

The purpose of this note is to prove a theorem which we believe is fundamental to the geometric theory of the conditionally convergent integral defined by Kurzweil (1957) and Henstock (1968).

The very beginning of a geometric integration theory is, of course, a workable higher dimensional integral. While the Henstock-Kurzweil definition extends trivially to intervals of any dimension, the integration over intervals is not adequate. It has been well established that the geometry of higher dimensional spaces is related to simplexes and their simplicial subdivisions, rather than to intervals and their usual subdivisions. The reason why intervals work in the real line is that one-dimensional intervals are simplexes, and their usual subdivisions are simplicial.

To define the Henstock-Kurzweil integral over simplexes, we must establish first the existence of simplicial partitions compatible with an arbitrary a priori given positive function. As we are concerned with simplicial partitions only, the standard argument of Henstock (1968, Theorem 1) does not work. Instead a completely different and surprisingly intricate technique has to be employed.

The geometric nature of our investigation brings in many notions which are common in algebraic or piecewise linear topology but usually alien in analysis. In order to keep the exposition reasonably self-contained, we shall include several definitions which would have been superfluous if this paper were aimed at the audience of topologists alone.

2. Preliminaries

By $\mathbf{R}$ and $\mathbf{R}_+$ we denote, respectively, the set of all real and all positive real numbers. Throughout, $m \geqslant 1$ will be a fixed integer, and $\mathbf{R}^m$ will denote the m-dimensional Euclidean space. For a point $x = (\xi_1, \ldots, \xi_m)$ in $\mathbf{R}^m$, we let

$$|x| = \left(\sum_{i=1}^{m} \xi_i^2 \right)^{1/2}.$$

If $A \subset \mathbf{R}^m$, then A^- and $d(A)$ denote the closure and the diameter of A, respectively. If $B \subset A \subset \mathbf{R}^m$, then

$$\operatorname{int}_A B = A - (A - B)^-$$

is the interior of B relative to A. Given a family $\mathcal{Q}$ of subsets of $\mathbf{R}^m$, we let

$$\operatorname{mesh} \mathcal{Q} = \sup\{d(A) \colon A \in \mathcal{Q}\}.$$

For $x_0, \ldots, x_k$ in $\mathbf{R}^m$, $-1 \leqslant k \leqslant m$, set

$$\langle x_0 \cdots x_k \rangle = \left\{ \sum_{i=0}^{k} c_i x_i \in \mathbf{R}^m \colon \sum_{i=0}^{k} c_i = 1, c_i \geqslant 0, i = 0, \ldots, k \right\}.$$

If the vectors $x_i - x_0$, $i = 1, \ldots, k$, are linearly independent, we call $\langle x_0 \cdots x_k \rangle$ a *k-simplex*. Each l-simplex $\langle x_{i_0} \cdots x_{i_l} \rangle$ where $0 \leqslant i_0 < \cdots < i_l \leqslant k$, $-1 \leqslant l \leqslant k$, is called an *$l$-face* of the k-simplex $\langle x_0, \ldots, x_k \rangle$. The *vertices* of a k-simplex are its 0-faces.

A *k-complex* is a finite family $\mathcal{Q} = \{A_1, \ldots, A_r\}$ of k-simplexes such that $A_i \cap A_j$ is a common face of A_i and A_j for each i, $j = 1, \ldots, r$. The *body* of $\mathcal{Q}$ is the set $|\mathcal{Q}| = \bigcup_{i=1}^{n} A_i$. The collection $\mathcal{Q}^l$, $-1 \leqslant l \leqslant k$, of all l-faces of $A_1, \ldots, A_r$ is an l-complex called the *l-skeleton* of $\mathcal{Q}$.

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A subdivision of a k-complex $\mathcal{A} = \{A_1,\ldots,A_r\}$ is a k-complex $\mathcal{B} = \{B_1,\ldots,B_s\}$ such that $|\mathcal{A}|=|\mathcal{B}|$ and each A_i is a union of some B_j's. A subdivision $\mathcal{B}$ of $\mathcal{A}$ is called *proper* if for each $E, F \in \mathcal{A}^l$, $0 \leqslant l \leqslant m$, and each $B \in \mathcal{B}$ we have

$$(B \cap E \neq \varnothing \text{ and } B \cap F \neq \varnothing) \Rightarrow B \cap E \cap F \neq \varnothing.$$

Since the barycentric subdivision—see Spanier (1966, Chapter 3, Section 3, p. 123)—is proper, we see that each subdivision of $\mathcal{A}$ can be subdivided further to a proper subdivision of $\mathcal{A}$.

If $A = \langle x_0,\ldots,x_k\rangle$ is a k-simplex and $x \in A$, denote by $\operatorname{st}(x, A)$ the family of all k-simplexes among the sets $\langle x_0 \cdots x_{i-1}xx_{i+1} \cdots x_k\rangle$, $i = 0,\ldots,k$. Clearly, $\operatorname{st}(x, A)$ is a subdivision of $\{A\}$ which is proper if and only if $k = 1$ and x is not a vertex of A.

Throughout, the words "simplex" and "complex" will be used to denote an m-simplex and an m-complex, respectively. A *figure* is a set $A \subset \mathbf{R}^m$ which is the body of a complex.

The set

$$\mathcal{A} = \{A_1,\ldots,A_r, x_1,\ldots,x_r\}$$

is called a *partition* if $\{A_1,\ldots,A_r\}$ is a complex, the simplexes $A_1,\ldots,A_r$ are distinct, and x_i is a vertex of A_i, $i = 1,\ldots,r$. When no confusion is possible we shall denote by $\mathcal{A}$ both the partition $\{A_1,\ldots,A_r, x_1,\ldots,x_r\}$ and the complex $\{A_1,\ldots,A_r\}$. If A ia a figure containing $|\mathcal{A}|$, we call $\mathcal{A}$ a *partition in* A. If $\delta\colon |\mathcal{A}| \to \mathbf{R}_+$, we say that $\mathcal{A}$ is δ*-fine* whenever $d(A_i) < \delta(x_i)$, $i = 1,\ldots,r$.

DEFINITION 2.1. Let $\mathcal{A}$ be a complex. A partition

$$\mathcal{B} = \{B_1,\ldots,B_r; x_1,\ldots,x_r\}$$

is called a *partition of* $\mathcal{A}$ if $\mathcal{B}$ subdivides $\mathcal{A}$ and

$$E \subset \operatorname{int}_{|\mathcal{A}|} \bigcup \{B_i\colon x_i \in E\}$$

for each $E \in \mathcal{A}^k$, $k = 0,\ldots,m$.

Clearly, if $\mathcal{B}$ is a partition of $\mathcal{A}$, then $\mathcal{B}$ is a proper subdivision of $\mathcal{A}$.

Let $\mathcal{A}$ be a complex, and let $\delta\colon |\mathcal{A}| \to \mathbf{R}_+$ be such that

$$\delta(x) \leqslant \min\{|x - y| : y \in |\mathcal{A}^0| - (x)\}$$

for each $x \in |\mathcal{A}^0|$, and

$$\delta(x) \leqslant \inf\{|x - y| : y \in |\mathcal{A}^{k-1}|\}$$

for each $x \in |\mathcal{A}^k| - |\mathcal{A}^{k-1}|$, $k = 1,\ldots,m$. Then it is easy to see that a δ-fine partition $\mathcal{B}$ in $|\mathcal{A}|$ is a partition of $\mathcal{A}$ if and only if $|\mathcal{B}|=|\mathcal{A}|$.

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3. Main result

We begin with some observations about a positive function on a compact set. Let $Q \subset \mathbf{R}^m$ be a compact set; $f\colon Q \to \mathbf{R}_+ \cup \{+\infty\}$, and let

$$f^{\frown}(x) = \lim_{\varepsilon \to 0+} \sup\{f(y)\colon y \in Q, |x-y| < \varepsilon\},$$
$$f_{\smile}(x) = \lim_{\varepsilon \to 0+} \inf\{f(y)\colon y \in Q, |x-y| < \varepsilon\}$$

for each $x \in Q$. Clearly $f_{\smile} \leqslant f^{\frown}$, and it is easy to see that $f_{\smile}$ and $f^{\frown}$ are, respectively, lower and upper semicontinuous.

LEMMA 3.1. *Let $g = f^{\frown}$, and let $A \subset \mathbf{R}^m$ be closed. If $g_{\smile}(x) > 0$ for each $x \in A \cap Q$, then there is an $\varepsilon > 0$ and a dense subset D of $\operatorname{int}_Q(A \cap Q)$ such that $f(x) > \varepsilon$ for each $x \in D$.*

PROOF. Since $A \cap Q$ is compact and $g_{\smile}$ is lower semi-continuous, there is an $\varepsilon > 0$ such that

$$f^{\frown}(x) = g(x) \geqslant g_{\smile}(x) > \varepsilon$$

for all $x \in A \cap Q$. Thus to each $x \in A \cap Q$ there is a sequence $\{x_n\}$ in Q with $x_n \to x$ and $f(x_n) > \varepsilon$, $n = 1, 2, \ldots$. As $\operatorname{int}_Q(A \cap Q)$ is open in Q is suffices to let

$$D = \{x_n\colon x \in Q, n = 1, 2, \ldots\} \cap \operatorname{int}_Q(A \cap Q).$$

LEMMA 3.2. *Let $g = f^{\frown}$ and $Q^* = \{x \in Q\colon g_{\smile}(x) = 0\}$. Then Q^* is compact and nowhere dense in Q.*

PROOF. Since Q is compact and $g_{\smile} \geqslant 0$ is lower semi-continuous, Q^* is compact. Suppose there is an $x \in \operatorname{int}_Q Q^*$. Then

$$U = \{y \in Q\colon |x-y| < \varepsilon\} \subset Q^*$$

for some $\varepsilon > 0$. Using the Baire category theorem in the locally compact space U —see Dugundji (1966, Chapter XI, Theorem 10.3, page 250)—we can find a $c > 0$ such that

$$V = \operatorname{int}_U\left[\{z \in U\colon f(z) \geqslant c\}^{-} \cap U\right] = \operatorname{int}_Q\left[\{z \in U\colon f(z) \geqslant c\}^{-} \cap U\right]$$

is nonempty. As $g(y) \geqslant c$ for each $y \in \{z \in U\colon f(z) \geqslant c\}^{-}$, we have $g(y) \geqslant c$ for each $y \in V$. This implies that $g_{\smile}(y) \geqslant c$ for each $y \in V$; for V is open in Q. Since $\varnothing \neq V \subset Q^*$, we have obtained a contradiction.

THEOREM 3.3. *Let $\mathscr{Q}$ be a complex, and let $\delta\colon |\mathscr{Q}| \to \mathbf{R}_+$. Then there is a δ-fine partition of $\mathscr{Q}$.*

PROOF. Let $0 \leqslant k \leqslant m$, $A = |\mathcal{A}|$, and let $\mathcal{B}$ be a complex with $|\mathcal{B}| \subset A$. Denote by $\Phi_k(\mathcal{B})$ the collection of all subdivisions $\{C_1, \ldots, C_s\}$ of $\mathcal{B}$ such that $\{C_1, \ldots, C_q; y_1, \ldots, y_q\}$, $1 \leqslant q \leqslant s$, is a δ-fine partition in $|\mathcal{B}|$ with $\{y_1, \ldots, y_q\} \subset |\mathcal{B}^k|$, and

$$E \subset \operatorname{int}_{|\mathcal{B}|} \bigcup \{C_j : y_j \in E\}$$

for each $E \in \mathcal{B}^l$, $l = 0, \ldots, k$. We shall prove by induction that $\Phi_k(\mathcal{B}) \neq \varnothing$. This is true if $k = 0$; for $\Phi_0(\mathcal{B})$ contains every proper subdivision $\mathcal{C}$ of $\mathcal{B}$ with

$$\operatorname{mesh} \mathcal{C} < \min\{\delta(x) : x \in \mathcal{B}^0\}.$$

Assume it is true for $k - 1$ with $1 \leqslant k \leqslant m$. Clearly, it suffices to show that $\Phi_k(\mathcal{A}) \neq \varnothing$.

CLAIM (i). *Let $\mathcal{B}$ be a complex with $|\mathcal{B}| \subset A$, and let $\mathcal{B}'$ be a subdivision of $\mathcal{B}$. Then $\Phi_{k-1}(\mathcal{B}') \subset \Phi_{k-1}(\mathcal{B})$. In particular, given $\varepsilon > 0$, $\Phi_{k-1}(\mathcal{B})$ contains a complex $\mathcal{C}$ with mesh $\mathcal{C} < \varepsilon$.*

PROOF. Choose a complex $\mathcal{C} = \{C_1, \ldots, C_s\}$ in $\Phi_{k-1}(\mathcal{B}')$, and let $\{C_1, \ldots, C_q; y_1, \ldots, y_q\}$, $1 \leqslant q \leqslant s$, be the appropriate δ-fine partition in $|\mathcal{B}'| = |\mathcal{B}|$. After a suitable reordering we may assume that

$$\{y_1, \ldots, y_p\} = \{y_1, \ldots, y_q\} \cap |\mathcal{B}^{k-1}|,$$

$1 \leqslant p \leqslant q$. Considering the partition $\{C_1, \ldots, C_p; y_1, \ldots, y_p\}$, it is easy to see that $\mathcal{C} \in \Phi_{k-1}(\mathcal{B})$; for $\mathcal{B}'^l \cap \mathcal{B}^l$ subdivides $\mathcal{B}^l$, $l = 0, \ldots, m$. Now the rest of the claim follows from Spanier (1966, Chapter 3, Section 3, Theorem 14, page 125).

CLAIM (ii). *Let $\mathcal{B}$ be a complex with $B = |\mathcal{B}| \subset A$, let $Q \subset |\mathcal{B}^k|$, and let D be a dense subset of $Q - |\mathcal{B}^{k-1}|$. Suppose that $\delta(x) > \varepsilon$ for some $\varepsilon > 0$ and each $x \in D$. Then there is a proper subdivision $\{C_1, \ldots, C_s\}$ of $\mathcal{B}$ such that $\{C_1, \ldots, C_q; y_1, \ldots, y_q\}$, $1 \leqslant q \leqslant s$, is a δ-fine partition in B with $\{y_1, \ldots, y_q\} \subset |\mathcal{B}^k|$,*

$$E \subset \operatorname{int}_B \bigcup \{C_j : y_j \in E\}$$

for each $E \in \mathcal{B}^l$, $l = 0, \ldots, k - 1$, and

$$\{y_1, \ldots, y_q\} \cup Q \subset \operatorname{int}_B \bigcup_{j=1}^{q} C_j.$$

PROOF. By Claim (i), there is a proper subdivision $\mathcal{C} = \{C_1, \ldots, C_s\}$ of $\mathcal{B}$ with $\operatorname{mesh} \mathcal{C} < \varepsilon$ and such that $\{C_1, \ldots, C_p; y_1, \ldots, y_p\}$, $1 \leqslant p \leqslant s$, is a δ-fine partition in B with $\{y_1,\ldots,y_p\} \subset |\mathfrak{B}^{k-1}|$, and

$$E \subset \operatorname{int}_B \bigcup \{C_j : y_j \in E\}$$

for each $E \in \mathfrak{B}^l$, $l = 0,\ldots,k-1$. In particular, $C_j \cap |\mathfrak{B}^{k-1}| = \varnothing$ for $j = p+1,\ldots,s$. After a suitable reordering we may assume that $C_j \cap Q \neq \varnothing$ for $j = p+1,\ldots,q$, and $C_j \cap Q = \varnothing$ for $j = q+1,\ldots,s$; $p \leqslant q \leqslant s$. Thus

$$Q \subset \operatorname{int}_B \bigcup_{j=1}^{q} C_j.$$

If C_j, $p < j \leqslant q$, has no vertex in Q, find an $x \in C_j \cap Q$ and replace each C_i, $p < i \leqslant q$, containing x by $\operatorname{st}(x, C_i)$. By repeating this process finitely many times, we obtain a subdivision of $\{C_{p+1},\ldots,C_q\}$ such that each simplex of this subdivision has a vertex in Q. Thus with no loss of generality we may assume that each C_j, $j = p+1,\ldots,q$, already has a vertex in Q. Let $\{z_1,\ldots,z_n\}$ be all vertices of $C_1,\ldots,C_q$. If $z_i \in Q$ and z_i is a vertex of C_j with $p < j \leqslant q$, choose a $z_i' \in D$; otherwise let $z_i' = z_i$; $i = 1,\ldots,n$. If $C_j = \langle z_{i_0} \cdots z_{i_m} \rangle$, set $C_j' = \{z_{i_0}' \cdots z_{i_m}'\}$, $j = 1,\ldots,q$. Since each z_i with $z_i' \neq z_i$, $1 \leqslant i \leqslant n$, lies in

$$\left(\operatorname{int}_B \bigcup_{j=1}^{q} C_j\right) \cap \left(Q - |\mathfrak{B}^{k-1}|\right),$$

and since D is dense in $Q - |\mathfrak{B}^{k-1}|$, we can choose z_i' in $|\mathfrak{B}^k| - |\mathfrak{B}^{k-1}|$ and so close to z_i that $\mathcal{C}' = \{C_1',\ldots,C_q', C_{q+1},\ldots,C_s\}$ is a proper subdivision of $\mathfrak{B}$ with mesh $\mathcal{C}' < \varepsilon$. Moreover, each C_j', $p < j \leqslant q$, has a vertex $y_j' \in D$. It follows that

$$\{C_1',\ldots,C_q';\, y_1,\ldots,y_p,\, y_{p+1}',\ldots,y_q'\}$$

is the desired δ-fine partition in B, and the claim is proved.

Let $Q_0 = |\mathfrak{A}^k|$, and suppose that a compact set $Q_\alpha \subset |\mathfrak{A}^k|$ has been defined for each ordinal $\alpha < \beta$. If β is a limit ordinal, set $Q_\beta = \bigcap_{\alpha<\beta} Q_\alpha$. If $\beta = \alpha + 1$, let δ_α be the restriction of δ to Q_α, $\Delta^\alpha = \delta_\alpha^\sim$, and set

$$Q_\beta = \{x \in Q_\alpha : \Delta_\sim^\alpha(x) = 0\}.$$

By Lemma 3.2, $Q_{\alpha+1} \subsetneq Q_\alpha$ whenever $Q_\alpha \neq \varnothing$. Thus there is the first ordinal κ with $Q_\kappa = \varnothing$. From our construction it is clear that $\kappa = \gamma + 1$. It follows from Kuratowski (1966, Section 24, II, Theorem 2) that κ is a countable ordinal but we shall not need this.

Let Γ be the set of all ordinals $\alpha \leqslant \gamma$ for which there is a proper subdivision $\mathfrak{B} = \{B_1,\ldots,B_r\}$ of $\mathfrak{A}$ such that $\{B_1,\ldots,B_p; x_1,\ldots,x_p\}$, $1 \leqslant p \leqslant r$, is a δ-fine partition in A with $\{x_1,\ldots,x_p\} \subset |\mathfrak{A}^k|$,

$$E \subset \operatorname{int}_A \bigcup \{B_i : x_i \in E\}$$

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for each $E \in \mathfrak{B}^l$, $l = 0,\ldots,k-1$, and

$$\{x_1,\ldots,x_p\} \cup Q_\alpha \subset \operatorname{int}_A \bigcup_{i=1}^{p} B_i.$$

As the subdivision $\mathfrak{B}$ is proper our induction will be completed by showing that $0 \in \Gamma$.

It follows immediately from Lemma 3.1 and Claim (ii) that $\gamma \in \Gamma$.

Claim (iii). *If $\beta \in \Gamma$ is a limit ordinal, then there is an $\alpha \in \Gamma$ with $\alpha < \beta$.*

Proof. If $\{B_1,\ldots,B_p; x_1,\ldots,x_p\}$ is a δ-fine partition in A associated with $\beta \in \Gamma$, then

$$Q_\beta \subset G = \operatorname{int}_A \bigcup_{i=1}^{p} B_i.$$

Since $\{Q_\alpha - G: \alpha < \beta\}$ is a chain of compact sets and

$$\bigcap_{\alpha<\beta} (Q_\alpha - G) = Q_\beta - G = \varnothing,$$

we have $Q_\alpha \subset G$ for some $\alpha < \beta$. It follows that $\alpha \in \Gamma$.

Claim (iv). *If $\alpha + 1 \in \Gamma$, then $\alpha \in \Gamma$.*

Proof. There is a proper subdivision $\mathfrak{B} = \{B_1,\ldots,B_r\}$ of $\mathfrak{A}$ such that $\{B_1,\ldots,B_p; x_1,\ldots,x_p\}$, $1 \leqslant p \leqslant r$, is a δ-fine partition in A with $\{x_1,\ldots,x_p\} \subset |\mathfrak{A}^k|$,

$$E \subset \operatorname{int}_A \bigcup \{B_i: x_i \in E\}$$

for each $E \in \mathfrak{B}^l$, $l = 0,\ldots,k-1$, and

$$\{x_1,\ldots,x_p\} \cup Q_{\alpha+1} \subset \operatorname{int}_A \bigcup_{i=1}^{p} B_i.$$

Let $\mathfrak{D} = \{B_{p+1},\ldots,B_r\}$. By Lemma 3.1 there is an $\varepsilon > 0$ and a dense subset D of $|\mathfrak{D}| \cap Q_\alpha - |\mathfrak{D}^{k-1}|$ such that $\delta(x) > \varepsilon$ for each $x \in D$; for $\Delta^\alpha_\sim(x) > 0$ for every $x \in |\mathfrak{D}| \cap Q_\alpha$, and

$$|\mathfrak{D}| \cap Q_\alpha - |\mathfrak{D}^{k-1}| \subset \operatorname{int}_{Q_\alpha}(|\mathfrak{D}| \cap Q_\alpha).$$

Using Claim (ii), we can find a proper subdivision $\{C_1,\ldots,C_s\}$ of $\mathfrak{D}$ such that $\{C_1,\ldots,C_q; y_1,\ldots,y_q\}$, $1 \leqslant q \leqslant s$, is a δ-fine partition in $|\mathfrak{D}|$ with $\{y_1,\ldots,y_q\} \subset |\mathfrak{D}^k|$,

$$E \subset \operatorname{int}_{|\mathfrak{D}|} \bigcup \{C_j: y_j \in E\}$$

for each $E \in \mathcal{D}^l$, $l = 0, \ldots, k-1$, and

$$\{y_1, \ldots, y_q\} \cup (|\mathcal{D}| \cap Q_\alpha) \subset \operatorname{int}_{|\mathcal{D}|} \bigcup_{j=1}^{q} C_j.$$

Since $\{x_1, \ldots, x_p\} \subset \operatorname{int}_A \bigcup_{i=1}^{p} B_i$, there is a subdivision $\{P_1, \ldots, P_n\}$ of $\{B_1, \ldots, B_p\}$ such that each P_j, $1 \leqslant j \leqslant n$, has a vertex z_j in $\{x_1, \ldots, x_p\}$ and $\{P_1, \ldots, P_n, C_1, \ldots, C_s\}$ is a subdivision of $\mathcal{B}$. After a suitable reordering we may assume that

$$\{y_1, \ldots, y_t\} = \{y_1, \ldots, y_q\} \cap |\mathcal{Q}^k|,$$

$1 \leqslant t \leqslant q$. Considering the partition $\{P_1, \ldots, P_n, C_1, \ldots, C_t;\ z_1, \ldots, z_n,\ y_1, \ldots, y_t\}$ it is easy to see that $\alpha \in \Gamma$; for $\mathcal{B}^l \cap \mathcal{Q}^l$ subdivides $\mathcal{Q}^l$, $l = 0, \ldots, m$.

As every decreasing sequence of ordinals is finite—see Dugundji (1966, Chapter II, Theorem 6.4 (5), page 43)— it follows from Claims (iii) and (iv) that $0 \in \Gamma$, and the theorem is proved.

The next proposition is the main motivation for Definition 2.1. It shows that a δ-fine partition of a subcomplex can be always extended to a δ-fine partition of the whole complex.

Proposition 3.4. *Let $\mathcal{Q}$ be a complex, $\delta: |\mathcal{Q}| \to \mathbf{R}_+$, and let $\mathcal{B}$ be a subcomplex of $\mathcal{Q}$. If $\mathcal{B}'$ is a δ-fine partition of $\mathcal{B}$, then there is a δ-fine partition $\mathcal{C}'$ of $\mathcal{C} = \mathcal{Q} - \mathcal{B}$ such that $\mathcal{B}' \cup \mathcal{C}'$ is a δ-fine partition of $\mathcal{Q}$.*

Proof. Without loss of generality, we may assume that $\mathcal{C}$ consists of a single m-simplex C. Let $\mathcal{B}' = \{B_1, \ldots, B_p;\ x_1, \ldots, x_p\}$, and for $k = 0, \ldots, m-1$, let $\mathcal{P}_k = \{C_1, \ldots, C_q;\ y_1, \ldots, y_q\}$ be a δ-fine partition in C with the following properties:

(i) $\mathcal{P}_k \cup \mathcal{B}'$ is a complex, and $\{y_1, \ldots, y_q\} \subset \{x_1, \ldots, x_p\}$;

(ii) $E \subset \operatorname{int}_C \bigcup \{C_j: y_j \in E\}$ for each $E \in \mathcal{C}^l \cap \mathcal{B}^l$, $l = 0, \ldots, k$;

(iii) $\mathcal{P}_k$ can be extended to a proper subdivision of $\mathcal{C}$;

(iv) the $(m-1)$-dimensional face E_j of C_j opposite to y_j, $1 \leqslant j \leqslant q$, is perpendicular to each $E \in \mathcal{C}^l$, $1 \leqslant l < m$, for which $E \cap E_j \neq \varnothing$.

It is easy to see that $\mathcal{P}_0$ exists. Assuming the existence of $\mathcal{P}_k$, $0 \leqslant k \leqslant m-2$, we shall prove that $\mathcal{P}_{k+1}$ exists.

Given $E \in \mathcal{C}^{k+1} \cap \mathcal{B}^{k+1}$, choose a $B \in \mathcal{B}$ with $E \subset B$. After a suitable reordering, we may assume that $B_1, \ldots, B_t$, $0 \leqslant t < p$, are all elements from $\mathcal{B}'$ for which the associated vertices $x_1, \ldots, x_t$ belong to $E - |\mathcal{B}^k|$. Using (i) and (iv), we can construct a δ-fine partition $\mathcal{P}_E = \{D_1, \ldots, D_n;\ z_1, \ldots, z_n\}$ in C such that

(a) $\mathcal{P}_E \cup \mathcal{P}_k \cup \mathcal{B}'$ is a complex, and $\{z_1, \ldots, z_n\} = \{x_1, \ldots, x_t\}$;

(b) $E \subset \text{int}_C[(\cup\{C_j \in \mathcal{P}_k; y_j \in E\}) \cup (\cup_{i=1}^n D_i)]$;

(c) $\mathcal{P}_E$ can be extended into a proper subdivision of $\mathcal{C}$;

(d) the $(m-1)$-dimensional face F_i of D_i opposite to z_i, $1 \leq i \leq n$, is perpendicular to each $F \in \mathcal{C}^l$, $1 \leq l < m$, for which $F \cap F_i \neq \varnothing$.

Roughly speaking, $\mathcal{P}_E$ is obtained by an appropriate squashing and refining of the partition $\{B_1,\ldots,B_t; x_1,\ldots,x_t\}$ flipped across E into C. Now it follows from (ii) that

$$\mathcal{P}_{k+1} = \mathcal{P}_k \cup \left[\cup\left\{\mathcal{P}_E: E \in \mathcal{C}^{k+1} \cap \mathcal{B}^{k+1}\right\}\right].$$

$\mathcal{P}_{m-1} = \{P_1,\ldots,P_r; u_1,\ldots,u_r\}$. By (iii), there is a complex $\mathcal{D}$ such that $\mathcal{D} \cup \mathcal{P}_{m-1}$ is a proper subdivision of $\mathcal{C}$. Applying Theorem 3.3, we can find a δ-fine partition $\{S_1,\ldots,S_s; v_1,\ldots,v_s\}$ of $\mathcal{D}$. Since

$$\{u_1,\ldots,u_r\} \subset \text{int}_C \bigcup_{i=1}^{r} P_r,$$

there is a subdivision $\{Q_1,\ldots,Q_l\}$ of $\mathcal{P}_{m-1}$ such that each Q_j, $1 \leq j \leq l$, has a vertex w_j in $\{u_1,\ldots,u_r\}$, and $\{Q_1,\ldots,Q_l, S_1,\ldots,S_s\}$ subdivides $\mathcal{C}$. The proof is completed by letting

$$\mathcal{C}' = \{Q_1,\ldots,Q_l, S_1,\ldots,S_s; w_1,\ldots,w_l, v_1,\ldots,v_s\}.$$

Let $\mathcal{Q}$ be a complex, and $\delta: |\mathcal{Q}| \to \mathbf{R}_+$. It is natural to investigate whether there is a δ-fine partition $\{A_1,\ldots,A_n; x_1,\ldots,x_n\}$ of $\mathcal{Q}$ whose simplexes satisfy some regularity condition—see Whitney (1957, Chapter IV, Section 4). In particular, one would like to know whether the solid angle of A_i at x_i, $i = 1,\ldots,n$, is bounded away from zero uniformly with respect to δ. The following example shows that this cannot be achieved.

EXAMPLE 3.5. Set $m = 2$, and let A be a 2-simplex in $\mathbf{R}^2$ such that $\{x \in R^2: |x| < 1\} \subset A$. Denote by 0 the origin in $\mathbf{R}^2$, and choose $0 < \varepsilon < \frac{1}{2}$. For $x \in X$, set

$$\delta_\varepsilon(x) = \begin{cases} \varepsilon|x| & \text{if } x \neq 0, \\ \varepsilon & \text{if } x = 0, \end{cases}$$

and let $\{A_1,\ldots,A_n; x_1,\ldots,x_n\}$ be a δ-fine partition of $\{A\}$. By the choice of δ_ε, $n \geq 2$ and after a suitable reordering, $A_1 = \langle 0xy\rangle$, $x_1 = 0$, and $A_2 = \langle xyz\rangle$ for some x, y, z in A. With no loss of generality, we may assume that $x_2 = x$ or $x_2 = z$. If $x_2 = z$, then $|x - y| < \varepsilon|z|$ and $|z - x| < \varepsilon|z|$. As $|z| \leq |x| + |z - x|$, we have $|z| < |x| + \varepsilon|z|$, and consequently

$$|x - y| < \varepsilon|z| < \frac{\varepsilon}{1-\varepsilon}|x| < 2\varepsilon|x|.$$

However, if $x_2 = x$, then again

$$|x - y| < \varepsilon|x| < 2\varepsilon|x|.$$

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From $|x| - |x-y| \leqslant |y| \leqslant |x| + |x-y|$ we obtain

$$1 - 2\varepsilon < 1 - \frac{|x-y|}{|x|} \leqslant \frac{|y|}{|x|} \leqslant 1 + \frac{|x-y|}{|x|} < 1 + 2\varepsilon.$$

If α is the angle of A_1 at 0, then

$$\cos\alpha = \frac{1}{2|x||y|}\left(|x|^2 + |y|^2 - |x-y|^2\right)$$

$$= \frac{1}{2}\left(\frac{|x|}{|y|} + \frac{|y|}{|x|} - \frac{|x-y|^2}{|x|^2}\cdot\frac{|x|}{|y|}\right) \geqslant 1 - 2\varepsilon.$$

It follows that $\alpha \to 0$ as $\varepsilon \to 0$.

4. An application

For $x \in \mathbf{R}^m$ and a sequence $\{B_n\}$ of simplexes, we write $B_n \to x$ whenever x is a common vertex of each B_n, $n = 1, 2, \ldots$, and $d(B_n) \to 0$. If A is a simplex, denote by σ_A the family of all simplexes contained in A. Finally, let λ be the m-dimensional Lebesgue measure in $\mathbf{R}^m$.

Let A be a simplex, $x \in A$, and let $\phi\colon \sigma_A \to \mathbf{R}$. The extended real number

$${}_*\phi(x) = \inf\left[\liminf \frac{\phi(B_n)}{\lambda(B_n)}\right]$$

where the infimum is taken over all sequences $\{B_n\}$ from σ_A with $B_n \to x$, is called the *lower derivate* of ϕ at x.

DEFINITION 4.1. Let A be a simplex. A function $\phi\colon \sigma_A \to \mathbf{R}$ is called *superadditive* if

$$\phi(A) \geqslant \Sigma\{\phi(B)\colon B \in \mathfrak{B}\}$$

for each complex $\mathfrak{B}$ with $|\mathfrak{B}| = A$.

We note that Definition 4.1 is substantially *weaker* than the usual definition of superadditivity.

THEOREM 4.2. *Let A be a simplex, and let $\phi\colon \sigma_A \to \mathbf{R}$ be a superadditive function. If ${}_*\phi(x) \geqslant 0$ for each $x \in A$, then $\phi(A) \geqslant 0$.*

PROOF. Choose an $\varepsilon > 0$, and let $\psi = \phi + \varepsilon\lambda$. Then $\psi\colon \sigma_A \to \mathbf{R}$ is superadditive, and

$${}_*\psi(x) \geqslant {}_*\phi(x) + \varepsilon > 0$$

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for each $x \in A$. Thus given $x \in A$, there is a $\delta(x) > 0$ such that $\psi(B) > 0$ for each $B \in \sigma_A$ with a vertex x and $d(B) < \delta(x)$. By Theorem 3.3, we can find a δ-fine partition $\{B_1, \ldots, B_p; x_1, \ldots, x_p\}$ of $\{A\}$. Since

$$\phi(A) + \varepsilon\lambda(A) = \psi(A) \geqslant \sum_{i=1}^{p} \psi(B_i) > 0,$$

the theorem follows from the arbitrariness of ε.

Theorem 4.2 is actually *equivalent* to the following weaker version of Theorem 3.3:

Given a simplex A and a $\delta: A \to \mathbf{R}_+$, there is a δ-fine partition $\mathcal{Q} = \{A_1, \ldots, A_n; x_1, \ldots, x_n\}$ in A with $|\mathcal{Q}| = A$.

To see this, suppose that there is a $\delta: A \to \mathbf{R}_+$ such that no such δ-fine partition $\mathcal{Q}$ in A exists. For $B \in \sigma_A$, let $\phi(B) = 0$ if B has a vertex x with $d(B) < \delta(x)$, and $\phi(B) = -1$ otherwise. Then $\phi: \sigma_A \to \mathbf{R}$ is superadditive, $\phi(A) = -1$, and ${}_*\phi(x) = 0$ for each $x \in A$.

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