TYPICALLY-REAL FUNCTIONS

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1. Introduction. A function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

a_n real, is called typically-real of order one in the closed region $|z| \leqq R$ if it satisfies the following conditions **(6)**.

(1) $f(z)$ is regular in $|z| \leqq R$.

(2) $\mathscr{I}\{f(z)\} > 0$ if and only if $\mathscr{I}\{z\} > 0$.

The same function is called typically-real of order p, p a positive integer greater than one, if it satisfies condition (1) above and in addition the following condition **(4; 5)**:

(2′) there exists a constant ρ, $0 < \rho < R$, such that on every circle $|z| = r$, $\rho < r < R$, $\mathscr{I}\{f(z)\}$ changes sign exactly $2p$ times.

We shall denote the class of functions which are typically-real of order p in the open disc $|z| < R$ by $T_p^*(R)$ while those which are typically-real of order p in the closed disc $|z| \leqq R$ will be denoted by $T_p(R)$.

In the proofs which follow we assume that all functions belong to $T_p(1)$. The results will remain valid for the larger class $T_p^*(1)$ by noting that if $f(z) \in T_p^*(1)$ then $f(rz)/r \in T_p(1)$ for all $\rho < r < 1$ **(2)**.

The problem to be considered in this paper is that of determining a positive expresssion R_p depending upon the first p coefficients of $f(z)$ with the following property:

$$f(z) \in T_p(1) \Rightarrow f(z) \in T_1(R_p).$$

In §§ 3 and 4 we will develop a recursion relationship for R_p, $p = 1, 2, 3, \ldots$, and in § 6 we will show that our definition of R_p is sharp for the class of functions, $\bigcup_p T_p(1)$ in the sense that for $p = 2$ it is the best possible bound.

2. A Representation theorem. We shall first develop an integral representation for functions of class $T_p(1)$, $p > 1$.

THEOREM 2.1. *If there exists a function R_{p-1} $(c_2, c_3, \ldots, c_{p-1}) > 0$ with the property that for any function*

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in T_{p-1}(1)$$

we have $g(z) \in T_1(R_{p-1})$, then given an arbitrary function

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$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in T_p(1)$$

we can write

$$f(R_{p-1}\omega) = \frac{2}{\pi}\int_0^{\pi} P(\omega, \nu, \phi)\, d\alpha(\phi), \qquad |\omega| < 1 \tag{2.1}$$

where

$$P(\omega, \nu, \phi) = \frac{\omega^3 + (R_{p-1}b_1 - 2\cos\phi)\omega^2 + \omega}{(1 - 2\omega\cos\phi + \omega^2)(1 - 2R_{p-1}\omega\cos\nu + R_{p-1}^2\omega^2)},$$

$b_1 = a_2 - 2\cos\nu \neq 0$, $0 < \nu < \pi$, $c_1, \ldots, c_{p-1}$ are given by (2.2), *and* $d\alpha(\phi) > 0$ for $0 < \phi < \pi$.

Proof. Since $f(z) \in T_p(1)$ we have from **(3)** that

$$g(z) = \frac{1 - 2z\cos\nu + z^2}{b_1 z} f(z) - \frac{1}{b_1} = z + \sum_{n=2}^{\infty} c_n z^n \in T_{p-1}(1), \tag{2.2}$$

where ν is chosen subject to the following conditions:

(1) $0 < \nu < \pi$.
(2) $\mathscr{I}\{f(z)\}$ changes sign at $z = e^{i\nu}$.
(3) $b_1 = a_2 - 2\cos\nu \neq 0$.
(4) $b_1 > 0$ if $p = 2$.

It follows then from the hypotheses of Theorem 2.1 that $g(z) \in T_1(R_{p-1})$. It should also be noted that from (2.2) it follows that the R_{p-1} of Theorem 2.1 is a function of $a_2, \ldots, a_p$ and ν.

Let us now compute the coefficients c_n of $g(z)$ by integrating over the path C: $|z| = R_{p-1}$. This yields

$$\begin{aligned} c_n &= \frac{1}{2\pi i}\int_C \frac{g(z)}{z^{n+1}}\, dz \\ &= \frac{1}{2\pi R_{p-1}^n}\int_0^{2\pi} g(R_{p-1}e^{i\phi})e^{-in\phi}\, d\phi \qquad \text{where } z = \rho e^{i\phi}. \end{aligned} \tag{2.3}$$

Adding to (2.3) the expression

$$\frac{1}{2\pi R_{p-1}^n}\int_0^{2\pi} g(R_{p-1}e^{i\phi})e^{in\phi}\, d\phi = 0$$

we obtain

$$c_n = \frac{-i}{\pi R_{p-1}^n}\int_0^{2\pi} g(R_{p-1}e^{i\phi}) \sin n\phi\, d\phi. \tag{2.4}$$

If we now let $g(R_{p-1}e^{i\phi}) = u(R_{p-1}e^{i\phi}) + iv(R_{p-1}e^{i\phi})$ we have, since the c_n are real,

$$c_n = \frac{1}{\pi R_{p-1}^n}\int_0^{2\pi} v(R_{p-1}e^{i\phi}) \sin n\phi\, d\phi. \tag{2.5}$$

Since, however, $v(R_{p-1}e^{i\phi}) > 0$ for all $0 < \phi < \pi$ and since $v(R_{p-1}e^{i\phi}) = - v(R_{p-1}e^{-i\phi})$ we have

$$c_n = \frac{2}{\pi R_{p-1}^n} \int_0^\pi v(R_{p-1}e^{i\phi}) \sin n\phi \, d\phi$$

where

(2.6) $$\frac{2}{\pi R_{p-1}} \int_\pi^0 v(R_{p-1}e^{i\phi}) \sin \phi \, d\phi = 1.$$

Thus

$$\begin{aligned} g(z) &= \sum_{n=1}^{\infty} \left[\frac{2}{\pi R_{p-1}^n} \int_0^\pi v(R_{p-1}e^{i\phi}) \sin \phi \, d\phi \right] z^n. \\ &= \frac{2}{\pi} \int_0^\pi \left[v(R_{p-1}e^{i\phi}) \sin \phi \sum_{n=1}^{\infty} \frac{\sin n\phi}{\sin \phi} \left(\frac{z}{R_{p-1}} \right)^n \right] d\phi \\ &= \frac{2}{\pi} \int_0^\pi \frac{R_{p-1} z \, v(R_{p-1}e^{i\phi}) \sin \phi \, d\phi}{R_{p-1}^2 - 2R_{p-1} z \cos \phi + z^2}, \qquad |z| < R_{p-1}. \end{aligned}$$

Thus

(2.7) $$f(z) = \frac{2}{\pi} \int_0^\pi \frac{b_1 R_{p-1} z^2 \, d\alpha(\phi)}{(R_{p-1}^2 - 2R_{p-1} z \cos \phi + z^2)(1 - 2z \cos \nu + z^2)} + \frac{z}{1 - 2z \cos \nu + z^2}$$

where $d\alpha(\phi) \equiv v(R_{p-1}e^{i\phi}) \sin \phi > 0$ for all $0 < \phi < \pi$.

Using (2.6) we can rewrite (2.7) in the form

(2.8) $$f(z) = \frac{2}{\pi R_{p-1}} \int_0^\pi \frac{[z^3 + R_{p-1}(R_{p-1} b_1 - 2 \cos \phi) z^2 + R_{p-1}{}^2 z]}{(R_{p-1}^2 - 2R_{p-1} z \cos \phi + z^2)(1 - 2z \cos \nu + z^2)} \, d\alpha(\phi), \qquad |z| < R_{p-1}.$$

The transformation of variable $z = R_{p-1}\omega$ now gives (2.1).

From (2.1) it follows that

(2.9) $$\mathscr{I}\{P\} = 4r^3(1 - r^2 R_{p-1})^2 \cdot D^{-2} \cdot \sin \theta \, (\cos^2\theta + B \cos \theta + C)$$

where $\omega = re^{i\theta}$,

(2.10) $$\begin{aligned} D^2 &\equiv D^2(a_2, \ldots, a_p; r, \nu, \phi) \\ &= |(1 - 2\omega \cos \phi + \omega^2)(1 - 2R_{p-1}\omega \cos \nu + R_{p-1}{}^2\omega^2)|^2 > 0 \end{aligned}$$

for all $R_{p-1} < 1$, $|\omega| < 1$,

(2.11) $$\begin{aligned} B &\equiv B(a_2, \ldots, a_p; r, \nu, \phi) \\ &= \frac{K(1 + r^2)(1 - r^2 R_{p-1}^2) - b_1 r^2 R_{p-1}(1 - R_{p-1}^2)}{2r(1 - r^2 R_{p-1}^2)}. \end{aligned}$$

(2.12) $$C \equiv C(a_2, \ldots, a_p; r, \nu, \phi) = \frac{- R_{p-1}^2 r^6 + [K R_{p-1}^3 b_1 - K^2 R_{p-1}^2 + 2R_{p-1}^2 + 2b_1 R_{p-1}^2 \cos \nu + 1] r^4 - [K b_1 R_{p-1} - K^2 + R_{p-1}^2 + 2b_1 R_{p-1}^2 \cos \nu + 2] r^2 + 1}{4r^2(1 - r^2 R_{p-1}^2)}$$

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and

(2.13) $$K \equiv K(a_2, \ldots, a_p; r, \nu, \phi) = R_{p-1}b_1 - 2\cos\phi.$$

3. Definition of $\tilde{R}_n(a_2, \ldots, a_p)$, $p > 1$. Given any function of the class $T_p(1)$ consider the equation

(3.1) $$\frac{\partial P}{\partial \omega} \equiv P'(\omega) = 0 \qquad (a_2, \ldots, a_p \text{ fixed.})$$

Find all of the real roots $\omega_1(\nu, \phi)$, $\omega_2(\nu, \phi), \ldots, \omega_k(\nu, \phi)$, $1 \leqslant k \leqslant 6$ of (3.1). We then define

(3.2) $$\tilde{R}_p \equiv \tilde{R}_p(a_2, \ldots, a_p) = \min_{\gamma,\phi} |\omega_i(\nu, \phi)|, \qquad i = 1, \ldots, k,$$

where the minimum is taken over all ν, ϕ satisfying $0 \leqslant \phi \leqslant \pi$, $0 \leqslant \nu \leqslant \pi$. We note here that $P'(0) = 1$,

(3.3) $$P'(1) = \frac{[R_{p-1}b_1 + 2(1 - \cos\phi)](R_{p-1}^2 - 4R_{p-1}\cos\nu + 3)}{2(1 - \cos\phi)(1 - 2R_{p-1}\cos\nu + R_{p-1}^2)^2},$$

and

(3.4) $$P'(-1) = \frac{[-R_{p-1}b_1 + 2(1 - \cos\phi)](1 - R_{p-1}^2)}{2(1 + \cos\phi)(1 + 2R_{p-1}\cos\nu + R_{p-1}^2)^2}.$$

It is clear then that for $0 < R_{p-1} < 1$ if $R_1b_1 > 0$ then there exists a ϕ such that $P'(-1) < 0$ while if $R_1b_1 < 0$ then there exists a ν and ϕ such that $P'(1) < 0$. Thus if $0 < R_{p-1} < 1$

(3.5) $$\tilde{R}_p < 1.$$

4. The main theorem. From our definition of P in (2.1) and from (2.9) and (2.10) it is clear that any variation in the sign of $\mathscr{I}\{P\}$ for $0 < \theta < \pi$ must result from a variation in the sign of the factor $(\cos^2\theta + B\cos\theta + C)$. Thus, for any $0 \leqq r \leqq 1$ the functions P must be members of one of the three classes $T_i(r)$, $i = 1, 2, 3$. It should also be noted here that if for a particular value of r a function belongs to $T_1(r)$, then that function belongs to $T_1(r)$ for all smaller values of r.

Next we note that if for $0 < r_1 < r_2 < 1$ and fixed $a_2, a_3, \ldots, a_p, \nu$, and ϕ we have $P \in T_2(r_2)$, $P \in T_1(r_1)$ and $P \notin T_3(r)$ for any r satisfying $r_1 < r < r_2$ then there must exist an r satisfying $r_1 \leqslant r \leqslant r_2$ for which either $P'(r) = 0$ or $P'(-r) = 0$. This follows directly from the relation $\mathscr{I}\{P(\omega)\} = \mathscr{I}\{P(\bar{\omega})\}$, $|\omega| \leqq 1$, and the analyticity of all the P in $|\omega| < 1$.

From the definition of $\tilde{R}_p$ in § 3 and from the preceding paragraph it is clear that for fixed $a_2, a_3, \ldots, a_p$ and $r < \tilde{R}_p$, no function P can change directly from the class $T_2(r)$ to $T_1(r)$.

If, then, we are able to show that for any choice of $a_2, a_3, \ldots, a_p, \nu, \phi$ there exists no r, $0 < r < \tilde{R}_p$, for which we have both $B^2 - 4 \leqslant 0$ and $B^2 - 4C \geqslant 0$ we will have shown that for $r < \tilde{R}_p$ we cannot have $P \in T_3(r)$ and with the result of paragraph (4.3) will have established the

MAIN THEOREM. *If* $f(z) \in T_p(1)$, p *a positive integer greater than* 1, *then* $f(z) \in T_1(r)$ *for all* r *satisfying* $0 < r < R_p\,(a_2, \ldots, a_p) = R_{p-1}\,\tilde{R}_p$, *where* $R_1 \equiv 1$ *and* $\tilde{R}_p$ *is as defined in* § 2.

In the proofs which follow in this section we will assume that $f(z) \in T_p(1)$, $p > 2$, and that $R_{p-1} < 1$. The case $p = 2$ will be treated separately in § 5. In § 5 we also show that $R_2 < 1$. This, then, justifies the assumption $R_{p-1} < 1$, $p > 2$.

The proof of the Main Theorem will depend upon four lemmas. In the proof of these we will fix $a_2, \ldots, a_p, \nu$ in the expressions $B^2 - 4 = 0$ and $B^2 - 4C = 0$ and plot r against K. The lemmas will be used to prove that the general geometric configuration is that of Figure 1.

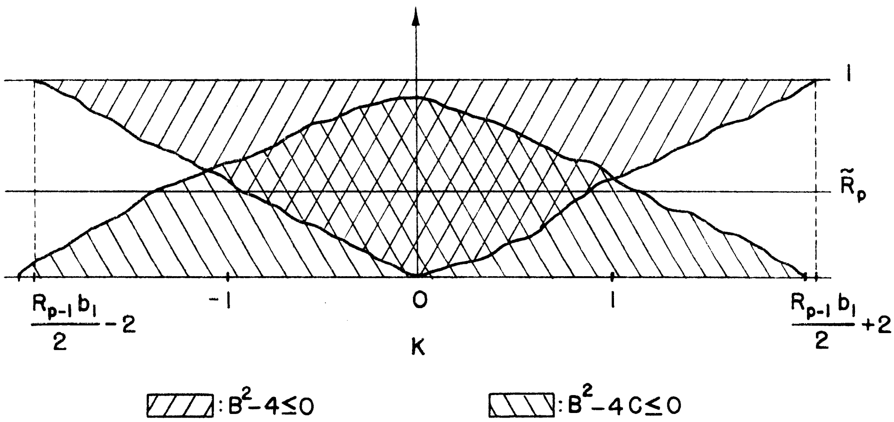

FIGURE 1

The lemmas to be proved are:

LEMMA 4.1. *The set of points* (r, K) *for which* $B^2 - 4 \leqslant 0$ *and* $0 \leqslant r \leqslant 1$, *is convex in the direction of the* K*-axis and in the direction of the* r*-axis.*

LEMMA 4.2. *The set of points* (r, K) *for which* $B^2 - 4C \leqslant 0$ *and* $0 \leqslant r \leqslant 1$ *is convex in the direction of the* K*-axis.*

LEMMA 4.3. *The set of points* (r, K) *for which* $B^2 - 4C \leqslant 0$, $B^2 - 4 \leqslant 0$, *and* $0 \leqslant r \leqslant 1$ *is convex in the direction of the* r*-axis.*

LEMMA 4.4. *If for any fixed* $a_2. a_3, \ldots, a_p$, *there exists a* K *and an* $r = \alpha$, $0 < \alpha < 1$, *for which both* $B^2 - 4C = 0$ *and* $B^2 - 4 = 0$, *then* $\alpha \geqq \tilde{R}_p$.

It should be noted that the continuity of the boundaries of the regions in Figure 1 follows directly from the continuity of the functions $B^2 - 4C$ and $B^2 - 4$ in the two variables r and K, where $0 < r \leqslant 1$, $0 < R_{p-1} < 1$.

Proof of Lemma 4.1. In the proof of this lemma we assume that $b_1 > 0$. The lemma remains valid for $b_1 < 0$ with obvious modifications in the argument.

(a) From (2.11) and (2.13) we note that if $K = 0$ then

$$B = \frac{-r\cos\phi(1 - R_{p-1}^2)}{(1 - r^2R_{p-1})^2}$$

and therefore $B^2 - 4 \leqslant 0$ for all $0 \leqslant r \leqslant 1$.

(b) For fixed $a_2, \ldots, a_p: \nu, r$ we have

$$\frac{dB}{dK} = \frac{1 + r^2}{r}.$$

From (a) and (b) the convexity in the direction of the K-axis is immediate.

(c) If $r = 1$ then $B = 2$ if

$$K = 2 + \frac{R_{p-1}b_1}{2}$$

and $B = -2$ if

$$K = -2 + \frac{R_{p-1}b_1}{2}.$$

Thus from (b) we have $B^2 - 4 < 0$ for all

$$-2 + \frac{R_{p-1}b_1}{2} < K < 2 + \frac{R_{p-1}b_1}{2}, \qquad r = 1.$$

(d) For $K > 0$,

$$\lim_{r\to 0} B = +\infty.$$

(e) For fixed $a_2, \ldots, a_p: \nu, K$

$$\frac{dB}{dr} = -\frac{[K(1 - r^2)(1 - r^2R_{p-1}^2)^2 + b_1R_{p-1}r^2(1 - R_{p-1}^2)(1 + r^2R_{p-1}^2)]}{2r^2(1 - r^2R_{p-1}^2)^2}$$
$$\equiv \frac{N(r)}{Q(r)}.$$

From (c), (d), and (e) we obtain the convexity in the direction of the r-axis of the set of points (r, K) for which $K \geqslant 0$, $B^2 - 4 \leqslant 0$, $0 \leqslant r \leqslant 1$.

(f) $dB/dr = 0$ implies that

$$(4.4)\quad N(r) \equiv KR_{p-1}^4r^6 - (KR_{p-1}^4 + 2KR_{p-1}^2 + b_1R_{p-1}^3 - b_1R_{p-1}^5)r^4 + (2KR_{p-1}^2 - b_1R_{p-1} + b_1R_{p-1}^3 + K)r^2 - K = 0.$$

From (3.4) we have $N(0) = K$, $N(1) = b_1R_{p-1}(1 - R^4{}_{p-1})$, and the product of the roots of $N(r)$ is

$$\frac{1}{R_{p-1}^4} > 1.$$

Thus, if $K < 0$ we see that $N(r)$ has but one root in the interval $0 < r < 1$.

(g) When $K < 0$ we also have the following relations: $B < 0$; $dB/dr < 0$ for r sufficiently small, $dB/dr < 0$ for $r = 1$, and $\lim_{r\to 0}B = -\infty$.

From (c), (f), and (g), we obtain the convexity in the direction of the r-axis of the sets of points (r, K) satisfying $K \leqslant 0$, $B^2 - 4C \leqslant 0$, $0 \leqslant r \leqslant 1$.

Proof of Lemma 4.2. (a) First we note that $\lim_{r\to 0} B^2 - 4C < 0$ if and only if $|K| < 2$, while for $r = 1$, $B^2 - 4C > 0$ for all K.

(b) For $|K| = 2$, $B^2 - 4C > 0$ for all $0 \leqslant r \leqslant 1$.

(c) Then, since $B^2 - 4C$ is a quadratic in K it follows that for fixed $a_2, \ldots, a_p, \nu, r$ there exist at most two values of K for which $B^2 - 4C = 0$.

Proof of Lemma 4.3. This is immediate since if for a particular choice of $a_2, \ldots, a_p, \nu, \phi, r$, we have $B^2 - 4C < 0$ then the P under consideration is a member of $T_1(r)$ and from paragraph (4.2) we see that $B^2 - 4C$ cannot be greater than zero for any smaller r unless we have $B^2 - 4 \geqslant 0$.

Proof of Lemma 4.4. For fixed $a_2, \ldots, a_p: \nu, \phi$ let $P = u(r, \theta) + iv\ (r, \theta)$. Then, from the definition of P we have $v(r, 0) = 0$ and $v(r, \pi) = 0$ for all $0 \leqslant r \leqslant 1$. Thus, $v_r(r, 0)$ and $v_r(r, \pi) = 0$ for all $0 \leqslant r \leqslant 1$. Now if we rewrite (1.9) as $Q(a_2, \ldots, a_p: r, \nu, \phi, \theta)\ (\cos^2\theta + B\cos\theta + C) = Q(\cos^2\theta + B\cos\theta + C)$, we have, since $C > 0$, $Q(a_2, \ldots, a_p, r, \nu, \phi, 0) = 0$ and $Q(a_2, \ldots, a_p: r, \nu, \phi, \pi) = 0$.

Any solution of the system $\{B^2 - 4C = 0,\ B^2 - 4 = 0\}$ is also a solution of the equivalent system $\{B^2 - 4 = 0,\ C = 1\}$. Let $r = \alpha$, $0 < \alpha < 1$ be a solution of this system for some particular $a_2, \ldots, a_p: \nu, \phi$. We have

$$\begin{aligned} v_\theta(r, \theta) &= (Q)(-B\sin\theta - 2\sin\theta\cos\theta) + (Q_\theta)(\cos^2\theta + B\cos\theta + C) \\ v_r(r, \theta) &= (Q)(B_r\cos\theta + C_r) \qquad\qquad\qquad + (Q_r)\ (\cos^2\theta + B\cos\theta + C) \end{aligned}$$

and, therefore, we have

$$\begin{aligned} v_\theta(\alpha, 0) &= (Q_\theta)(1 + B + C)]_{\theta=0} & v_r(\alpha, 0) &= 0, \\ v_\theta(\alpha, \pi) &= (Q_\theta)(1 - B - C)]_{\theta=\pi} & v_r(\alpha, \pi) &= 0. \end{aligned}$$

Thus for $r = \alpha$ either $v_\theta(\alpha, 0) = 0$ or $v_\theta(\alpha, \pi) = 0$, since B and C are independent of θ. This, however, implies that either $P'(\alpha) = 0$ or $P'(-\alpha) = 0$ for this choice of $a_2, \ldots, a_p: \nu, \phi$. Thus $\alpha \geqq \tilde{R}_p$ follows from (3.2).

Proof of the Main Theorem. From Lemmas 4.1 through 4.4 it is clear that for any choice of $a_2, \ldots, a_p$ no function P can belong to $T_3(r)$ if $r < \tilde{R}_p$. The proof then follows directly from the first two paragraphs of § 4, and formula (2.1).

5. The Class $T_2(1)$. Because of the discontinuity of the functions (2.11) and (2.12) at $r = 1$, $R_{p-1} = 1$ the derivation of the R_p, $p > 2$ employed in § 4 is not valid for the case $p = 2$ in which $R_{p-1} \equiv R_1 \equiv 1$. We present, therefore, in this section a rather simple proof of the validity for $p = 2$ of the Main Theorem. This proof is a modification of the proof found in the author's paper **(1)**.

When $p = 2$ we must have $b_1 > 0$ if statement (2.2) is to be compatible with Rogosinski's definition of the class $T_1(1)$, **(6)**.

(5.1) $$B = \frac{K(1+r^2)}{2r},$$

(5.2) $$C = \left(\frac{1-r^2}{2r}\right) - \frac{K(\cos\phi + \cos\nu)}{2} - \cos\phi\cos\nu,$$

and

(5.3) $$K = b_1 - 2\cos\phi = a_2 - 2\cos\nu - 2\cos\phi.$$

Equation (3.1) takes the form

(5.4) $$\left(\frac{\omega^2+1}{2\omega} + \frac{K}{2}\right)^2 - \left(\cos\phi\cos\nu + \frac{Ka_2}{4}\right) = 0, \qquad 0 < |\omega| < 1.$$

Solving (5.4) for ω, we obtain

(5.5) $$\omega_1 = a + (a^2-1)^{\frac{1}{2}},\ \omega_2 = a - (a^2-1)^{\frac{1}{2}},\ \omega_3 = b + (b^2-1)^{\frac{1}{2}},\ \omega_4 = b - (b^2-1)^{\frac{1}{2}},$$

where

(5.6) $$a = \frac{-K}{2} + (\cos\phi\cos\nu + \tfrac{1}{4}Ka_2)^{\frac{1}{2}}$$

and $$b = \frac{-K}{2} - (\cos\phi\cos\nu + \tfrac{1}{4}Ka_2)^{\frac{1}{2}}.$$

From (5.5) and (5.6) it is evident that to obtain $\tilde{R}_2(a_2)$ we need only minimize the expression $|a| - (a^2-1)^{\frac{1}{2}}$, $|a| \geqq 1$, since $|a|$ and $|b|$ have the same maximum value.

The minimum of $|a| - (a^2-1)^{\frac{1}{2}}$ occurs when $|a|$ is maximum, that is, when $\phi = \nu = \pi$, $a_2 > 0$ or $\phi = \nu = 0$, $a_2 < 0$. Thus

(5.7) $$\tilde{R}_2(a_2) = (|a_2| + 3) - ((|a_2| + 3)^2 - 1)^{\frac{1}{2}}.$$

We do not establish the validity of the Lemmas 4.1 to 4.4 for $p = 2$ since from (5.1) we have for fixed a_2, ν, and ϕ that

(5.8) $$\frac{d|B|}{dr} = \left|\frac{K}{2}\right| \left(\frac{r^2-1}{r^2}\right) < 0$$

for all $0 < r < 1$ and

(5.9) $$\frac{d(B^2-4C)}{dr} = \left(1 - \frac{K^2}{4}\right)\left(\frac{1-r^2}{4r^3}\right) > 0$$

for all $|K| < 2$, $0 < r < 1$.

From (5.7) we see that

$$\max_{|a_2|} \tilde{R}_2(a_2) = 3 - 2\sqrt{2}.$$

If $r = 3 - 2\sqrt{2}$ and $|B| \leqq 2$ we have from (5.1) that $|K| \leqq 2/3$. Then from (5.8) we see that for $|K| > 2/3$ and $r < 3 - 2\sqrt{2}$ we have $|B| > 2$.

Next, from (5.1) and (5.2) we have for $r = 3 - 2\sqrt{2}$,

$$B^2 - 4C = 8(\tfrac{1}{4}K^2 - 1) + (\tfrac{1}{2}K + \cos\phi)(\tfrac{1}{2}K + \cos\nu)$$

which is readily seen to be negative if $|K| \leqslant 2/3$. Then from (5.9) we see that for $r < 3 - 2\sqrt{2}$ and $|K| \leqslant 2/3$ we have $B^2 - 4C < 0$.

Thus, as in § 4, it follows that for any fixed a_2, ν, $0 < \nu < \pi$, and all $r < \tilde{R}_2(a_2)$ we have $P \in T_1(r)$. This establishes the Main Theorem for $p = 2$.

6. Sharpness. To show that our result is sharp over $\cup_p T_p(1)$ we give a function of class $T_2^*(1)$ which is typically real of order one for and only for $|z| < R_2(a_2) \equiv R(a_2)$ as defined in (5.7).

Consider the function

$$f(z) = \frac{z^3 + (a_2 + 4)z^2 + z}{(z+1)^4}; \qquad a_2 > 0,\ |z| < 1. \tag{6.1}$$

This function is a member of $T_2^*(1)$ and

$$f'(z) = \frac{(z-1)[z^2 + (2a_2 + 6)z + 1]}{(z+1)^5}. \tag{6.2}$$

From (5.2) it is readily seen that $f(z)$ cannot belong to $T_1(r)$ for any r greater than $(a_2 + 3) - ((a_2 + 3)^2 - 1)^{\frac{1}{2}} \equiv \tilde{R}_2(a_2) \equiv R(a_2)$.

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