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SOME CHARACTERIZATIONS OF π-REGULAR RINGS WITH NO INFINITE TRIVIAL SUBRING

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Dedicated to Professor Hiroyuki Tachikawa on his 60th birthday

It is shown that a ring R is a π-regular ring with no infinite trivial subring if and only if R is a subdirect sum of a strongly regular ring and a finite ring. Some other characterizations of such a ring are given. Similar result is proved for a periodic ring. As a corollary, it is shown that every δ-ring is a subdirect sum of a finite ring and a commutative ring. This was conjectured by Putcha and Yaqub.

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In [7] Laffey proved that if R is an infinite periodic ring which is orthogonally finite and with all trivial subrings finite, then R has a commutative ideal I such that R/I is finite. Recently Armendariz [1] showed that such a ring is described as follows: $R=F\oplus S$, where F is a finite ring and S is a finite direct sum of periodic fields. He also gave an analogue result for strongly π-regular rings. More recently, in [6] we removed the assumption of orthogonal finiteness, and proved that a π-regular ring (resp. periodic ring) R with no infinite trivial subring has a strongly regular ideal (resp. commutative ideal) M such that R/M is finite. In this paper, we will show that a ring R is a π-regular ring (resp. periodic ring) with no infinite trivial subring if and only if R is a subdirect sum of a strongly regular ring (resp. J-ring) and a finite ring. We will also provide several characterizations of π-regular rings (resp. periodic rings) with no infinite trivial subring. As a result, we will solve the open problem of Putcha and Yaqub in [8] affirmatively.

The Jacobson radical of a ring R will be denoted by $J(R)$, and the prime radical by $P(R)$. The ring of integers will be denoted by $\mathbf{Z}$.

A ring R is called π-regular if for each x in R there exists a positive integer n (depending on x) and an element y of R such that $x^n=x^nyx^n$. A π-regular ring R for which the n in the above can be taken to be 1 for all x is called von Neumann regular. A strongly regular ring is a von Neumann regular ring with no nonzero nilpotent element. By a trivial subring of a ring R we mean a subring S of R with $S^2=0$.

Theorem 1. *The following conditions are equivalent for a ring R.*

(1) R is a π-regular ring with no infinite trivial subring.

(2) *R has a strongly regular ideal M such that R/M is a finite ring.*

(3) *R is a subdirect sum of a strongly regular ring and a finite ring.*

(4) *R has a finite ideal I such that R/I is strongly regular.*

(5) *P(R) is finite and R/P(R) is the direct sum of a strongly regular ring and a finite ring.*

(6) *R is a π-regular ring with only finitely many nilpotent elements.*

(7) *There exists a finite subset S of R such that for each $x \in R$, there is $a \in R$ such that $ax = xa$ and $x - x^2 a \in S$.*

Proof. (1)⇒(2). This was proved in [6].

(2)⇒(3). Take an ideal I of R which is maximal with respect to the property that $I \cap M = 0$. We claim that R/I has no nonzero nilpotent element. Let a be an element of R such that $a^2 \in I$ and let $m \in M$. Then we see $(ama)^2 \in M \cap I = 0$. Since $ama \in M$ and since M has no nonzero nilpotent element, we get $ama = 0$. Then we obtain $ma = 0$, because $ma \in M$ and $(ma)^2 = 0$. Thus we get $Ma = 0$. Similarly we obtain $aM = 0$. Now let $A = M \cap (Ra + aR + \mathbf{Z}a + I)$. Then we see that $A^2 \subset M(Ra + aR + \mathbf{Z}a + I) \subset M \cap I = 0$. Since $A \subset M$, we get $A = 0$. By the choice of I, we conclude that $a \in I$. This proves that R/I has no nonzero nilpotent element. Let b be an arbitrary element of R. Since R/M is a finite ring, it is π-regular, and so there exists a positive integer n and $x \in R$ such that $b^n - b^n x b^n \in M$. Since M is strongly regular, there exists $y \in M$ such that $b^n - b^n x b^n = (b^n - b^n x b^n) y (b^n - b^n x b^n)$. Then it holds that $b^n = b^n z b^n$, where $z = x + y - x b^n y - y b^n x + x b^n y b^n x$. Therefore R, and hence R/I, is π-regular. As shown above, R/I has no nonzero nilpotent element, and hence R/I is strongly regular. Thus R is a subdirect sum of the strongly regular ring R/I and the finite ring R/M.

(3)⇒(4). By hypothesis, there exist ideals I and M with $I \cap M = 0$ such that R/I is strongly regular and R/M is finite. Then I can be embedded in the finite ring R/M, and so I is finite.

(4)⇒(5). Since every element of $P(R)$ is nilpotent and since R/I has no nonzero nilpotent element, we get $P(R) \subset I$. Hence $P(R)$ is finite. Let $R' = R/P(R)$, and $I' = I/P(R)$. We claim that I' is a semisimple ring. Suppose, to the contrary, that $J(I') \neq 0$. Since I' is a finite ring, $J(I')$ is nilpotent. Hence $I'J(I')$ is a nilpotent left ideal of R'. Since $P(R') = 0$, we get $I'J(I') = 0$. Thus $(R'J(I'))^2 = R'J(I')R'J(I') \subset I'J(I') = 0$, and hence $J(I') = 0$. Thus the finite semisimple ring I' has an identity e (see e.g. [5, Corollary 2, p. 30]). Since I' is an ideal of R', we can easily see that e is a central idempotent of R'. Hence we have the decomposition $R' = I' \oplus S$, where $S = \{r - re \mid r \in R'\}$. Since S is isomorphic to R/I, S is strongly regular.

(5)⇒(6). By hypothesis R has a finite ideal I $(\supset P(R))$ such that R/I is strongly regular. Since R/I has no nilpotent element, all nilpotent elements of R are contained in the finite ideal I. Hence those forms a finite set. To prove the π-regularity of R, take an arbitrary prime ideal of R. Then, by hypothesis, R/P is either a division ring or a finite simple ring. Obviously R is of bounded index. Therefore, by Corollary 2.2 of [4], we conclude that R is π-regular.

(6)⇒(7). Since R is of bounded index, R is strongly π-regular by Theorem 5 of [2]. Let S be the set of all nilpotent elements of R and let x be any element of R. By Theorem 3 and Lemma 4 of [2] there exists $a \in R$ such that $xa = ax$ and $x - x^2 a \in S$.

(7)⇒(1). It is easy to see that all trivial subrings of R are contained in S. Hence R has no infinite trivial subring. To prove the π-regularity of R, take an arbitrary element x of R. By hypothesis there exists $a_1 \in R$ such that $a_1 x = x a_1$ and $x - x^2 a_1 \in S$. Let us set $b_1 = x - x^2 a_1$. Inductively, if $b_{n-1} \in S$ is defined, then we obtain $a_n \in R$ such that $a_n b_{n-1}^2 = b_{n-1}^2 a_n$ and $b_{n-1}^2 - b_{n-1}^4 a_n \in S$, and then we define $b_n = b_{n-1}^2 - b_{n-1}^4 a_n$. Since S is finite, there are positive integers $k < m$ such that $b_k = b_m$. Then we get y, $z \in R$ such that $x^{2^{k-1}} = x^{2^{k-1}+1} y = z x^{2^{k-1}+1}$. If we set $n = s^{k-1}$, $a = y^n$, and $b = z^n$, then we have $x^n = x^{2n} a = b x^{2n}$. Hence we obtain $x^{2n} = x^{2n} a b x^{2n}$. This proves that R is π-regular.

Since the condition (7) of Theorem 1 is inherited by every homomorphic image of R, we have the following:

Corollary 1. *If R is a π-regular ring with no infinite trivial subring, then so is every homomorphic image of R.*

The following example shows that, in the condition (3) of Theorem 1, "subdirect sum" can not be replaced by "direct sum",

Example. Let K be a finite field, I an infinite set, and let K^I denote the direct product of the I-copies of K. Take a maximal ideal M of K^I containing the direct sum $K^{(I)}$ of the I-copies of K. Let n be the order of K. Then the field K^I/M satisfies the identity $X^n - X = 0$. Hence we conclude that K^I/M is isomorphic to the field K. Let R be the additive abelian group $K^I \oplus (K^I/M)$ and define multiplication in R by $(a,r)(b,s) = (ab, as + rb)$. Then R is a ring satisfying the equivalent conditions of Theorem 1. In fact, $(K^I/M)^* = \{(0,r) \mid r \in K^I/M\}$ is a finite ideal of R, and $R/(K^I/M)^* (\simeq K^I)$ is strongly regular, and hence R satisfies the condition (4) of Theorem 1. It is easy to see that R cannot be represented as the direct sum of a strongly regular ring and a finite ring.

A ring R is said to be periodic if for each $x \in R$, there exist distinct positive integers m, n for which $x^m = x^n$. A ring R is called a J-ring if for each $x \in R$, there exists an integer $n > 1$ such that $x = x^n$. We can easily see that a ring R is a J-ring if and only if R is a periodic ring with no nonzero nilpotent element. Finally, following [8], a ring R is called a δ-ring if R has a finite subset S such that for each $x \in R$, there exists a polynomial $f(X) \in \mathbf{Z}[X]$ such that $x - x^2 f(x) \in S$.

Theorem 2. *The following conditions are equivalent for a ring R.*

(1) R is a periodic ring with no infinite trivial subring.

(2) R has a J-ideal M such that R/M is a finite ring.

(3) R is a subdirect sum of a J-ring and a finite ring.

(4) R has a finite ideal I such that R/I is a J-ring.

(5) $P(R)$ is finite and $R/P(R)$ is the direct sum of a strongly regular ring and a finite semisimple ring.

(6) R is a periodic ring with only finitely many nilpotent elements.

(7) R is a δ-ring.

In preparation for the proof of Theorem 2, we state two lemmas the first of which is Proposition 2 of [**3**].

Lemma 1 ([**3**]). *Let R be a ring. Suppose that for each $x \in R$, there exists a positive integer n and a polynomial $p(X) \in \mathbf{Z}[X]$ for which $x^n = x^{n+1}p(x)$. Then R is periodic.*

Lemma 2. *Let I be an ideal of a ring R. Then R is periodic if and only if both I and R/I are periodic.*

Proof. It suffices to prove the "if" part. Let x be an arbitrary element of R. Since R/I is periodic, there exist positive integers $n<m$ such that $x^n - x^m \in I$. Since I is periodic, there exist positive integers $k<h$ such that $(x^n - x^m)^k = (x^n - x^m)^h$. This equation can be rewritten in the form $x^{nk} = x^{nk+1}p(x)$ for some $p(X) \in \mathbf{Z}[X]$. Hence, by Lemma 1, R is periodic.

Now we can prove Theorem 2. The proof proceeds parallel to the proof of Theorem 1.

Proof of Theorem 2. (1)$\Rightarrow$(2). This was shown in the proof of Corollary 1 of [**6**].

(2)$\Rightarrow$(3). As shown in the proof of (2)$\Rightarrow$(3) of Theorem 1, there exists an ideal I of R with $I \cap M = 0$ such that R/I has no nonzero nilpotent element. By virtue of Lemma 2, we can easily see that R/I is a J-ring. Therefore R is a subdirect sum of the J-ring R/I and the finite ring R/M.

(3)$\Rightarrow$(4) and (4)$\Rightarrow$(5) can be shown by the similar way as in the corresponding parts of the proof of Theorem 1.

(5)$\Rightarrow$(6). Using Lemma 2, we can prove that R is periodic. By the same argument as in the proof of (5)$\Rightarrow$(6) of Theorem 1, we also see that R has only finitely many nilpotent elements.

(6)$\Rightarrow$(7). Let N be the set of all nilpotent elements of R and let x be an arbitrary element of R. Since R is periodic, there exist positive integers m and n such that $x^m = x^{m+n}$. Then we have that $x - x^{n+1} \in N$. By hypothesis, N is finite, and hence R is a δ-ring.

(7)$\Rightarrow$(1). It was proved in Lemma 2 of [**8**] that R satisfies the hypothesis of Lemma 1. Hence R is periodic. Since every subring of a δ-ring is a δ-ring, R has no infinite trivial subring by Theorem 1 of [**8**].

The following corollary is the answer to the question of Putcha and Yaqub [**8**, p. 20].

Corollary 2. *Every δ-ring is a subdirect sum of a finite ring and a commutative ring.*

Proof. A well known theorem of Jacobson says that every J-ring is commutative (see [5, Theorem 3.1.2]). Hence our assertion follows from the equivalence of the conditions (3) and (7) of Theorem 2.

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