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COMPOSITION WITH A NONHOMOGENEOUS BOUNDED HOLOMORPHIC FUNCTION ON THE BALL

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1. Introduction. For an integer $n > 1$, the letters U and B_n denote the open unit disc in $\mathbf{C}$ and the open euclidean unit ball in $\mathbf{C}^n$, respectively. It is known that the homogeneous polynomials

$$\pi_A(z) = n^{\frac{n}{2}} z_1 z_2 \cdots z_n, \qquad [\mathbf{1}]$$

$$\pi_R(z) = z_1^2 + z_2^2 + \cdots + z_n^2, \qquad [\mathbf{8}]$$

$$\pi_{AR}(z) = b_\alpha z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_p^{\alpha_p}, \quad 1 \leqq p \leqq n, \qquad [\mathbf{2}]$$

where b_α is chosen so that $\pi_{AR}(B_n) = U$, have the following pull-back property:

If $g \in \mathcal{B}(U)$, the Bloch space, then $g \circ \pi \in BMOA(B_n)$, the space of holomorphic functions on B_n of bounded mean oscillation, for $\pi = \pi_A$, π_R and π_{AR}.

In this paper we show that the nonhomogeneous map

$$\pi_{n,m}(z) = \frac{z_{m+1}^2 + z_{m+2}^2 + \cdots + z_n^2}{1 - (z_1^2 + z_2^2 + \cdots + z_m^2)}, \quad 1 \leqq m \leqq n-1,$$

pulls the Bloch space $\mathcal{B}(U)$ back to the $\bigcap_{0<p<\infty} H^p(B_n)$. It should be noted that unlike π_A, π_R and π_{AR}, the map $\pi_{n,m}$ has a large set of singularities on ∂B_n which is

$$V = \{z \in \partial B_n : z_1^2 + z_2^2 + \cdots + z_m^2 = 1\},$$

an $m-1$ dimensional sphere $\mathbf{S}^{m-1}$ imbedded in ∂B_n, and

$$\pi_{n,m}^{-1}(\partial U) = W \backslash V,$$

where

$$W = \{z \in \partial B_n : |1 - (z_1^2 + z_2^2 + \cdots + z_m^2)| = |z_{m+1}^2 + z_{m+2}^2 + \cdots + z_n^2|\}$$

(which is easily verified to be homeomorphic to $\mathbf{S}^{n-1}\partial U$), is also an n-dimensional submanifold of ∂B_n as $\pi_A^{-1}(\partial U)$ and $\pi_R^{-1}(\partial U)$. The authors do not know whether $\pi_{n,m}$ pulls $\mathcal{B}(U)$ back to $BMOA(B_n)$ or not. The second author

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2. Definitions and preliminaries. Let σ_n be the Lebesgue measure on ∂B_n normalized so that $\sigma_n(\partial B_n) = 1$ and ν_n the Lebesgue measure on B_n nomalized so that $\nu_n(B_n) = 1$. The Hardy space $H^p(B_n)$ is the class of holomorphic functions f on B_n for which

$$\|f\|_p^p = \sup_{0<r<1} \int_{\partial B_n} |f(r\xi)|^p d\sigma_n(\xi) < \infty.$$

For $f \in H^2(B_n)$ we say that $f \in BMOA(B_n)$ if there exists a constant C such that for all $F \in H^2(B_n)$ we have

$$\left| \int_{\partial B_n} F\bar{f} d\sigma_n \right| \leqq C\|F\|_1.$$

$BMOA(B_n)$ serves as the dual of $H^1(B_n)$. For more intrinsic descriptions on $BMOA(B_n)$ see [**4**].

Next we describe some function spaces on U. If μ is a positive measure on U, then $A^p(d\mu)$ will denote the space of holomorphic functions in $L^p(d\mu)$, $0 < p < \infty$. When

$$d\mu(r,\theta) = (1-r)^\alpha dr d\theta, \quad \alpha > -1,$$

we use the notation $A^p(d\mu) = A^p_\alpha(U)$. Finally we say that g is a Bloch function, $g \in \mathcal{B}(U)$, if

$$\|g\|_{\mathcal{B}} = \sup_{|z|<1}(1-|z|)|g'(z)| < \infty.$$

Any unexplained notations are as in [**7**].

For the integrations with respect to $d\nu_n$ and $d\sigma_n$ we have the following formulas.

$$(2.1) \qquad \int_{\partial B_n} f(z) d\nu_n(z) = 2n \int_0^1 r^{2n-1} dr \int_{\partial B_n} f(r\zeta) d\sigma_n(\zeta)$$

for $f \in L^1(d\nu_n)$. See [**7**].

$$(2.2) \qquad \int_{\partial B_n} f(\zeta) d\sigma_n(\zeta) = \frac{1}{mB(m, n-m)} \times \int_{B_m} \int_{\partial B_{n-m}} f(\xi, (1-|\xi|^2)^{\frac{1}{2}}\eta) d\sigma_{n-m}(\eta)(1-|\xi|^2)^{n-m-1} d\nu_m(\xi)$$

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for $f \in L^1(d\sigma_n)$.

For $m = 1$, (2.2) is proved in [**6**]. This general form can be proved exactly the same way.

If $\pi(z) = z_1^2 + z_2^2 + \cdots + z_m^2$ with $m \geqq 2$ the following formula is proved in [**8**].

$$(2.3)\qquad \int_{\partial B_m} f \circ \pi d\sigma_m = \frac{m-1}{2\pi} \int_0^1 \int_0^{2\pi} f(re^{i\theta}) d\theta (1-r^2)^{\frac{m-3}{2}} r dr d\theta$$

for continuous functions f on U.

Finally we have the following orthogonality relations for the monomials

$$(2.4)\qquad \int_{\partial B_n} \zeta^\alpha \bar{\zeta}^\beta d\sigma_n(\zeta) = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!} & \text{if } \alpha = \beta \end{cases}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\alpha_1! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. See [**7**].

3. Pull back to $\cap_{0<p<\infty} H^p(B_n)$. The results of this paper are based on the following theorem.

THEOREM 1. *For each integer $n > 1$ and $1 \leqq m \leqq n-1$, there exists a continuous function $w_{n,m} : (0,1) \longrightarrow [0, \infty)$ such that*

$$\text{(i)}\qquad \int_0^1 w_{n,m}(r) dr = \frac{1}{2\pi} < \infty,$$

$$\text{(ii)}\qquad 0 < \lim_{r \to 1} w_{n,m}(r)(1-r)^{\frac{(3-n)}{2}} < \infty,$$

(iii) *if g is a continuous complex-valued function defined on $\bar{U}$ then*

$$\int_{\partial B_n} g \circ \pi_{n,m} d\sigma_n = \int_0^1 \int_0^{2\pi} g(re^{i\theta}) d\theta w_{n,m}(r) dr.$$

Proof. (i) is a consequence of (iii) by taking $g \equiv 1$. We divide the proof into four cases (a) $n = 2$; so $m = 1$, (b) $n > 2$ and $m = 1$, (c) $n > 2$ and $m = n-1$, and (d) $n > 3$ and $1 < m < n-1$.

For the case (a), let

$$\pi(z) = \pi_{2,1}(z) = z_2^2/(1-z_1^2).$$

We have to show the existence of $w = w_{2,1}$ with the properties (ii) and (iii):

$$(3.1)\qquad \int_{\partial B_2} g\left(\frac{\zeta_2^2}{1-\zeta_1^2}\right) d\sigma_2(\zeta) = \int_0^1 \int_0^{2\pi} g(re^{i\theta}) d\theta w(r) dr.$$

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The left hand side of (3.1) can be written as

$$\int_{\partial B_2} g \circ \pi d\sigma_2 = \frac{1}{2\pi^2}\int_U \int_0^{2\pi} g\left(\frac{(1-\rho^2)e^{2i\theta}}{1-\rho^2 e^{2i\varphi}}\right) d\theta \rho d\rho d\varphi. \tag{3.2}$$

If we write

$$G(R) = \int_0^{2\pi} g(Re^{i\theta})d\theta, \quad 0 \leqq R \leqq 1, \tag{3.3}$$

the right hand side of (3.2) becomes

$$\begin{aligned} &\frac{1}{2\pi^2}\int_0^1 \int_0^{2\pi} G\left(\frac{1-\rho^2}{|1-\rho^2 e^{2i\varphi}|}\right)\rho d\rho d\varphi \\ &= \frac{1}{\pi^2}\int_0^1 \int_0^{\frac{\pi}{2}} G\left(\frac{1-\rho}{\sqrt{(1-\rho)^2+4\pi\sin^2\varphi}}\right) d\rho d\varphi \end{aligned} \tag{3.4}$$

by the symmetry of $\sin^2\varphi$ and a change of variable in the part of $d\rho$-integral. By the successive changes of variables

$$\sin\varphi = \frac{1-\rho}{2\sqrt{\rho}} \quad \text{and} \quad \frac{1}{\sqrt{1+u^2}} = r$$

and by the interchange of order of integration, the right hand side of (3.4) becomes successively

$$\begin{aligned} &\frac{1}{\pi^2}\int_0^1 (1-\rho)d\rho \int_0^{\frac{2\sqrt{\rho}}{1-\rho}} G\left(\frac{1}{\sqrt{1+u^2}}\right)\frac{du}{\sqrt{4\rho-(1-\rho)^2u^2}} \\ &= \frac{1}{\pi^2}\int_0^1 (1-\rho)d\rho \int_{\frac{1-\rho}{1+\rho}}^1 G(r)\frac{1}{r\sqrt{1-r^2}}\,\frac{dr}{\sqrt{4\rho r^2-(1-\rho)^2(1-r^2)}} \\ &= \int_0^1 G(r)\left[\frac{1}{\pi^2}\,\frac{1}{r\sqrt{1-r^2}}\int_{\frac{1-r}{1+r}}^2 \frac{(1-\rho)d\rho}{\sqrt{4\rho r^2-(1-\rho)^2(1-r^2)}}\right]dr. \end{aligned} \tag{3.5}$$

If $w(r)$ denotes the expression in the bracket in (3.5), (iii) is satisfied. If we make a change of variable $1-\rho = 2rt/(1+r)$, we have

$$w(r) = \frac{2}{\pi^2(1+r)^2\sqrt{1-r}}\int_0^1 \frac{tdt}{\sqrt{1-t^2+r(1-t)^2}}. \tag{3.6}$$

By the dominated convergence theorem, we have

$$\lim_{r\to 1} w(r)(1-r)^{\frac{1}{2}} = \frac{\sqrt{2}}{3\pi^2}.$$

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Therefore (ii) is satisfied for $w(r)$. We note that the integral in (3.6) can be evaluated by an easy calculation. In fact, $w(r)$ can be expressed as

$$w(r) = \frac{2}{\pi^2(1+r)^2\sqrt{1-r}} \cdot \frac{1}{1-r}\left\{\sqrt{1+r} - \frac{r}{\sqrt{1-r}}\left(\frac{\pi}{2} - \sin^{-1} r\right)\right\}.$$

The proof for the case (d) is much more complicated. We write

$$\zeta = (\zeta_1, \cdots, \zeta_m; \zeta_{m+1}, \cdots, \zeta_n) = (\xi; \sqrt{1-|\xi|^2}\eta).$$

By (2.2), we have

$$\begin{aligned}(3.7)\qquad &\int_{\partial B_n} g \circ \pi_{n,m}(\zeta) d\sigma_n(\zeta) \\ &= \frac{1}{mB(m,, n-m)} \int_{B_m} \int_{\partial B_{n-m}} g\left(\frac{(1-|\xi|^2)(\eta_1^2 + \cdots + \eta_{n-m}^2)}{1-(\xi_1^2+\cdots+\xi_m^2)}\right) d\sigma_{n-m}(\eta) \\ &\times (1-|\xi|^2)^{n-m-1} d\nu_m(\xi).\end{aligned}$$

If we apply (2.3) to the inner integral, we have

$$\begin{aligned}(3.8)\qquad &\int_{\partial B_{n-m}} g\left(\frac{(1-|\xi|^2)(\eta_1^2+\cdots+\eta_{n-m}^2)}{1-(\xi_1^2+\cdots+\xi_m^2)}\right) d\sigma_{n-m}(\eta) \\ &= \frac{n-m-1}{2\pi}\int_0^1\int_0^{2\pi} g\left(\frac{(1-|\xi|^2)se^{i\theta}}{1-(\xi_1^2+\cdots+\xi_m^2)}\right) d\theta (1-s^2)^{\frac{n-m-3}{2}} s\,ds \\ &= \frac{n-m-1}{2\pi}\int_0^1 G\left(\frac{(1-|\xi|^2)s}{|1-(\xi_1^2+\cdots+\xi_m^2)|}\right)(1-s^2)^{\frac{n-m-3}{2}} s\,ds,\end{aligned}$$

where

$$G(R) = \int_0^{2\pi} g(Re^{i\theta})d\theta.$$

Interchanging the order of integration in (3.7) the $d\nu_m$-integral on G can be written by (2.1) and (2.3) as

$$\begin{aligned}(3.9)\qquad &\int_{B_m} G\left(\frac{(1-|\xi|^2)s}{|1-(\xi_1^2+\cdots+\xi_m^2)|}\right)(1-|\xi|^2)^{n-m-1} d\nu_m(\xi) \\ &= 2m\int_0^1 t^{2m-1}(1-t^2)^{n-m-1}dt \\ &\times \int_{\partial B_m} G\left(\frac{(1-t^2)s}{|1-t^2(\tau_1^2+\cdots+\tau_m^2)|}\right) d\sigma_m(\tau) \\ &= m\int_0^1 t^{m-1}(1-t)^{n-m-1}dt \\ &\times \frac{m-1}{2\pi}\int_0^1\int_0^{2\pi} G\left(\frac{(1-t)s}{|1-t\rho e^{i\varphi}|}\right) d\varphi (1-\rho^2)^{\frac{m-3}{2}} \rho\, d\rho.\end{aligned}$$

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If we note

$$|1 - t\rho e^{i\varphi}| = \sqrt{(1-t\rho)^2 + 4t\rho \sin^2\left(\frac{\varphi}{2}\right)}$$

and set

$$\sin\frac{\varphi}{2} = \frac{1-t\rho}{2\sqrt{t\rho}}u,$$

the $d\varphi$-integral on the right hand side of (3.9) becomes by the symmetry of $\sin^2(\frac{\varphi}{2})$

$$(3.10)\quad \int_0^{2\pi} G\left(\frac{(1-t)s}{|1-t\rho e^{i\varphi}|}\right) d\varphi$$
$$= 4\int_0^{\frac{2\sqrt{t\rho}}{1-t\rho}} G\left(\frac{1-t}{1-t\rho}\,\frac{s}{\sqrt{1+u^2}}\right) \frac{(1-t\rho)du}{\sqrt{4t\rho-(1-t\rho)^2u^2}}.$$

If we combine (3.7), (3.8), (3.9) and (3.10) we have

$$(3.11)\quad \int_{\partial B_n} g\circ\pi_{n,m}(\zeta)d\sigma_n(\zeta)$$
$$= C(n,m)\int_0^1 (1-s^2)^{\frac{n-m-3}{2}} sds \int_0^1 t^{m-1}(1-t)^{n-m-1}dt$$
$$\times \int_0^1 (1-\rho^2)^{\frac{m-3}{2}}\rho d\rho$$
$$\times \int_0^{\frac{2\sqrt{t\rho}}{1-t\rho}} G\left(\frac{1-t}{1-t\rho}\,\frac{s}{\sqrt{1+u^2}}\right) \frac{(1-t\rho)du}{\sqrt{4t\rho-(1-t\rho)^2u^2}},$$

where

$$C(n,m) = \frac{(m-1)(n-m-1)}{\pi^2 B(m,n-m)}.$$

We have to make judicious changes of variables and interchanges of the order of integration successively. For example we make a series of changes of variables:

$$t\rho = v \quad (t \text{ fixed}),$$
$$1-t = (1-v)w \quad (v \text{ fixed}),$$
$$1/\sqrt{1+u^2} = R,$$
$$Rw = t \quad (w \text{ fixed}),$$
$$ts = r \quad (s \text{ fixed}),$$
$$1-w = \left(1-\frac{r}{s}\right)u \quad (r,s \text{ fixed}).$$

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We then have

$$\int_{\partial B_n} g \circ \pi_{n,m} d\sigma_n = \int_0^1 G(r)w(r)dr \tag{3.12}$$

where

$$\begin{aligned}
w(r) &\equiv w_{n,m}(r) \\
&= C(n,m)\frac{1}{r}\int_r^1 \frac{(1-s^2)^{\frac{n-m-3}{2}}(s-r)^{\frac{m-2}{2}}}{s^{n-4}} ds \\
&\times \int_0^1 \frac{u^{\frac{m-3}{2}}}{\sqrt{1-u}} \frac{(s-su+ru)^{n-m+1}du}{\sqrt{s-su+r+ru}} \\
&\times \int_{\frac{(s-r)(1-u)}{(s-su+r+ru)}}^1 \frac{v(1-v)^{n-\frac{m+1}{2}}\{2sv+(1-v)(s-r)u\}^{\frac{m-3}{2}}dv}{\sqrt{4r^2v-(1-v)^2(s-r)(1-u)(s-su+r+ru)}}.
\end{aligned}$$

If we make further changes of variables

$$1-s = (1-r)t \quad (r \text{ fixed})$$

and

$$1-v = \left(1 - \frac{(1-r)(1-t)(1-u)}{(1-t+rt)(1-u)+r+ru}\right)s, \quad (s \text{ new})$$

$w(r)$ then has the form

$$\begin{aligned}
w(r) &= C(n,m)2^{n-2}(1-r)^{\frac{n-3}{2}} r^{n-\frac{m+3}{2}} \\
&\times \int_0^1 t^{\frac{n-m-3}{2}}(1-t)^{\frac{m-2}{2}} \frac{(2-t+rt)^{\frac{n-m-3}{2}}dt}{(1-t+rt)^{n-4}} \\
&\times \int_0^1 \frac{u^{\frac{m-3}{2}}}{\sqrt{1-u}} \frac{[(1-u)(1-t)+r\{u+t(1-u)\}]^{n-m+1}du}{[(1-u)(1-t)+r\{1+u+t(1-u)\}]^n} \\
&\times \int_0^1 \frac{s^{n-\frac{m+1}{2}}}{\sqrt{1-s}} \frac{N ds}{\sqrt{(1-u)(1-t)(1+s)+r[(1-s)+\{u+t(1-u)\}(1+s)]}}
\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}
N &= \{(1-u)(1-t)+r(1+u+t(1-u)-2s)\} \\
&\times [(1-t+rt)\{(1-u)(1-t)+r(1+u+t(1-u)-2s)\} \\
&+ 2rs(1-r)(1-t)u]^{\frac{m-3}{2}}.
\end{aligned}$$

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We can easily check that as $r \longrightarrow 1$ the integrand is dominated by

$$\text{constant} \cdot t^{\frac{n-m-3}{2}} \cdot \frac{u^{\frac{m-3}{2}}}{\sqrt{1-u}} \cdot \frac{s^{n-\frac{m+1}{2}}}{\sqrt{1-s}}$$

which is integrable with respect to $dt \cdot du \cdot ds$. We now apply the dominated convergence theorem to have

$$\begin{aligned}
(3.14)\quad & \lim_{r \longrightarrow 1} w(r)(1-r)^{\frac{3-n}{2}} \\
&= C(n,m)2^{\frac{n-9}{2}} \int_0^1 t^{\frac{n-m-3}{2}}(1-t)^{\frac{m-2}{2}} dt \\
&\times \int_0^1 (1-u)^{-\frac{1}{2}} u^{\frac{m-3}{2}} du \int_0^1 (1-s)^{\frac{m-2}{2}} s^{n-\frac{m+1}{2}} ds \\
&= 2^{\frac{n-9}{2}} C(n,m) B\left(\frac{n-m-1}{2}, \frac{m}{2}\right) B\left(\frac{m-1}{2} \cdot \frac{1}{2}\right) B\left(n - \frac{m-1}{2}, \frac{m}{2}\right).
\end{aligned}$$

(3.12), (3.13) and (3.14) show that $w(r)$ in (3.13) satisfies (ii) and (iii).

The computations for the cases (b) and (c) are very much similiar to that for the case (d) but a little simpler. We omit the details. One form for $w_{n,1}(r)$ is given by

$$\begin{aligned}
w_{n,1}(r) &= \frac{2^{n-1}(n-1)(n-2)}{\pi^2}(1-r)^{\frac{n-3}{2}} r^{n-2} \\
&\times \int_0^1 \frac{u^{\frac{n-4}{2}}}{\sqrt{1-u}} \frac{(2-u+ru)^{\frac{n-4}{2}}(1-u+ru)^3 du}{\{1-u+r(1+u)\}^n} \\
&\times \int_0^1 \frac{s^{n-1}}{\sqrt{1-s}} \frac{ds}{\sqrt{(1-u)(1+s)+r\{(1-s)+u(1+s)\}}}.
\end{aligned}$$

If we apply the dominated convergence theorem we have

$$\begin{aligned}
& \lim_{r \longrightarrow 1} w_{n,1}(r)(1-r)^{\frac{3-n}{2}} \\
&= \frac{2^{\frac{n-7}{2}}(n-1)(n-2)}{\pi^2} B\left(\frac{n-2}{2}, \frac{1}{2}\right) B\left(n, \frac{1}{2}\right).
\end{aligned}$$

One form of $w_{n,n-1}(r)$ is given by

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$$
\begin{aligned}
w_{n,n-1}(r) = {} & \frac{2^{n-2}(n-1)(n-2)}{\pi^2}(1-r)^{\frac{n-3}{2}} r^{\frac{n-2}{2}} \\
& \times \int_0^1 \frac{u^{\frac{n-4}{2}}}{\sqrt{1-u}} \frac{(1-u+ru)^2 du}{\{1-u+r(1+u)\}^n} \\
& \times \int_0^1 \frac{s^{\frac{n}{2}}}{\sqrt{1-s}} \{(1-u)(1-r)+2r(1-s)\} \\
& \times \frac{\{(1-u)(1-r)+2r(1-s)+rs(1-r)u\}^{\frac{n-4}{2}} ds}{\sqrt{(1-u)(1+s)+r\{(1-s)+u(1+s)\}}},
\end{aligned}
$$

for which we have, by the use of dominated convergence theorem again

$$
\begin{aligned}
& \lim_{r \to 1} w_{n,n-1}(r)(1-r)^{\frac{3-n}{2}} \\
& = \frac{2^{\frac{n-7}{2}}(n-1)(n-2)}{\pi^2} B\left(\frac{n-2}{2}, \frac{1}{2}\right) B\left(\frac{n+2}{2}, \frac{n-1}{2}\right).
\end{aligned}
$$

This completes the proof.

If g is continuous on $\bar{U}$ and we apply Theorem 1 to $|g|^p$, we have

$$\int_{\partial B_n} |T_{n,m} g|^p d\sigma = \int_U |g|^p d\mu_{n,m}$$

where

$$T_{n,m} g = g \circ \pi_{n,m} \quad \text{and} \quad d\mu_{n,m}(r, \theta) = w_{n,m}(r) dr d\theta.$$

It is now clear that $T_{n,m}$ extends uniquely to be an isometry of $L^p(d\mu_{n,m})$ into $L^p(d\sigma_n)$. If g is holomorphic, then it is obvious from Theorem 1 that $g \in L^p(d\mu_{n,m})$ if and only if

$$g \in A^p_{\frac{n-3}{2}}(U).$$

Also if g is holomorphic, then so is $T_{n,m} g$. Hence we have the following

COROLLARY 2. *$T_{n,m}$ is a bounded, linear, one-to-one map of $A^p_{\frac{n-3}{2}}(U)$ into $H^p(B_n)$.*

The following lemma can be obtained by an easy computation, but we give a proof for the completeness.

LEMMA 3. *If $g \in B(U)$, then $g \in A^p_\alpha(U)$ for every $\alpha > -1$ and $0 < p < \infty$.*

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Proof. Without loss of generality we may assume $g(0) = 0$. We have the following well known property:

$$|g(re^{i\theta})| \leqq \|g\|_{\mathcal{B}} \log \frac{1}{1-r} \quad (0 \leqq r < 1). \quad [\mathbf{5}]$$

Hence we have

$$\int_0^{2\pi} \int_0^1 |g(re^{i\theta})|^p (1-r)^\alpha dr d\theta$$
$$\leqq 2\pi \|g\|_{\mathcal{B}}^p \int_0^1 \left(\log \frac{1}{1-r}\right)^p (1-r)^\alpha dr.$$

We know that

$$\int_0^1 \left(\log \frac{1}{1-r}\right)^p (1-r)^\alpha dr < \infty.$$

This completes the proof.

Combining Corollary 2 and Lemma 3, we have the following theorem.

THEOREM 4. *If $g \in \mathcal{B}(U)$ then*

$$g \circ \pi_{n,m} \in \cap_{0<p<\infty} H^p(B_n).$$

Proof. Let $g \in \mathcal{B}(U)$. Then by Lemma 3, we have $g \in A^p_\alpha(U)$ for every $\alpha > -1$ and $0 < p < \infty$. In particular $g \in A^p_{\frac{n-3}{2}}(U)$ for every $p(0 < p < \infty)$. Therefore

$$g \circ \pi \in \cap_{0<p<\infty} H^p(B_n)$$

by Corollary 2. This completes the proof.

We do not know whether $\pi_{n,m}$ pulls $\mathcal{B}(U)$ back to $BMOA(B_n)$. That the methods of Ahern [**1**] and of Ahern-Rudin [**2**] do not work for this $\pi_{n,m}$ was pointed out by Professors P. R. Ahern and B. R. Choe.

Remark (a). As to the method of Ahern-Rudin $\pi_{n,m}$ does not satisfy the Cauchy Integral Equalities (CIE) of [**3**]. Suppose $\pi = \pi_{2,1}$ satisfies CIE:

$$C[\pi^{k+1}\bar{\pi}] = \gamma_k \pi^k \quad (k = 0, 1, 2, \cdots),$$

for some sequence γ_k depending on π, where $C[\pi^{k+1}\bar{\pi}]$ is the Cauchy integral of $\pi^{k+1}\bar{\pi}$ on B_2. Then for every $h \in H^2(B_2)$ we should have

$$\langle \pi^2\bar{\pi}, h\rangle = \langle C[\pi^2\bar{\pi}], h\rangle = \gamma_2\langle \pi, h\rangle.$$

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If we take

$$h = \xi_1^{2m}\xi_2^2, \quad m = 0, 1, 2, \cdots,$$

by a routine calculation using the series expansion and the integral formulas of [7] we have

$$\langle \pi^2\bar{\pi}, \xi_1^{2m}\xi_2^2\rangle = \sum_{j=0}^{\infty}(j+m+1)\frac{\Gamma(2j+2m+1)\Gamma(5)}{\Gamma(2j+2m+6)}$$

and

$$\langle \pi, \xi_1^{2m}\xi_2^2\rangle = \frac{2}{(2m+3)(2m+2)(2m+1)}.$$

Therefore

$$\gamma_2 = (2m+3)(2m+2)(2m+1)\cdot 3! \times \sum_{j=0}^{\infty}\frac{1}{(2j+2m+5)(2j+2m+4)(2j+2m+3)(2j+2m+1)}$$

for $m = 0, 1, 2, \cdots$.

For $m = 0$

$$\gamma_2 = \frac{9}{4}\sum_{j=0}^{\infty}\frac{1}{(j+\frac{5}{2})(j+2)(j+\frac{3}{2})(j+\frac{1}{2})} > \frac{9}{4}\left(\frac{1}{\frac{5}{2}\cdot 2\cdot\frac{3}{2}\cdot\frac{1}{2}} + \frac{1}{\frac{7}{2}\cdot 3\cdot\frac{5}{2}\cdot\frac{3}{2}} + \frac{1}{\frac{9}{2}\cdot 4\cdot\frac{7}{2}\cdot\frac{5}{2}}\right) = \frac{47}{70}.$$

For $m = 1$

$$\begin{aligned}\gamma_2 &= \frac{45}{2}\sum_{j=0}^{\infty}\frac{1}{(j+\frac{7}{2})(j+3)(j+\frac{5}{2})(j+\frac{3}{2})}\\ &= \frac{45}{2}\left(\sum_{j=0}^{\infty}\frac{1}{(j+\frac{5}{2})(j+2)(j+\frac{3}{2})(j+\frac{1}{2})} - \frac{1}{\frac{5}{2}\cdot 2\cdot\frac{3}{2}\cdot\frac{1}{2}}\right)\\ &= \frac{45}{2}\left(\frac{4}{9}\gamma_2 - \frac{1}{\frac{5}{2}\cdot 2\cdot\frac{3}{2}\cdot\frac{1}{2}}\right).\end{aligned}$$

Therefore

$$\gamma_2 = \frac{2}{3} < \frac{47}{70}.$$

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This is a contradiction.

(b). From the integration formula (iii) of Theorem 1, we know that $\{\pi^k\}_{k=0}^{\infty}$ is orthogonal in $H^2(B_2)$. By (a) and Proposition 5.1 of [**3**] we know that $T_{2,1}^*$ (= the adjoint of the operator $T_{2,1}\colon L^2(\mu_{2,1}) \longrightarrow L^2(d\sigma_2)$) do not map $H^2(B_2)$ to

$$A^2(d\mu_\pi)(= H(U) \cap L^2(d\mu_\pi)).$$

Therefore the method of Ahern [**1**] can not be applied for this $\pi_{2,1}$.

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