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AN INDUCTION THEOREM FOR UNITS OF p-ADIC GROUP RINGS

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ABSTRACT. Let G be a finite group and let $\mathcal{C}$ be the family of cyclic subgroups of G. We show that the normal subgroup H of $U = U(\mathbb{Z}_pG)$ generated by $U(\mathbb{Z}_pC), C \in \mathcal{C}$, where $\mathbb{Z}_p$ is the ring of p-adic integers, is of finite index in U.

1. **Introduction.** Let $U(RG)$ be the group of units of the group ring RG of a finite group G over a commutative ring R. Recently Kleinert [3] has shown that, with a few restrictions on G, the normal subgroup of $U(\mathbb{Z}G)$ generated by the units of the integral group rings of cyclic subgroups of G is of finite index in $U(\mathbb{Z}G)$ (see also [7]). In this note we show that the corresponding result is true for the group rings of finite groups over $\mathbb{Z}_p$, the ring of p-adic integers, without any restriction on G. We prove

THEOREM 1.1. *Let G be a finite group and let $\mathcal{C}$ be the family of cyclic subgroups of G. Then the normal subgroup H of $U = U(\mathbb{Z}_pG)$ generated by $U(\mathbb{Z}_pC), C \in \mathcal{C}$ is of finite index in U.*

The proof is based on the p-adic version of a theorem of Bass due to Lam, a theorem of Borel and some arguments of Kleinert.

There is no counterexample known to the global version of the theorem.

2. **Notations and Preliminaries.** Let $\mathbb{Q}_p$ and $\mathbb{Z}_p$ denote respectively the field and ring of p-adic integers. Let Λ be a fixed maximal order of $\mathbb{Q}_pG$ containing $\mathbb{Z}_pG$. From the semi-simplicity of $\mathbb{Q}_pG$ and [5, Corollary 17.4] it follows that

$$\Lambda \approx \prod_{i=1}^{t} M_{n_i}(\mathfrak{o}_i),$$

where $\mathfrak{o}_i$ is the maximal $\mathbb{Z}_p$-order in a division $\mathbb{Q}_p$ -algebra $\mathcal{D}_i$ for which $\mathbb{Q}_pG \approx \prod_{i=1}^{t} M_{n_i}(\mathcal{D}_i)$. We put $\Lambda_i = M_{n_i}(\mathfrak{o}_i)$. Let K_i be the centre of $\mathcal{D}_i$ and let R_i be the integers of K_i. Then R_i is the centre of Λ_i. Let Λ^* be the group of units of Λ and let γ be its group of central units. We have

$$\Lambda^* \approx \Lambda_1^* \times \Lambda_2^* \times \cdots \times \Lambda_t^*$$
$$\gamma \approx \gamma_1 \times \gamma_2 \times \cdots \times \gamma_t$$

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where γ_i is the centre of $\Lambda_i^* = GL_{n_i}(\mathfrak{o}_i)$ which is the group of units of $M_{n_i}(\mathfrak{o}_i)$. We denote by $SL_{n_i}(\mathfrak{o}_i)$ the subgroup of $GL_{n_i}(\mathfrak{o}_i)$ consisting of all elements having reduced norm 1. Finally, let π_i be the projection of $\mathbb{Q}_pG$ onto $M_{n_i}(\mathcal{D}_i)$.

We shall need a theorem of Borel which adapted to our situation reads as

THEOREM 2.1 [6, P. 523]. *Let $\mathcal{D}$ be a finite dimensional division algebra over a field F which is a finite extension of the p-adic field $\mathbb{Q}_p$. Then any non-central normal subgroup of an open (in the topology induced from the field) subgroup of $SL_n(\mathcal{D})$, $n \geq 2$, is open.*

For a non-commutative finite dimensional division algebra $\mathcal{D}$ over a p-adic field F of characteristic zero, the group $SL_1(\mathcal{D})$ is not almost simple. However, the Lie algebra associated to $SL_1(\mathcal{D})$ is

$$\mathfrak{sl}(\mathcal{D}) = \{x \in \mathcal{D} \mid \text{red. tr}\,(x) = 0\},$$

which is simple. Also, for fields of characteristic zero, the kernel of the *Ad* map is equal to the centre of $SL_1(\mathcal{D})$ ([1, p. 133]). Thus the proof of Borel's theorem, as given in [6], can be applied to $SL_1(\mathcal{D})$ to yield

THEOREM 2.2. *Let $\mathcal{D}$ be a non-commutative finite dimensional division algebra over a field F which is a finite extension of the p-adic field $\mathbb{Q}_p$. Then every non-central normal subgroup of an open subgroup of $SL_1(\mathcal{D})$ is open.*

If $\mathfrak{o}$ is the maximal order in $\mathcal{D}$, then the group $SL_n(\mathfrak{o}) = GL_n(\mathfrak{o}) \cap SL_n(\mathcal{D})$, $n \geq 1$, is a compact open subgroup of $SL_n(\mathcal{D})$. Hence every open subgroup of $SL_n(\mathfrak{o})$ is of finite index and from Theorem 2.1 and 2.2 we deduce

COROLLARY 2.3. *Let $\mathcal{D}$ be as in Theorem 2.1 (for $n \geq 2$) and as in Theorem 2.2 (for $n = 1$). Then every non-central normal subgroup of an open subgroup of $SL_n(\mathfrak{o})$, $n \geq 1$, is of finite index.*

As the reduced norm embeds $GL_{n_i}/SL_{n_i}(\mathfrak{o}_i)\gamma_i$ into $R_i^*/R_i^{*\ell_i}$ where $\ell_i = n_i[D_i : K_i]$ and since $R_i^*/R_i^{*\ell_i}$ is finite [2, Section 15.5], it follows that $\left(GL_{n_i}(\mathfrak{o}_i) : \gamma_i SL_{n_i}(\mathfrak{o}_i)\right) < \infty$. Thus $\left(\Lambda^* : \gamma \prod_{i=1}^{t} SL_{n_i}(\mathfrak{o}_i)\right) < \infty$.

We also have

LEMMA 2.4 [4, SECTION 4.2]. *The images of $U(\mathbb{Z}_pC)$, $C \in \mathcal{C}$, under the natural map j: $U(\mathbb{Z}_pG) \longrightarrow K_1(\mathbb{Z}_pG)$ generate a subgroup of finite exponent.*

Now, let $u \in \zeta = \zeta\left(U(\mathbb{Z}_pG)\right)$, the centre of U. By Lemma 2.4 there is a power u^k and an $h \in H$ such that $j(u^k) = j(h)$ or $u^k h^{-1} \in \ker j$. Hence

$$\begin{pmatrix} u^k h^{-1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

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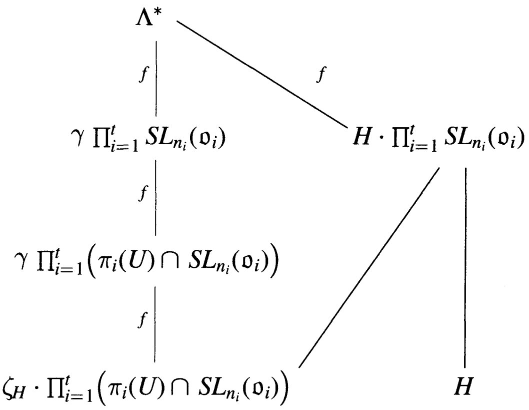
is a commutator in $GL_n(\mathbb{Z}_pG)$ and it follows that $nr\big(\pi_i(u^kh^{-1})\big) = 1$ for all i. Since by [2, Section 15.5] ζ/ζ^k is finite, it follows that a subgroup $\zeta_H = \zeta^k \subseteq \zeta$ of finite index is contained in $H \cdot \prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i)$. Also, since ζ centralizes G, it is contained in γ and $(\gamma : \zeta) < \infty$. Thus, we have the following diagram (where an f indicates that the appropriate piece is of finite index).



Since $(\prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i) : H \cap \prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i)\big) < \infty \Rightarrow \big(H \cdot \prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i) : H\big) < \infty$, therefore, we have

LEMMA 2.5. *To show* $(U : H) < \infty$, *it suffices to show that* $\big(\prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i) : H \cap \prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i)\big) < \infty$.

3. **Proof of Theorem 1.1.** We shall need the following

PROPOSITION 3.1. *Let N be a normal subgroup of finite index in $SL_{n_i}(\mathfrak{o}_i)$, $n_i \geq 1$. Then $(SL_{n_i}(\mathfrak{o}_i) : [N,N]) < \infty$ and hence $(SL_{n_i}(\mathfrak{o}_i) : [SL_{n_i}(\mathfrak{o}_i), N]) < \infty$.*

PROOF. We first consider the case when $n_i \geq 2$. Clearly $[N,N]$ is non-central as it contains the non-diagonal matrix $\left[\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}\right]$ with $q = (SL_{n_i}(\mathfrak{o}_i) : N)$. Since $[N,N]$ is normal in $SL_{n_i}(\mathfrak{o}_i)$, Corollary 2.3 yields the result.

We now consider the case when $n_i = 1$. We may clearly assume that $\mathcal{D}_i$ is not commutative. We use the results of [6] in this case.

If $t_i = [\mathcal{D}_i : K_i]$, then $\zeta\big(SL_1(\mathcal{D}_i)\big)$ is the group of t_i^{th} roots of 1 lying in K_i ([6, Theorem 7 (ii)]). Since N is of finite index, by [6, Theorem 7(ii), Theorems 15 and 20] it is not central and contains H_s for some s, where $H_s = \{x \in SL_1(\mathfrak{o}_i) | x \equiv 1 (\operatorname{mod} \mathfrak{p}_i^s)\}$ with $\mathfrak{p}_i$ being the maximal ideal of $\mathfrak{o}_i$. Since $\cap H_r = \{1\}$ ([6, p. 521]), by choosing s large enough we may assume that H_s does not contain any root of unity except 1 and that $\mathfrak{p}_i^s$ is non-central. Thus the centrality of $[N,N]$ will imply that H_s is abelian, which is impossible as $[1 + x\mathfrak{p}_i^s, 1 + \mathfrak{p}_i^s] \neq 1$ if $x \in \mathfrak{o}_i$ does not commute with $\mathfrak{p}_i^s$. The proof is now complete in view of Corollary 2.3.

The following proposition allows us to reduce the question to the simple components of $\mathbb{Q}_pG$.

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PROPOSITION 3.2. *If $H_i = \pi_i(H) \cap SL_{n_i}(\mathfrak{o}_i)$ is of finite index in $SL_{n_i}(\mathfrak{o}_i)$ for all i, then $H \cap \big(\prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i)\big)$ is of finite index in $\prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i)$.*

PROOF. Let $V = U \cap \prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i)$ where $U = U(\mathbb{Z}_p G)$. We have $H \cap \prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i) \triangleleft V < \prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i)$ and $\big(\prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i) : V\big) < \infty$. Let $\tilde{V} = \bigcap_{y \in \prod SL_{n_i}(\mathfrak{o}_i)} V^y$. Then $\tilde{V}$ is a normal subgroup of finite index in $\prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i)$. We put

$$\hat{V} = \bigoplus_i [\tilde{V}, SL_{n_i}(\mathfrak{o}_i)]$$

and note that $[\tilde{V}, SL_{n_i}(\mathfrak{o}_i)] = [\pi_i(\tilde{V}), SL_{n_i}(\mathfrak{o}_i)] \subseteq \tilde{V} \cap SL_{n_i}(\mathfrak{o}_i)$ and $\hat{V} = \oplus_i \pi_i(\tilde{V})$.

We shall prove that

(i) $(\prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i) : \hat{V}) < \infty$ and

(ii) $(\hat{V} : A) < \infty$ where $A = \hat{V} \cap H$.

In order to prove (i), it is enough to prove that $\big(SL_{n_i}(\mathfrak{o}_i) : \pi_i(\tilde{V}) < \infty\big)$ for all i. Now, $\pi_i(\tilde{V})$ is a normal subgroup of finite index in $SL_{n_i}(\mathfrak{o}_i)$; therefore, by Proposition 3.1,

$$\big(SL_{n_i}(\mathfrak{o}_i) : [\pi_i(\tilde{V}), SL_{n_i}(\mathfrak{o}_i)]\big) < \infty.$$

To prove (ii), we observe that A contains $\oplus_i [A, \pi_i(\tilde{V})]$. Therefore, it suffices to prove that $\big(\pi_i(\hat{V}) : [A, \pi_i(\hat{V})]\big) < \infty$ for all i. Writing $A_i = \pi_i(A)$ we have to show that $\big(\pi_i(\hat{V}) : [A_i, \pi_i(\hat{V})]\big) < \infty$.

Since, due to (i), $\big(H \cap \prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i) : A\big) < \infty$, we get that $\big(\pi(H \cap \prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i)) : A_i\big) < \infty$ and it follows that $\big(\pi_i(\hat{V}) \cap H_i : A_i\big) < \infty$ (as $\pi_i(\hat{V}) \cap H_i \subset \pi_i\big(H \cap \prod_{i=1}^t SL_{n_i}(\mathfrak{o}_i)\big)$. Also, because of the hypothesis, $\big(\pi_i(\hat{V})H_i : H_i\big) = \big(\pi_i(\hat{V}) : \pi_i(\hat{V}) \cap H_i\big) < \infty$. Hence we get that $\big(\pi_i(\hat{V}) : A_i\big) < \infty$. Considering

$$\pi_i(\hat{V}) \supset A_i \supset [A_i, \pi_i(\hat{V})]$$

and factoring by $[A_i, \pi(\hat{V})]$, we apply a theorem of Schur ([8, p. 39]) to conclude that $\big([\pi_i(\hat{U}), \pi_i(\hat{V})] : [A_i, \pi_i(\hat{V})]\big) < \infty$. Moreover, by Proposition 3.1, we know that $\big(\pi_i(\hat{V}) : [\pi_i(\hat{V}), \pi_i(\hat{V})]\big) < \infty$. Hence $\big(\pi_i(\hat{V}) : [A_i, \pi_i(\hat{V})]\big) < \infty$, as desired.

PROOF OF THEOREM 1.1. In view of Lemma 2.5 and the above proposition, we need only to prove that $(SL_{n_i}(\mathfrak{o}_i) : H_i) < \infty$ for all i. In case $n_i = 1$ and the Wedderburn component $\mathcal{D}_i$ is a field, then $SL_1(\mathcal{D}_i) = \{1\}$. Thus to show $(SL_{n_i}(\mathfrak{o}_i) : H_i) < \infty$ we may suppose that either $n_i \geq 2$ or the corresponding Wedderburn component is a non-commutative division ring $\mathcal{D}_i$.

$H_i = \pi_i(H) \cap SL_{n_i}(\mathfrak{o}_i)$ is a normal subgroup of $\pi_i(U) \cap SL_{n_i}(\mathfrak{o}_i)$. We claim that H_i is non-central. Now, $\pi_i(G) \cap SL_{n_i}(\mathfrak{o}_i) \subset H_i$ and the group $nr\big(\pi_i(G)\big)$ being a finite subgroup of K_i^* is cyclic. Thus, if H_i is central then $\pi_i(G)/\,\pi_i(G) \cap SL_{n_i}(\mathfrak{o}_i) \approx nr\big(\pi_i(G)\big)$ would imply that $\pi_i(G)$ is abelian which is not possible (as $\pi_i(G)$ spans $M_{n_i}(\mathcal{D}_i)$ over K_i).

Next, we observe that $\pi(U) \cap SL_{n_i}(\mathfrak{o}_i)$ is open in $SL_{n_i}(\mathfrak{o}_i)$. To see this, let

$$J = \{x \in \mathbb{Q}G \mid \Lambda x \subseteq \mathbb{Z}_p G\}$$

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be the (right) conductor of $\mathbb{Z}_pG$ in Λ. Thus J is a 2-sided Λ-ideal contained in $\mathbb{Z}_pG$ and J splits into the direct sum $J = \oplus J_i$ ([5, p. 380]). Since $\pi_i(U) \cap SL_{n_i}(\mathfrak{o}_i)$ contains the subgroup $\{ u \in SL_{n_i}(\mathfrak{o}_i) \mid u \equiv 1 \pmod{J_i} \}$, it is open. Hence, by Corollary 2.3, we get that $(SL_{n_i}(\mathfrak{o}_i) : H_i) < \infty$. This completes the proof of the theorem.

REFERENCES

1. A. Borel, *Linear algebraic groups*. W. A. Benjamin, Inc., New York, 1969.
2. H. Hasse, *Number theory*. Springer-Verlag, Berlin, 1980.
3. E. Kleinert, *A theorem of units of integral group rings*, J. Pure and Applied Algebra **49**(1987), 161–171.
4. T. Y. Lam, *Induction theorems for Grothendieck groups and Whitehead groups of finite groups*, Ann. Sci. École Norm. Sup. (4)**1**(1968), 91–148.
5. I. Reiner, *Maximal orders*. Academic Press, London, 1975.
6. C. Reihm, *The norm 1 group of p-adic division algebra*, Amer. J. Math. **92**(1970), 499–523.
7. J. Ritter and S. K. Sehgal, *Certain normal subgroups of units in group rings*, J. Reine Angew. Math. **381**(1987), 214–220.
8. S. K. Sehgal, *Topics in group rings*. Marcel-Dekker, N.Y., 1978.

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