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ON KAEHLER IMMERSIONS

KOICHI OGIUE

1. Introduction. Let $\tilde{M}$ be an $(n + p)$-dimensional Kaehler manifold of constant holomorphic sectional curvature $\tilde{c}$. B. O'Neill [**3**] proved the following result.

PROPOSITION A. *Let M be an n-dimensional complex submanifold immersed in $\tilde{M}$. If $p < \frac{1}{2}n(n + 1)$ and if the holomorphic sectional curvature of M with respect to the induced Kaehler metric is constant, then M is totally geodesic.*

He also gave the following example: There is a Kaehler imbedding of an n-dimensional complex projective space of constant holomorphic sectional curvature $\frac{1}{2}$ into an $\{n + \frac{1}{2}n(n + 1)\}$-dimensional complex projective space of constant holomorphic sectional curvature 1. This shows that Proposition A is the best possible.

The purpose of this paper is to prove the following theorems.

THEOREM 1. *Let M be an n-dimensional complex submanifold immersed in an $(n + p)$-dimensional Kaehler manifold $\tilde{M}$ of constant holomorphic sectional curvature $\tilde{c}$ $(\tilde{c} > 0)$. If $p \geqq \frac{1}{2}n(n + 1)$ and if the holomorphic sectional curvature of M with respect to the induced Kaehler metric is a constant c, then either $c = \tilde{c}$ (i.e., M is totally geodesic) or $c \leqq \frac{1}{2}\tilde{c}$.*

THEOREM 2. *Let M be an n-dimensional complex submanifold immersed in an $(n + p)$-dimensional Kaehler manifold $\tilde{M}$ of constant holomorphic sectional curvature $\tilde{c}$. If*

(i) $p \geqq \frac{1}{2}n(n + 1)$,

(ii) *the holomorphic sectional curvature of M with respect to the induced Kaehler metric is a constant c, and*

(iii) *the second fundamental form of the immersion is parallel, then either $c = \tilde{c}$ (i.e., M is totally geodesic) or $c = \frac{1}{2}\tilde{c}$, the latter case arising only when $\tilde{c} > 0$.*

2. Preliminaries. Let J (respectively $\tilde{J}$) be the complex structure of M (respectively $\tilde{M}$) and g (respectively $\tilde{g}$) be the Kaehler metric of M (respectively $\tilde{M}$). We denote by ∇ (respectively $\tilde{\nabla}$) the covariant differentiation with respect to g (respectively $\tilde{g}$). Then the second fundamental form σ of the immersion is given by

$$\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$$

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and it satisfies

$$\sigma(X, JY) = \sigma(JX, Y) = \tilde{J}\sigma(X, Y).$$

Let R be the curvature tensor field of M. Then the equation of Gauss is

$$\begin{aligned} g(R(X, Y)Z, W) = {} & \tilde{g}(\sigma(X, W), \sigma(Y, Z)) - \tilde{g}(\sigma(X, Z), \sigma(Y, W)) \\ & + \tfrac{1}{4}\tilde{c}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \\ & + g(JX, W)g(JY, Z) - g(JX, Z)g(JY, W) \\ & + 2g(X, JY)g(JZ, W)\}. \end{aligned}$$

Let $\xi_1, \ldots, \xi_p, \tilde{J}\xi_1, \ldots, \tilde{J}\xi_p$ be local fields of orthonormal vectors normal to M. If we set, for $i = 1, \ldots, p$,

$$\sigma(X, Y) = \sum g(A_iX, Y) \cdot \xi_i + \sum g(A_{i*}X, Y) \cdot \tilde{J}\xi_i,$$

then $A_1, \ldots, A_p, A_{1*}, \ldots, A_{p*}$ are local fields of symmetric linear transformations. We can easily see that $A_{i*} = JA_i$ and $JA_i = -A_iJ$ so that, in particular, $\operatorname{tr} A_i = \operatorname{tr} A_{i*} = 0$. The equation of Gauss can be written in terms of A_i's as

$$\begin{aligned} g(R(X, Y)Z, W) = {} & \sum \{g(A_iX, W)g(A_iY, Z) - g(A_iX, Z)g(A_iY, W) \\ & + g(JA_iX, W)g(JA_iY, Z) - g(JA_iX, Z)g(JA_iY, W)\} \\ & + \tfrac{1}{4}\tilde{c}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \\ & + g(JX, W)g(JY, Z) - g(JX, Z)g(JY, W) \\ & + 2g(X, JY)g(JZ, W)\}. \end{aligned}$$

Let S be the Ricci tensor of M. Then we have

$$S(X, Y) = \tfrac{1}{2}(n + 1)\tilde{c}g(X, Y) - 2g(\sum A_i^2X, Y). \tag{1}$$

We can see that the sectional curvature K of M determined by orthonormal vectors X and Y is given by

$$K(X, Y) = \tfrac{1}{4}\tilde{c}\{1 + 3g(X, JY)^2\} + \tilde{g}(\sigma(X, X), \sigma(Y, Y)) - ||\sigma(X, Y)||^2. \tag{2}$$

In particular, the holomorphic sectional curvature H of M determined by a unit vector X is given by

$$H(X) = \tilde{c} - 2||\sigma(X, X)||^2. \tag{3}$$

Let $||\sigma||$ be the length of the second fundamental form σ of the immersion so that $||\sigma||^2 = 2 \sum \operatorname{tr} A_i^2$.

Let ∇' be the covariant differentiation with respect to the connection in (tangent bundle of M) $\oplus$ (normal bundle) induced naturally from $\tilde{\nabla}$. Then we have

$$(\nabla_X'\sigma)(Y, Z) = (\tilde{\nabla}_X \cdot \sigma(Y, Z))^\perp - \sigma(\nabla_XY, Z) - \sigma(Y, \nabla_XZ),$$

where $\perp$ denotes the normal component.

We know that the second fundamental form σ satisfies a differential equation, that is,

LEMMA 1 [2]. *We have*

(4) $$\tfrac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 - 8\,\mathrm{tr}(\sum A_i^2)^2 - \sum (\mathrm{tr}\, A_\alpha A_\beta)^2 + \tfrac{1}{2}(n+2)\tilde{c}\|\sigma\|^2,$$

where Δ *denotes the Laplacian and* $\alpha, \beta = 1, \ldots, p, 1^*, \ldots, p^*$.

3. Proof of theorems. First we note that $c \leqq \tilde{c}$.

Since $H = c$, we have from (1)

(5) $$\sum A_i^2 = \tfrac{1}{4}(n+1)(\tilde{c} - c)I,$$

where I denotes the identity transformation. From (5) we have

(6) $$\|\sigma\|^2 = n(n+1)(\tilde{c} - c).$$

Moreover, from (3) we have

(7) $$\|\sigma(X, X)\|^2 = \tfrac{1}{2}(\tilde{c} - c)$$

for every unit vector X.

On the other hand, $H = c$ implies $K(X, Y) = K(X, JY) = \frac{1}{4}c$ provided that X, Y and JY are orthonormal. Therefore from (2) we have

(8) $$\|\sigma(X, Y)\|^2 = \tfrac{1}{4}(\tilde{c} - c)$$

for orthonormal X, Y and JY.

Let $e_1, \ldots, e_n, Je_1, \ldots, Je_n$ be local fields of orthonormal vectors on M. Then we have the following

LEMMA 2 [3]. *The* $n(n+1)$ *local fields of vectors* $\sigma(e_a, e_b)$, $\tilde{J}\sigma(e_a, e_b)$ $(1 \leqq a \leqq b \leqq n)$ *are orthogonal.*

This, together with (7) and (8), implies that $\sigma(e_a, e_b)$, $\tilde{J}\sigma(e_a, e_b)$ $(1 \leqq a \leqq b \leqq n)$ are linearly independent at each point provided $c \neq \tilde{c}$.

If $c = \tilde{c}$ then M is totally geodesic in $\tilde{M}$. From now on we may therefore assume that $c \neq \tilde{c}$.

Let $\xi_1, \ldots, \xi_p, \tilde{J}\xi_1, \ldots, \tilde{J}\xi_p$ be local fields of orthonormal vectors normal to M such that

$$\xi_a = \left[\frac{2}{\tilde{c} - c}\right]^{\frac{1}{2}} \sigma(e_a, e_a), \quad \text{for } 1 \leqq a \leqq n$$

$$\xi_r = \frac{2}{(\tilde{c} - c)^{\frac{1}{2}}} \sigma(e_a, e_b), \quad \text{for } 1 \leqq a < b \leqq n$$

$$\text{and } r = a + \tfrac{1}{2}(b - a)(2n + 1 + a - b).$$

Then we can see that the corresponding A_i's are as follows:

$$A_1 = \left[\begin{array}{c|c} \begin{matrix}\theta & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0\end{matrix} & \\ \hline & \begin{matrix}-\theta & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0\end{matrix}\end{array}\right], \ldots, \quad A_n = \left[\begin{array}{c|c} \begin{matrix}0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \theta\end{matrix} & \\ \hline & \begin{matrix}0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & -\theta\end{matrix}\end{array}\right],$$

$$A_{n+1} = \left[\begin{array}{c|c} \begin{matrix}0 & \psi & & & \\ \psi & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0\end{matrix} & \\ \hline & \begin{matrix}0 & -\psi & & & \\ -\psi & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0\end{matrix}\end{array}\right], \ldots, \quad A_{2n-1} = \left[\begin{array}{c|c} \begin{matrix}0 & & & \\ & \ddots & & \\ & & 0 & \psi \\ & & \psi & 0\end{matrix} & \\ \hline & \begin{matrix}0 & & & \\ & \ddots & & \\ & & 0 & -\psi \\ & & -\psi & 0\end{matrix}\end{array}\right],$$

$$A_{2n} = \left[\begin{array}{c|c} \begin{matrix}0 & 0 & \psi & & \\ 0 & 0 & 0 & & \\ \psi & 0 & 0 & & \\ & & & \ddots & \\ & & & & 0\end{matrix} & \\ \hline & \begin{matrix}0 & 0 & -\psi & & \\ 0 & 0 & 0 & & \\ -\psi & 0 & 0 & & \\ & & & \ddots & \\ & & & & 0\end{matrix}\end{array}\right], \ldots, \quad A_{3n-3} = \left[\begin{array}{c|c} \begin{matrix}0 & & & & \\ & \ddots & & & \\ & & 0 & 0 & \psi \\ & & 0 & 0 & 0 \\ & & \psi & 0 & 0\end{matrix} & \\ \hline & \begin{matrix}0 & & & & \\ & \ddots & & & \\ & & 0 & 0 & -\psi \\ & & 0 & 0 & 0 \\ & & -\psi & 0 & 0\end{matrix}\end{array}\right],$$

$\ldots,$

$$A_{\frac{1}{2}n(n+1)} = \left[\begin{array}{c|c} \begin{matrix}0 & \cdots & 0 & \psi \\ \vdots & & & 0 \\ 0 & & 0 & \vdots \\ \psi & 0 & \cdots & 0\end{matrix} & \\ \hline & \begin{matrix}0 & \cdots & 0 & -\psi \\ \vdots & & & 0 \\ 0 & & 0 & \vdots \\ -\psi & 0 & \cdots & 0\end{matrix}\end{array}\right],$$

and $A_\alpha = 0$ for $\alpha > \frac{1}{2}n(n+1)$, where $\theta = (\frac{1}{2}(\tilde{c} - c))^{\frac{1}{2}}$ and $\psi = \frac{1}{2}(\tilde{c} - c)^{\frac{1}{2}}$. Hence we have

$$(9) \qquad \sum (\operatorname{tr} A_\alpha A_\beta)^2 = 2 \sum (\operatorname{tr} A_i^{\,2})^2 = n(n+1)(\tilde{c} - c)^2.$$

Therefore, from (4), (5), (6) and (9), we have

$$||\nabla'\sigma||^2 = n(n+1)(n+2)(\tilde{c} - c)(\tfrac{1}{2}\tilde{c} - c),$$

from which our theorems follow immediately.

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4. Remark. We consider the special case $n = p = 1$. We have the following

LEMMA. *Let M be a complex curve immersed in a 2-dimensional Kaehler manifold $\tilde{M}$ of constant holomorphic sectional curvature $\tilde{c}$. If $\sigma \neq 0$ everywhere on M, then*

$$\Delta \log ||\sigma||^2 = 3(\tilde{c} - ||\sigma||^2).$$

For the proof see Corollary 1.7 in [**1**]. As an immediate consequence of this Lemma, we have the following result which is an improvement of Theorem 2 for the case $n = p = 1$.

PROPOSITION. *Let M be a complex curve immersed in a 2-dimensional Kaehler manifold $\tilde{M}$ of constant holomorphic sectional curvature $\tilde{c}$. If the curvature of M with respect to the induced Kaehler metric is a constant c, then either $c = \tilde{c}$* (i.e., *M is totally geodesic) or $c = \frac{1}{2}\tilde{c}$, the latter case arising only when $\tilde{c} > 0$.*

The proof is clear from the fact that $||\sigma||^2 = 2(\tilde{c} - c)$.

Added in proof. A generalization of this proposition is published in J. Math. Soc. Japan *24* (1972), 518–526.

REFERENCES

1. K. Ogiue, *Differential geometry of algebraic manifolds*, Differential Geometry, in honor of K. Yano, 355–372 (Kinokuniya, Tokyo, 1972).

2. ——— *Positively curved complex submanifolds immersed in a complex projective space* (to appear in J. Differential Geometry).

3. B. O'Neill, *Isotropic and Kaehler immersions*, Can. J. Math. *17* (1965), 907–915.

Tokyo Metropolitan University,
Tokyo, Japan