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GEOMETRY OF G_2 ORBITS AND ISOPARAMETRIC HYPERSURFACES

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Abstract. We characterize the adjoint G_2 orbits in the Lie algebra $\mathfrak{g}$ of G_2 as fibered spaces over S^6 with fibers given by the complex Cartan hypersurfaces. This combines the isoparametric hypersurfaces of case $(g,m)=(6,2)$ with case $(3,2)$. The fibrations on two singular orbits turn out to be diffeomorphic to the twistor fibrations of S^6 and $G_2/SO(4)$. From the symplectic point of view, we show that there exists a 2-parameter family of Lagrangian submanifolds on every orbit.

§1. Introduction

The exceptional compact Lie group G_2 plays an important role in various fields of geometry. Here we are concerned with the adjoint orbits of G_2 in S^{13}, where G_2 acts on its Lie algebra $\mathfrak{g} \cong \mathbb{R}^{14}$ as an isometry with respect to the bi-invariant metric. They are the unique isoparametric hypersurfaces with six principal curvatures of multiplicity 2 (see [M4]). Those with multiplicity 1 are obtained by the inverse image of the real Cartan hypersurfaces $C^3_{\mathbb{R}}$ in S^4 under the Hopf fibration $\pi : S^7 \to S^4$ (see [M1]). The purpose of this paper is to characterize the multiplicity 2 case in conjunction with the complex Cartan hypersurfaces $C^6_{\mathbb{C}}$ in S^7 (the dimension of a hypersurface is always given in real). The difference is, however, that there is no fibration between S^{13} and S^7. On the other hand, since $\pi^{-1}(C^3_{\mathbb{R}}) \cong C^3_{\mathbb{R}} \times S^3$, by interchanging the fiber and the base manifold we succeed in obtaining the following theorems.

THEOREM 1.1. *Let M be a principal G_2 orbit in S^{13}, and let $M_\pm$ be the singular orbits. Then M is diffeomorphic to G_2/T^2, and $M_\pm$ are both diffeomorphic to $\mathbb{Q}^5 = G_2/U(2)$, the complex quadratic. Each orbit has a*

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Kähler structure with respect to the induced metric and, moreover, has a Kähler fibration:

(i) *$M \to S^6$ with the fiber $C^6_{\mathbb{C}} = SU(3)/T^2$, the complex Cartan hypersurface;*

(ii) *$M_+ \to S^6$ with the fiber $\mathbb{C}P^2 = SU(3)/S(U(2)\times U(1))$, a fibration that is diffeomorphic to the twistor fibration on S^6;*

(iii) *$M_- \to G_2/SO(4)$ with the fiber $\mathbb{C}P^1 = SU(2)/S(U(1)\times U(1))$, which fibration is diffeomorphic to the twistor fibration on the quaternionic Kähler manifold $G_2/SO(4)$.*

Hence, M_+ is not congruent to M_- in S^{13}, but the fibrations are converted from one to the other through the fibration on the principal orbits.

THEOREM 1.2. *Let M and $M_\pm$ be as in Theorem 1.1. Then at each point of M, there exists a 2-parameter family of Lagrangian submanifolds transferred from an $SO(4)$ suborbit $N^6 \cong C^3_{\mathbb{R}} \times S^3$, which collapses into $N^5_\pm \cong \mathbb{R}P^2 \times S^3$ on $M_\pm$. These are minimal Lagrangian submanifolds of $M_\pm$ and of M_0, where the latter is the minimal principal orbit.*

Theorem 1.1 is not a formal factorization of a homogeneous space but has a significant application, say, a reduction of analysis on M to that on the factored spaces (see [MO]).

Isoparametric hypersurfaces in a real space form $\overline{M}$ are hypersurfaces with constant principal curvatures. They consist of a 1-parameter family of parallel hypersurfaces which sweeps out $\overline{M}$ with focal submanifold(s) at the end. There are rich examples in $\overline{M} = S^n$, where the number of principal curvatures g takes values in $\{1,2,3,5,6\}$ (see [Mü]). Typical examples are given by homogeneous hypersurfaces which have been classified as the linear isotropy orbits of rank 2 symmetric spaces (see [HL]). Other than hyperspheres ($g=1$) and the Clifford hypersurfaces ($g=2$), those with $g=3$ were found by Cartan and called the *Cartan hypersurfaces* $C_{\mathbb{F}}$ (see [C]). They are tubes over the standard embedding of $\mathbb{F}P^2$ in S^{3d+1}, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and $\mathbb{H}$ are Cayley numbers and $d = 1,2,4,8$, respectively. The case $g=4$ is exceptional, as there *exist* infinitely many nonhomogeneous isoparametric hypersurfaces (see [OT], [FKM]) where the classification problem (see [Y]) still remains open (see [CCJ], [I]).

When $g=6$, the multiplicity of each principal curvature coincides, which takes values $m = 1,2$ (see [A]). For $m=1$, the hypersurfaces are homogeneous and given by the isotropy orbits of $G_2/SO(4)$ (see [DN], [M2]). Homogeneous hypersurfaces M^{12} with $(g,m) = (6,2)$ are unique; that is, the G_2 orbits (see [HL]). Dorfmeister and Neher [DN] conjectured that the isoparametric hypersurfaces with $(g,m)=(6,2)$ are homogeneous (see [M4] for the affirmative answer).

The paper is organized as follows. In Section 2, we review some basic facts of isoparametric hypersurfaces, and in Section 3, we compute basic data of G_2 orbits in terms of the root and root vectors. Finally, we prove our theorem in a refined way in Section 4.

§2. Preliminaries

We refer readers to [Th] for a survey of isoparametric hypersurfaces. Here we review fundamental facts and the notation of [M1] and [M3]. Let M be an isoparametric hypersurface in the unit sphere S^{n+1}. Let ξ be a unit normal vector field. We denote the Riemannian connection on S^{n+1} by $\tilde{\nabla}$, and we denote the induced connection on M by ∇. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the principal curvatures of M, and let $D_\lambda(p)$ be the curvature distribution of $\lambda \in \{\lambda_1, \ldots, \lambda_n\}$ with multiplicity m_λ. Then D_λ is completely integrable, and a leaf L_λ is an m_λ-dimensional sphere of S^{n+1}. Choose a local orthonormal frame $e_1, \ldots, e_n$ consisting of unit principal vectors corresponding to $\lambda_1, \ldots, \lambda_n$. We express

$$\tilde{\nabla}_{e_\alpha} e_\beta = \Lambda^\sigma_{\alpha\beta} e_\sigma + \lambda_\alpha \delta_{\alpha\beta}\xi, \quad \Lambda^\gamma_{\alpha\beta} = -\Lambda^\beta_{\alpha\gamma} \tag{1}$$

where $1 \leq \alpha, \beta, \sigma \leq n$, using the Einstein convention. From the equation of Codazzi, we obtain for distinct $\lambda_\alpha, \lambda_\beta, \lambda_\gamma$,

$$\Lambda^\gamma_{\alpha\beta}(\lambda_\beta - \lambda_\gamma) = \Lambda^\beta_{\gamma\alpha}(\lambda_\alpha - \lambda_\beta) = \Lambda^\alpha_{\beta\gamma}(\lambda_\gamma - \lambda_\alpha). \tag{2}$$

Because λ_α is constant on M, we can see that

$$\Lambda^\gamma_{aa} = 0 = \Lambda^\gamma_{ab}, \quad \text{if } \lambda_a = \lambda_b \neq \lambda_\gamma \text{ and } a \neq b. \tag{3}$$

Now, consider the case $(g,m)=(6,2)$. As is well known, we can express

$$\lambda_i = \cot\Big(\theta_1 + \frac{(i-1)\pi}{6}\Big), \quad 0 < \theta_1 < \frac{\pi}{6}, 1 \leq i \leq 6. \tag{4}$$

Note that if we choose $\theta_1 = \pi/12 = -\theta_6$, we have a minimal case with

$$\lambda_1 = -\lambda_6 = 2+\sqrt{3}, \qquad \lambda_2 = -\lambda_5 = 1, \qquad \lambda_3 = -\lambda_4 = 2-\sqrt{3}. \tag{5}$$

Denote $D_i = D_{\lambda_i}$. We take a local frame field $e_1, e_{\bar{1}}, \ldots, e_6, e_{\bar{6}}$, where $e_i, e_{\bar{i}}$ is an orthonormal frame of D_i. For convenience, we put $\lambda_{\bar{i}} = \lambda_i$, and $\underline{i}$ always stands for i or $\bar{i}$. Each leaf $L_i = L_i(p)$ of D_i is a 2-sphere, and M has a structure of an iterated S^2 bundle over S^2. For $a = 6$ or 1, define the focal map $f_a\colon M \to S^{13}$ by

$$f_a(p) = \cos\theta_a p + \sin\theta_a \xi_p,$$

which makes $L_a(p)$ collapse into a point $\bar{p} = f_a(p)$. Then we have

$$df_a(e_j) = \sin\theta_a(\lambda_a - \lambda_j)e_j \qquad \text{and} \qquad df_a(e_{\bar{j}}) = \sin\theta_a(\lambda_a - \lambda_j)e_{\bar{j}}, \tag{6}$$

where the right-hand sides are considered as vectors in $T_{\bar{p}}S^{13}$ by a parallel translation of S^{13}. In the following, we always use such identification. The rank of f_a is constant, and we obtain the focal submanifold M_a of M:

$$M_a = \{\cos\theta_a p + \sin\theta_a \xi_p \mid p \in M\}.$$

We denote $M_+ = M_6$ and $M_- = M_1$. It follows that $T_{\bar{p}}M_a = \bigoplus_{j\neq a} D_j(q)$ from (6) for any $q \in {f_a}^{-1}(\bar{p})$. An orthonormal basis of the normal space of M_a at $\bar{p}$ is given by

$$\eta_q = -\sin\theta_a q + \cos\theta_a \xi_q, \quad \zeta_q = e_a(q) \text{ and } \bar{\zeta}_q = e_{\bar{a}}(q),$$

for any $q \in L_a(p) = f_a^{-1}(\bar{p})$. By a standard argument, we obtain the following (see [M2], [M4]).

Lemma 2.1. *When we identify $T_{\bar{p}}M_a$ with $\bigoplus_{j=1}^{5} D_{a+j}(p)$, where the indices are modulo 6, the shape operators B_{η_p}, B_{ζ_p}, and $B_{\bar{\zeta}_p}$ at $\bar{p}$ with respect to the basis $e_{a+1}, e_{\overline{a+1}}, \dots, e_{a+5}, e_{\overline{a+5}}$ at p are given, respectively, by the symmetric matrices*

$$B_{\eta_p} = \begin{pmatrix} \sqrt{3}I & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}}I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{3}}I & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3}I \end{pmatrix},$$

$$B_{\zeta_p} = \begin{pmatrix} 0 & B_{a+1\,a+2} & B_{a+1\,a+3} & B_{a+1\,a+4} & B_{a+1\,a+5} \\ B_{a+2\,a+1} & 0 & B_{a+2\,a+3} & B_{a+2\,a+4} & B_{a+2\,a+5} \\ B_{a+3\,a+1} & B_{a+3\,a+2} & 0 & B_{a+3\,a+4} & B_{a+3\,a+5} \\ B_{a+4\,a+1} & B_{a+4\,a+2} & B_{a+4\,a+3} & 0 & B_{a+4\,a+5} \\ B_{a+5\,a+1} & B_{a+5\,a+2} & B_{a+5\,a+3} & B_{a+5\,a+4} & 0 \end{pmatrix},$$

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$$B_{\bar{\zeta}_p} = \begin{pmatrix} 0 & \bar{B}_{a+1\,a+2} & \bar{B}_{a+1\,a+3} & \bar{B}_{a+1\,a+4} & \bar{B}_{a+1\,a+5} \\ \bar{B}_{a+2\,a+1} & 0 & \bar{B}_{a+2\,a+3} & \bar{B}_{a+2\,a+4} & \bar{B}_{a+2\,a+5} \\ \bar{B}_{a+3\,a+1} & \bar{B}_{a+3\,a+2} & 0 & \bar{B}_{a+3\,a+4} & \bar{B}_{a+3\,a+5} \\ \bar{B}_{a+4\,a+1} & \bar{B}_{a+4\,a+2} & \bar{B}_{a+4\,a+3} & 0 & \bar{B}_{a+4\,a+5} \\ \bar{B}_{a+5\,a+1} & \bar{B}_{a+5\,a+2} & \bar{B}_{a+5\,a+3} & \bar{B}_{a+5\,a+4} & 0 \end{pmatrix},$$

where I (resp., zero) is the 2×2 unit (resp., zero) matrix and

$$\text{(7)} \qquad \begin{aligned} B_{jk} &= \frac{1}{\sin\theta_a(\lambda_j - \lambda_a)} \begin{pmatrix} \Lambda^k_{ja} & \Lambda^{\bar{k}}_{ja} \\ \Lambda^k_{\bar{j}a} & \Lambda^{\bar{k}}_{\bar{j}a} \end{pmatrix} = {}^t B_{kj}, \\ \bar{B}_{jk} &= \frac{1}{\sin\theta_a(\lambda_j - \lambda_a)} \begin{pmatrix} \Lambda^k_{j\bar{a}} & \Lambda^{\bar{k}}_{j\bar{a}} \\ \Lambda^k_{\bar{j}\bar{a}} & \Lambda^{\bar{k}}_{\bar{j}\bar{a}} \end{pmatrix} = {}^t \bar{B}_{kj}. \end{aligned}$$

In particular, we have $B_{\eta_p}(e_j) = \mu_j e_j$, where

$$\text{(8)} \qquad \mu_j = \frac{1 + \lambda_j \lambda_a}{\lambda_a - \lambda_j} \in \left\{ \pm\sqrt{3}, \pm\frac{1}{\sqrt{3}}, 0 \right\}.$$

Since any unit normal can be expressed as η_q for some $q \in L_6(p)$, *all the shape operators have the same eigenvalues with multiplicity 2.*

§3. Geometric data of G_2 orbits

In this section, we investigate an adjoint G_2 orbit M in S^{13}, which is the same as an isotropy orbit of the symmetric space $G_2 \times G_2/G_2$. Here, G_2 is the automorphism group of the Cayley numbers $\mathcal{C}$. Let $\mathcal{C}$ be generated by $\{e_0, e_1, \ldots, e_7\}$ satisfying

$$\begin{cases} e_0 = 1, \\ e_i^2 = -1, \qquad 1 \le i \le 7, \\ e_i e_j = -e_j e_i = e_k, \end{cases}$$

where (i, j, k) is a triple on some segment or a circle of Figure 1 put in the order shown by its arrows. The automorphism group G_2 of $\mathcal{C}$ is a subgroup of $SO(7)$, where the metric on $\mathcal{C}$ is given by

$$(x, y) = \Re(x\bar{y}) = \sum_{i=0}^{7} x^i y^i, \quad \text{for } x = \sum_{i=0}^{7} x^i e_i \text{ and } y = \sum_{i=0}^{7} y^i e_i.$$

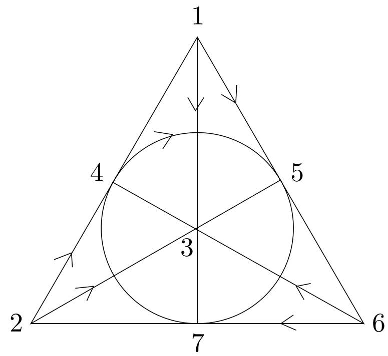

Figure 1

The Lie algebra $\mathfrak{g}$ of G_2 is given as follows (see [OT]). Let E_{ij} be the standard basis of 7×7 matrices with $\mathbb{R}$-coefficients. Put $G_{ij} = E_{ij} - E_{ji}, i, j = 1, \ldots, 7$, and put

$$\mathfrak{g}_i = \left\{ \eta_1 G_{i+1 i+3} + \eta_2 G_{i+2 i+6} + \eta_3 G_{i+4 i+5} \,\middle|\, \eta_j \in \mathbb{R}, \sum_{j=1}^{3} \eta_j = 0 \right\},$$

for $1 \leq i \leq 7$. Then $\mathfrak{g}$ is given by

$$\mathfrak{g} = \sum_{i=1}^{7} \mathfrak{g}_i, \tag{9}$$

which satisfies $[\mathfrak{g}_i, \mathfrak{g}_i] = 0$ and $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_k$, where (i, j, k) is as before. Note that $[G_{ij}, G_{jk}] = G_{ik}$ for any $1 \leq i, j, k \leq 7$. Note also that (9) is an orthogonal decomposition with respect to the metric $(\ ,\ )$ on $\mathfrak{g}$ given by

$$(X, Y) = -\frac{1}{2} \operatorname{Tr} XY.$$

For later use, we decompose $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where

$$\mathfrak{k} = \mathfrak{g}_3 + \mathfrak{g}_4 + \mathfrak{g}_6,$$
$$\mathfrak{p} = \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_5 + \mathfrak{g}_7.$$

Let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$, and let τ be the involutive automorphism of $\mathfrak{g}^{\mathbb{C}}$ given by $\tau(X) = \bar{X}$. Then $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$ is the Cartan decomposition. We identify $\sqrt{-1}\mathfrak{g}$ with $\mathfrak{g}$ by $\sqrt{-1}X \mapsto X$. Take a maximal abelian

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subspace $\mathfrak{a} = \mathfrak{g}_1 = \{\xi_1 G_{24} + \xi_2 G_{37} + \xi_3 G_{56} \mid \xi \in \mathbb{R}, \sum_{i=1}^{3} \xi_i = 0\}$ of $\mathfrak{g}$, whose dimension, called the *rank of* $(\mathfrak{g}^{\mathbb{C}}, \tau)$, is 2. Let α be a linear form on $\mathfrak{a}$, and put

$$\mathfrak{k}_\alpha = \left\{X \in \mathfrak{k} \,\middle|\, (\mathrm{ad}H)^2(X) = -\alpha(H)^2 X \quad \text{for all } H \in \mathfrak{a}\right\},$$
$$\mathfrak{p}_\alpha = \left\{X \in \mathfrak{p} \,\middle|\, (\mathrm{ad}H)^2(X) = -\alpha(H)^2 X \quad \text{for all } H \in \mathfrak{a}\right\}.$$

Note that for $H \in \mathfrak{a}$ and a linear form α on $\mathfrak{a}$ such that $\alpha(H) \neq 0$, $\mathrm{ad}H$ maps $\mathfrak{k}_\alpha$ (resp., $\mathfrak{p}_\alpha$) isomorphically onto $\mathfrak{p}_\alpha$ (resp., $\mathfrak{k}_\alpha$) (see (12) below). Selecting a suitable ordering in the dual space of $\mathfrak{a}$, let Σ_+ be the set of positive roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$, and let $\Sigma_+^* = \left\{\alpha \in \Sigma_+, \frac{\alpha}{2} \notin \Sigma_+\right\}$. We have

$$\text{(10)} \qquad \begin{aligned} \Sigma_+^* = \{&\alpha_1 = -\xi_2, \alpha_2 = \xi_1 - \xi_2, \alpha_3 = \xi_1, \\ &\alpha_4 = \xi_1 - \xi_3, \alpha_5 = -\xi_3, \alpha_6 = \xi_2 - \xi_3\}, \end{aligned}$$

and the root vectors $X_i \in \mathfrak{k}_{\alpha_i}$ and $T_i \in \mathfrak{p}_{\alpha_i}$ are given by

$$\text{(11)} \qquad \begin{aligned} X_1 &= G_{46} + G_{52} - 2G_{71}, & X_4 &= G_{46} - G_{52} && \in \mathfrak{g}_3 \\ X_2 &= G_{72} - G_{34}, & X_5 &= G_{72} + G_{34} - 2G_{15} && \in \mathfrak{g}_6 \\ X_3 &= G_{57} + G_{63} - 2G_{12}, & X_6 &= G_{57} - G_{63} && \in \mathfrak{g}_4 \\ T_1 &= G_{26} + G_{45} - 2G_{13}, & T_4 &= G_{26} - G_{45} && \in \mathfrak{g}_7 \\ T_2 &= G_{23} + G_{47}, & T_5 &= G_{47} - G_{23} - 2G_{16} && \in \mathfrak{g}_5 \\ T_3 &= G_{35} + G_{67} + 2G_{14}, & T_6 &= -G_{35} + G_{67} && \in \mathfrak{g}_2. \end{aligned}$$

We have immediately

$$\text{(12)} \qquad \mathrm{ad}H(X_i) = \alpha_i(H) T_i, \qquad \mathrm{ad}H(T_i) = -\alpha_i(H) X_i.$$

Note that any two of the above vectors are mutually orthogonal.

Now, let $H = \xi_1 G_{24} + \xi_2 G_{37} + \xi_3 G_{56}$ be a regular element of $\mathfrak{a}$, and let $H^\perp = (\xi_3 - \xi_2) G_{24} + (\xi_1 - \xi_3) G_{37} + (-\xi_1 + \xi_2) G_{56}$ be an element of $\mathfrak{a}$ orthogonal to H. For a hypersurface $M = \mathrm{Ad}\, G_2(\tilde{H})$, where $\tilde{H} = H/\|H\|$, by using (12) and $\|H^\perp\| = \sqrt{3}\|H\|$ we can express the second fundamental tensor $A_{\tilde{H}^\perp}$ of M with respect to the unit normal vector $\tilde{H}^\perp = H^\perp/\|H^\perp\|$ at $\tilde{H}$ by (see [TT])

$$A_{\tilde{H}^\perp} X_i = -\tilde{\nabla}_{X_i} \tilde{H}^\perp = -\frac{1}{\alpha_i(\tilde{H})} \frac{d}{dt}\Big|_{t=0} (\mathrm{Ad} \exp t T_i) \tilde{H}^\perp$$

$$= -\frac{1}{\alpha_i(\tilde{H})}[T_i, \tilde{H}^\perp] = -\frac{\alpha_i(\tilde{H}^\perp)}{\alpha_i(\tilde{H})}X_i = -\frac{\alpha_i(H^\perp)}{\sqrt{3}\alpha_i(H)}X_i,$$

$$A_{\tilde{H}^\perp}T_i = -\tilde{\nabla}_{T_i}\tilde{H}^\perp = \frac{1}{\alpha_i(\tilde{H})}\frac{d}{dt}\Big|_{t=0}(\mathrm{Ad}\exp tX_i)\tilde{H}^\perp$$

$$= \frac{1}{\alpha_i(\tilde{H})}[X_i, \tilde{H}^\perp] = -\frac{\alpha_i(\tilde{H}^\perp)}{\alpha_i(\tilde{H})}T_i = -\frac{\alpha_i(H^\perp)}{\sqrt{3}\alpha_i(H)}T_i.$$

Thus, the principal curvatures of M are given by

$$\begin{aligned} \lambda_1 &= -\frac{\xi_1 - \xi_3}{\sqrt{3}\xi_2} = -\frac{1}{\lambda_4}, \\ \lambda_2 &= -\frac{\sqrt{3}\xi_3}{\xi_1 - \xi_2} = -\frac{1}{\lambda_5}, \\ \lambda_3 &= \frac{\xi_2 - \xi_3}{\sqrt{3}\xi_1} = -\frac{1}{\lambda_6}, \end{aligned} \tag{13}$$

and the unit principal vectors corresponding to λ_i are $X_i/\|X_i\|$ and $T_i/\|T_i\|$. Note that by $\lambda_1 > \cdots > \lambda_6$, (13) implies that $\xi_1 > 0 > \xi_2 > \xi_3$, and hence that

$$\alpha_i(H) > 0, \quad 1 \le i \le 6 \tag{14}$$

follows from (10). Now, putting $e_i = X_i/\|X_i\|, e_{\bar{i}} = T_i/\|T_i\|$, we calculate the structure constants $\Lambda_{\alpha\beta}^\gamma$ with respect to this basis of M. As before, using (12), we obtain $X_i = \big(\|H\|/(\alpha_i(H))\big)\frac{d}{dt}|_{t=0}\,\mathrm{Ad}(\exp tT_i)\tilde{H}$. Here we have

$$\nabla_{X_i}X_j = \frac{\|H\|}{\alpha_i(H)}\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}(\exp tT_i)X_j = \frac{\|H\|}{\alpha_i(H)}[T_i, X_j]. \tag{15}$$

Similarly, we have

$$\nabla_{X_i}T_j = \frac{\|H\|}{\alpha_i(H)}[T_i, T_j], \tag{16}$$

$$\nabla_{T_i}X_j = -\frac{\|H\|}{\alpha_i(H)}[X_i, X_j], \tag{17}$$

$$\nabla_{T_i}T_j = -\frac{\|H\|}{\alpha_i(H)}[X_i, T_j]. \tag{18}$$

Then, noting that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, we see that $\Lambda_{ij}^k = \Lambda_{i\bar{j}}^{\bar{k}} = \Lambda_{\bar{i}j}^{\bar{k}} = \Lambda_{\bar{i}\bar{j}}^k = 0$, $1 \le i, j, k \le 6$ (be careful for the indices with and without bars).

Moreover, by (11) and (12), we obtain $\Lambda_{\alpha\beta}^{\gamma} = 0$ if two of the indices—say, (α, β)—satisfy $\alpha \in [1]$ and $\beta \in [4]$, or $\alpha \in [2]$ and $\beta \in [5]$, or $\alpha \in [3]$ and $\beta \in [6]$, where $[i] = \{i, \bar{i}\}$. Thus, the possible nonzero $\Lambda_{\alpha\beta}^{\gamma}$ are

$$\begin{aligned}\big\{[\alpha], [\beta], [\gamma]\big\} = \{1,2,3\}, \{1,2,6\}, \{1,3,5\}, \{1,5,6\},\\ \{2,3,4\}, \{2,4,6\}, \{3,4,5\}, \{4,5,6\}.\end{aligned}$$

For computation of these, Table 1 shows $[A, B], A, B \in \mathfrak{g}$.

Table 1

	X_2	T_2	X_3	T_3	X_4	T_4	X_5	T_5	X_6	T_6
X_1	$-X_3$	$-T_3$	$2X_5+3X_2$	$2T_5+3T_2$			$-2X_3-3X_6$	$-2T_3-3T_6$	X_5	T_5
T_1	T_3	$-X_3$	$-2T_5+3T_2$	$2X_5-3X_2$			$-2T_3+3T_6$	$2X_3-3X_6$	T_5	$-X_5$
X_2			$-X_1$	T_1	$-X_6$	$-T_6$			X_4	T_4
T_2			$-T_1$	$-X_1$	T_6	$-X_6$			T_4	$-X_4$
X_3					X_5	T_5	$2X_1-3X_4$	$-2T_1-3T_4$		
T_3					$-T_5$	X_5	$2T_1-3T_4$	$2X_1+3X_4$		
X_4							X_3	$-T_3$	$-X_2$	T_2
T_4							T_3	X_3	$-T_2$	$-X_2$
X_5									$-X_1$	T_1
T_5									$-T_1$	$-X_1$

REMARK 3.1. M is a Kähler manifold with complex structure J defined by $JX_i = T_i, JT_i = -X_i$. This is a general theory, but the vanishing of the torsion N and ∇J can be shown directly from Table 1 and (15)–(18).

Here we may assume that $\lambda_1 = 2 + \sqrt{3} = -(\xi_1 - \xi_3)/\sqrt{3}\xi_2$, from which it follows that $\xi_1/\xi_2 = -(2+\sqrt{3})$. Thus, noting (14), we obtain

$$\begin{aligned}&\frac{\|H\|}{\alpha_1(H)} = \sqrt{3}(\sqrt{3}+1), \qquad \frac{\|H\|}{\alpha_2(H)} = 1, \qquad \frac{\|H\|}{\alpha_3(H)} = \sqrt{3}(\sqrt{3}-1),\\ &\frac{\|H\|}{\alpha_4(H)} = (\sqrt{3}-1), \qquad \frac{\|H\|}{\alpha_5(H)} = \sqrt{3}, \qquad \frac{\|H\|}{\alpha_6(H)} = (\sqrt{3}+1).\end{aligned} \tag{19}$$

Now, it follows that

$$\begin{pmatrix}\Lambda_{16}^{2} & \Lambda_{16}^{\bar{2}}\\ \Lambda_{\bar{1}6}^{2} & \Lambda_{\bar{1}6}^{\bar{2}}\end{pmatrix} = \begin{pmatrix}\Lambda_{1\bar{6}}^{2} & \Lambda_{1\bar{6}}^{\bar{2}}\\ \Lambda_{\bar{1}\bar{6}}^{2} & \Lambda_{\bar{1}\bar{6}}^{\bar{2}}\end{pmatrix} = \begin{pmatrix}\Lambda_{42}^{3} & \Lambda_{42}^{\bar{3}}\\ \Lambda_{\bar{4}2}^{3} & \Lambda_{\bar{4}2}^{\bar{3}}\end{pmatrix} = \begin{pmatrix}\Lambda_{4\bar{2}}^{3} & \Lambda_{4\bar{2}}^{\bar{3}}\\ \Lambda_{\bar{4}\bar{2}}^{3} & \Lambda_{\bar{4}\bar{2}}^{\bar{3}}\end{pmatrix} = 0, \tag{20}$$

$$\begin{pmatrix}\Lambda_{16}^{5} & \Lambda_{16}^{\bar{5}}\\ \Lambda_{\bar{1}6}^{5} & \Lambda_{\bar{1}6}^{\bar{5}}\end{pmatrix} = -\frac{\sqrt{3}(\sqrt{3}+1)}{\sqrt{2}} J, \qquad \begin{pmatrix}\Lambda_{1\bar{6}}^{5} & \Lambda_{1\bar{6}}^{\bar{5}}\\ \Lambda_{\bar{1}\bar{6}}^{5} & \Lambda_{\bar{1}\bar{6}}^{\bar{5}}\end{pmatrix} = -\frac{\sqrt{3}(\sqrt{3}+1)}{\sqrt{2}} I,$$

$$
(21)\quad
\begin{aligned}
&\begin{pmatrix}\Lambda^4_{26} & \Lambda^{\bar 4}_{26}\\ \Lambda^4_{\bar 2 6} & \Lambda^{\bar 4}_{\bar 2 6}\end{pmatrix} = -\frac{1}{\sqrt{2}}J, &&
\begin{pmatrix}\Lambda^4_{2\bar 6} & \Lambda^{\bar 4}_{2\bar 6}\\ \Lambda^4_{\bar 2\bar 6} & \Lambda^{\bar 4}_{\bar 2\bar 6}\end{pmatrix} = -\frac{1}{\sqrt{2}}I,\\
&\begin{pmatrix}\Lambda^3_{51} & \Lambda^{\bar 3}_{51}\\ \Lambda^3_{\bar 5 1} & \Lambda^{\bar 3}_{\bar 5 1}\end{pmatrix} = -\sqrt{2}J, &&
\begin{pmatrix}\Lambda^3_{5\bar 1} & \Lambda^{\bar 3}_{5\bar 1}\\ \Lambda^3_{\bar 5\bar 1} & \Lambda^{\bar 3}_{\bar 5\bar 1}\end{pmatrix} = -\sqrt{2}I,\\
&\begin{pmatrix}\Lambda^2_{31} & \Lambda^{\bar 2}_{31}\\ \Lambda^2_{\bar 3 1} & \Lambda^{\bar 2}_{\bar 3 1}\end{pmatrix} = \frac{\sqrt{3}(\sqrt{3}-1)}{\sqrt{2}}J, &&
\begin{pmatrix}\Lambda^2_{3\bar 1} & \Lambda^{\bar 2}_{3\bar 1}\\ \Lambda^2_{\bar 3\bar 1} & \Lambda^{\bar 2}_{\bar 3\bar 1}\end{pmatrix} = \frac{\sqrt{3}(\sqrt{3}-1)}{\sqrt{2}}I,\\
&\begin{pmatrix}\Lambda^3_{45} & \Lambda^{\bar 3}_{45}\\ \Lambda^3_{\bar 4 5} & \Lambda^{\bar 3}_{\bar 4 5}\end{pmatrix} = -\frac{\sqrt{3}-1}{\sqrt{2}}J, &&
\begin{pmatrix}\Lambda^3_{4\bar 5} & \Lambda^{\bar 3}_{4\bar 5}\\ \Lambda^3_{\bar 4\bar 5} & \Lambda^{\bar 3}_{\bar 4\bar 5}\end{pmatrix} = -\frac{\sqrt{3}-1}{\sqrt{2}}I,
\end{aligned}
$$

where

$$
I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad J = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.
$$

Then by (7) and $1/\sin\theta_a = \pm\sqrt{2}(\sqrt{3}+1)$ for $a = 1, 6$, respectively, we have B_ζ and $B_{\bar\zeta}$ of M_+ by Lemma 2.1:

$$
(22)\quad
\begin{aligned}
B_\zeta &= \begin{pmatrix}
0 & 0 & 0 & 0 & \sqrt{3}J\\
0 & 0 & 0 & \frac{1}{\sqrt{3}}J & 0\\
0 & 0 & 0 & 0 & 0\\
0 & -\frac{1}{\sqrt{3}}J & 0 & 0 & 0\\
-\sqrt{3}J & 0 & 0 & 0 & 0
\end{pmatrix},\\
B_{\bar\zeta} &= \begin{pmatrix}
0 & 0 & 0 & 0 & \sqrt{3}I\\
0 & 0 & 0 & \frac{1}{\sqrt{3}}I & 0\\
0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{\sqrt{3}}I & 0 & 0 & 0\\
\sqrt{3}I & 0 & 0 & 0 & 0
\end{pmatrix}.
\end{aligned}
$$

Similarly, denoting the shape operators of M_- by C_ζ and $C_{\bar\zeta}$, $\zeta, \bar\zeta \in D_1$, we can express these with respect to $D_2 \oplus \cdots \oplus D_6$ as

$$
(23)\quad
C_\zeta = \begin{pmatrix}
0 & J & 0 & 0 & 0\\
-J & 0 & 0 & -\frac{2}{\sqrt{3}}J & 0\\
0 & 0 & 0 & 0 & 0\\
0 & \frac{2}{\sqrt{3}}J & 0 & 0 & J\\
0 & 0 & 0 & -J & 0
\end{pmatrix},
$$

$$C_{\bar{\zeta}} = \begin{pmatrix} 0 & -I & 0 & 0 & 0 \\ -I & 0 & 0 & \frac{2}{\sqrt{3}}I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{3}}I & 0 & 0 & -I \\ 0 & 0 & 0 & -I & 0 \end{pmatrix}.$$

In particular, M_+ is not congruent to M_- in S^{13}.

§4. Geometry of G_2 orbits

In [M1, Proposition 2.1], we show that an isoparametric hypersurface N^6 in S^7 with $(g,m) = (6,1)$ is the inverse image of a real Cartan hypersurface $C^3_{\mathbb{R}} \cong SO(3)/(\mathbb{Z}_2 + \mathbb{Z}_2)$ under the Hopf fibration $\pi : S^7 \to S^4$. Since the restriction of the fibration to a proper subset of S^4 is trivial, we have a homeomorphism

$$N^6 \cong C^3_{\mathbb{R}} \times S^3. \tag{24}$$

Note that $C^3_{\mathbb{R}}$ is a principal orbit of the adjoint action of $SO(3)$ on the space of traceless symmetric matrices $\mathrm{Sym}^0(\mathbb{R},3)$. We can express (24) in terms of the decomposition of the tangent bundle of N^6 into two integrable distributions $TN = \mathcal{R} \oplus \mathcal{S}$ given by

$$\begin{aligned} \mathcal{R} &= \mathrm{span}\{e_2, e_4, e_6\}, \\ \mathcal{S} &= \mathrm{span}\{e_1 - \lambda_1 e_4, e_2 + \lambda_2 e_5, e_3 + \lambda_3 e_6\}, \end{aligned} \tag{25}$$

where $\mathcal{S}$ is the direction of the Hopf fiber (see [M1, p. 188, line 6]) and is totally geodesic. On the other hand, $\mathcal{R}$ corresponds to the Lie algebra $\mathfrak{so}(3)$ in $\mathfrak{so}(4)$, since $\Lambda^j_{24} = 0$, $\Lambda^j_{26} = 0$, $\Lambda^j_{46} = 0$ hold except for the indices consisting of $\{2,4,6\}$. Thus, $\mathcal{R}$ is also integrable. Note that N is an arbitrary principal orbit, and λ_i are given by (4) for some $\theta_1 \in (0, \pi/6)$.

In the case $(g,m) = (6,2)$, a parallel argument is *not* valid because of the lack of corresponding fibrations. Instead, since $SU(3)$ is a subgroup of G_2, and its Lie algebra is generated by $\mathfrak{a} \oplus \mathrm{span}\{X_2, T_2, X_4, T_4, X_6, T_6\}$, the subspace

$$\mathcal{R} = D_2 \oplus D_4 \oplus D_6 \tag{26}$$

defines an integrable distribution on M. The leaves are Cartan hypersurfaces $C^6_{\mathbb{C}} \cong SU(3)/T_2$, which are half-dimensional Kähler submanifolds of M^{12}. This defines a Kähler fibration $M \to S^6 \cong G_2/SU(3)$ with fiber $C^6_{\mathbb{C}}$.

We note that $\pi_1(M) = 1 = \pi_1(M_\pm)$ in these arguments. On M_+, the space D_6 collapses, and it is easy to see that $\mathfrak{a} \oplus \mathrm{span}\{X_6, T_6\} = \mathfrak{u}(2)$. Thus, M_+ is diffeomorphic to $G_2/U(2) = \mathbb{Q}^5$, the complex quadric. Moreover, (26) implies that the focal submanifold M_+ has a fibration $M_+ \to S^6$ with fiber $\mathbb{C}P^2 \cong SU(3)/S(U(2) \times U(1))$, which is tangent to $df_6(D_2 \oplus D_4)$. The total space $M_+ \cong \mathbb{Q}^5$ is diffeomorphic to the twistor space of $S^6 = G_2/SU(3)$ given by Bryant [B1].

Similarly, on M_-, the space D_1 collapses, and $\mathfrak{a} \oplus \mathrm{span}\{X_1, T_1\} = \mathfrak{u}(2)$ shows that $M_- = G_2/U(2) = \mathbb{Q}^5$; however, M_+ and M_- are not congruent as is seen from (22) and (23).* In fact, since D_4 is invariant along L_1 ($\Lambda^\alpha_{\underline{14}} = 0$), the image of the curvature surface L_4 under the focal map f_1 defines a totally geodesic $S^2 = \mathbb{C}P^1$ fibration on M_-. Here, total geodesity follows since $df_1(D_4)$ belongs to the kernel of all the shape operators (see (23)). It is easy to see that $\mathrm{span}\{H, X_1, T_1\}$ and $\mathrm{span}\{H^\perp, X_4, T_4\}$ are isomorphic to $\mathfrak{so}(3)$, where

$$H = G_{24} + G_{56} - 2G_{37}, \qquad H^\perp = G_{24} - G_{56},$$

and hence that the space $\mathfrak{a} \oplus \mathrm{span}\{X_1, T_1, X_4, T_4\}$ is isomorphic to $\mathfrak{so}(4)$. Therefore, the base manifold of this $\mathbb{C}P^1$ fibration is given by $G_2/SO(4)$, the quaternionic Kähler manifold. This implies that M_- is diffeomorphic to the twistor space of $G_2/SO(4)$ given in [B2].

On the other hand, since $\mathfrak{g}_3 \oplus \mathfrak{g}_4 \oplus \mathfrak{g}_6 = \mathrm{span}\{X_1, \dots, X_6\}$ generates another $\mathfrak{so}(4)$ (see [M1, p. 183]), we have a half-dimensional submanifold at each point of M given by this $SO(4)$ suborbit N^6. In fact, the tangent space of N^6 is spanned by

$$T_i = \frac{d}{dt}\Big|_{t=0} (\mathrm{Ad}\exp tX_i)\tilde{H}, \quad i = 1, \dots, 6.$$

From Remark 3.1, we see that N^6 is a Lagrangian submanifold of M^{12}. At each point of M^{12}, the tangent space of N^6 is expressed as $\{e_{\bar{1}}, e_{\bar{2}}, \dots, e_{\bar{6}}\}$; however, the direction of $e_{\bar{i}}$ can be replaced by a suitable combination of e_i and $e_{\bar{i}}$ in each D_i. In fact, $SO(4)$ is embedded in G_2 in a 2-parameter family, such as

$$\sin\varphi X_1 - \cos\varphi T_1, \qquad \sin\psi X_2 - \cos\psi T_2,$$

$$\text{(27)} \quad \cos(\varphi - \psi) X_3 - \sin(\varphi - \psi) T_3, \qquad \sin(2\psi - 3\varphi) X_4 - \cos(2\psi - 3\varphi) T_4,$$

*Here N^5_+ is not congruent to N^5_- in S^7 (see [M1, Proposition 2.5]).

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$$\sin(\psi-2\varphi)X_5-\cos(\psi-2\varphi)T_5, \qquad \cos(3\varphi-\psi)X_6-\sin(3\varphi-\psi)T_6.$$

By using Table 1, we can see that the Lie bracket closes in this space, which generates $\mathfrak{so}(4)$ for any fixed φ and ψ. The tangent space of the corresponding $SO(4)$ orbit is the 6-dimensional subspace of TM spanned by

$$\begin{aligned}&\cos\varphi e_1+\sin\varphi e_{\bar{1}}, \qquad \cos\psi e_2+\sin\psi e_{\bar{2}},\\ &\sin(\varphi-\psi)e_3+\cos(\varphi-\psi)e_{\bar{3}}, \qquad \cos(2\psi-3\varphi)e_4+\sin(2\psi-3\varphi)e_{\bar{4}},\\ &\cos(\psi-2\varphi)e_5+\sin(\psi-2\varphi)e_{\bar{5}}, \qquad \sin(3\varphi-\psi)e_6+\cos(3\varphi-\psi)e_{\bar{6}}.\end{aligned} \tag{28}$$

This reflects the fact that the isotropy subgroup T^2 of $M=G_2/T^2$ at $o=T^2$ acts on T_oM as an isometry. Thus, at $o=T^2\in M$ (and hence at each point of M), there exists a 2-parameter family of the $SO(4)$ orbits which are Lagrangian.

Note that the distribution $\mathcal{R}$ given in (26) and the tangent space of each $SO(4)$ orbit (e.g., spanned by $e_{\bar{1}},\dots,e_{\bar{6}}$) are *not* transversal; that is, they do not span TM. Now we have almost shown Theorems 1.1 and 1.2, which we restate in a refined way.

MAIN THEOREM. *On every G_2 orbit M_t, $t\in(-1,1)$, and $M_\pm$, which sweep out S^{13}, there exists a Kähler fibration:*

(i) *$M_t\cong G_2/T^2\to S^6=G_2/SU(3)$ with fiber $C^6_{\mathbb{C}}=SU(3)/T^2$ tangent to $D_2\oplus D_4\oplus D_6$;*

(ii) *$M_+\cong\mathbb{Q}^5\to S^6=G_2/SU(3)$ with fiber $\mathbb{C}P^2=SU(3)/S(U(2)\times U(1))$ tangent to $df_6(D_2\oplus D_4)$, where f_6 is the focal map, and which is diffeomorphic to the twistor fibration of S^6;*

(iii) *$M_-\cong\mathbb{Q}^5\to G_2/SO(4)$ with fiber $\mathbb{C}P^1=SU(2)/S(U(1)\times U(1))$ tangent to $df_1(D_4)$, where f_1 is the focal map, and which is diffeomorphic to the twistor fibration of the quaternionic Kähler manifold $G_2/SO(4)$.*

Note that M_+ is not congruent to M_- in S^{13}.

Moreover, at each point of M_t, there exists a 2-parameter family of Lagrangian submanifolds transferred from an $SO(4)$ suborbit N^6, which is tangent to $\operatorname{span}\{e_{\bar{i}},1\le i\le 6\}$*, a set of suitably chosen $e_{\bar{i}}\in D_i$. Here, $C^6_{\mathbb{C}}$ and N^6 are not transversal. Such N^6 collapses into $N^5_\pm\cong\mathbb{R}P^2\times S^3$ on $M_\pm$, where N_+ is tangent to* $\operatorname{span}\{df_6(e_i),1\le i\le 5\}$ *and where N_- is tangent to* $\operatorname{span}\{df_1(e_i),2\le i\le 6\}$*. In particular, these are minimal Lagrangian submanifolds on $M_\pm$ and on M_0, where the latter is the minimal principal orbit. However, they never define Lagrangian fibrations on M_t or on $M_\pm$.*

Proof. Here N^6 collapses into $N^5_\pm$ as D_6 and D_1 collapse on $M_\pm$, respectively. We denote by N_0 the minimal principal $SO(4)$ orbit lying in M_0. Because N_0 and $N_\pm$ are minimal in some totally geodesic 7-sphere of S^{13}, these are minimal in S^{13}, and hence minimal in $M_\pm$ and in M_0, respectively.

Nonexistence of a Lagrangian fibration follows because the topology of N^6 or $N^5_\pm$ is not that of a torus. □

Since there are 2-parameter isometric deformations of $N^5_\pm$ in $M_\pm$, and N^6_0 in M_0, we obtain the following.

COROLLARY 4.1. *The nullity of the Lagrangian minimal submanifold $N^5_\pm$ in $M_\pm$, and N^6_0 in M_0, respectively, is not less than 2.*

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