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A GENERALIZATION OF AN INEQUALITY OF BHATTACHARYA AND LEONETTI

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ABSTRACT. We show that bounded John domains and bounded starshaped domains with respect to a point satisfy the following inequality

$$\int_D F\Big(b\frac{|u(x)-u_D|}{\operatorname{dia}(D)}\Big)\,dx \le K\int_D F\big(|\nabla u(x)|\big)\,dx,$$

where $F\colon [0,\infty) \longrightarrow [0,\infty)$ is a continuous, convex function with $F(0)=0$, and u is a function from an appropriate Sobolev class. Constants b and K do depend at most on D. If $F(x)=x^p$, $1\le p<\infty$, this inequality reduces to the ordinary Poincaré inequality.

1. **Introduction.** Tilak Bhattacharya and Francesco Leonetti introduced the following version of the Poincaré inequality

$$\int_D F\Big(b\frac{|u(x)-u_D|}{\operatorname{dia}(D)}\Big)\,dx \le K\int_D F(|\nabla u(x)|)\,dx. \tag{1.1}$$

Here, $F\colon [0,\infty) \longrightarrow [0,\infty)$ is a convex, continuous function with $F(0)=0$, D is a bounded domain in R^n, u is a function from an appropriate Sobolev class, and u_D stands for the integral average of u over D. Constants $b\in(0,1]$ and $K>0$ depend at most on D. A domain D is an *F-Poincaré domain*, write $D\in\mathcal{P}(F)$, whenever there are constants $b=b(D)$ and $K=K(D)$ such that (1.1) holds for all $u\in W_1^1(D)$ and $F(|\nabla u|)\in L^1(D)$.

Bhattacharya and Leonetti proved that inequality (1.1) with $b=1$ holds for convex domains, [1, Lemma 1], and with additional assumptions of F for an annulus, [1, Theorem 2]. In this paper we show that John domains and starshaped domains are F-Poincaré domains. Further we consider a modification of (1.1) in Section 6.

If $F(x)=x^p$, $1\le p<\infty$, inequality (1.1) reduces to the ordinary Poincaré inequality

$$\int_D |u(x)-u_D|^p\,dx \le \kappa_p(D)^p\int_D|\nabla u(x)|^p\,dx,$$

whenever $u\in W_p^1(D)$. It is customary to write $D\in\mathcal{P}(p)$ and $\kappa_p(D)=K^{1/p}\operatorname{dia}(D)\,b^{-1}$ and to say D is a p-Poincaré domain. Starshaped domains as well as John domains are p-Poincaré domains for all p, $1\le p<\infty$, [4, Theorem 3.1], [3, Theorems 3.1 and 8.5].

We restate the result of Bhattacharya and Leonetti here.

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THEOREM 1.2 [1, Lemma 1]. *Let D be a convex, bounded subset of R^n, $n \geq 1$. Let $F\colon [0,\infty) \longrightarrow [0,\infty)$ be a continuous, convex function with $F(0) = 0$. If $u \in W_1^1(D)$ such that $F(|\nabla u|) \in L^1(D)$, then (1.1) holds with $b = 1$ and $K = (\frac{\omega_n \operatorname{dia}(D)^n}{|D|})^{1-1/n}$.*

A generalization of Theorem 1.2 is that John domains and starshaped domains are F-Poincaré domains, Theorems 4.1 and 5.1.

2. **Notation and definitions.**

Throughout this paper we let D and G be bounded domains of euclidean n-space R^n, $n \geq 2$.

If $x \in R^n$ and $r > 0$, then $B^n(x,r) = \{y \in R^n \mid |x-y| < r\}$ is an open ball in R^n and the sphere $S^{n-1}(x,r)$ is its boundary. We use the abbreviations $B^n(r) = B^n(0,r)$ and $S^{n-1}(r) = S^{n-1}(0,r)$.

The euclidean distance between sets A and B is written as $d(A,B)$, and $d(x,\partial A)$ denotes the distance from $x \in A$ to the boundary of A. We let $\operatorname{dia}(A)$ denote the diameter of A. We write tQ for the cube with the same center as Q and dilated by a factor $t > 1$.

The average of a function u is $u_A = \frac{1}{|A|}\int_A u(x)\,dx = \fint_A u(x)\,dx$ if $|A| > 0$; here $|A|$ stands for the n-dimensional Lebesgue measure of A. We write $|B^n(1)| = \omega_n$.

The L^p-norm of u in A is $\|u\|_{L^p(A)} = (\int_A |u(y)|^p\,dy)^{1/p}$. The Sobolev space $W_p^1(G)$, $1 \leq p < \infty$, is the space of functions $u \in L^p(G)$ whose first distributional partial derivatives belong to $L^p(G)$. In $W_p^1(G)$ we use the norm $\|u\|_{W_p^1(G)} = \|u\|_{L^p(G)} + \|\nabla u\|_{L^p(G)}$; here $\nabla u = (\partial_1 u, \ldots, \partial_n u)$ is the gradient of u.

We let $c(*,\ldots,*)$ denote a constant which depends only on the quantities appearing in the parentheses.

A domain D is called an (α,β)-*John domain*, $0 < \alpha \leq \beta < \infty$, if there is $x_0 \in D$ such that each $x \in D$ can be joined to x_0 by a rectifiable curve $\gamma\colon [0,\ell] \longrightarrow D$ parametrized by arc length with $\ell \leq \beta$ and

$$d\big(\gamma(t),\partial D\big) \geq \frac{\alpha}{\ell} t, \quad t \in [0,\ell].$$

Convex domains, Lipschitz domains, and bounded uniform domains are John domains. John domains form a proper subclass of domains satisfying a quasihyperbolic boundary condition. We refer to [2] for detailed discussion of these concepts.

A bounded domain in R^n is called *starshaped with respect to a point* $x_0 \in D$, if each ray starting from x_0 intersects ∂D exactly at one point. A starshaped domain is not necessarily a John domain: a simple example is

$$D = \{(x_1,x_2) \in B^2\big((1,0),1\big) : |x_2| < x_1^2\}.$$

The following chains and decompositions of a domain are essential to our sufficient condition in Theorem 3.1.

2.1. CHAINS. Sets D_i, $i = 0,1,\ldots,k$, in R^n form a *chain*, abbreviated $C(D_k) = (D_0,D_1,\ldots,D_k)$, if $D_i \cap D_j \neq \emptyset$ if and only if $|i-j| \leq 1$.

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2.2. DECOMPOSITIONS. Let $\mathcal{W}$ be a family of domains $D \in \mathcal{P}(F)$ with $0 < b_0 \leq b(D)$ and $K(D) \leq c_0 < \infty$ such that $D \in \mathcal{P}(1)$ with $\kappa_1(D) \leq c_1 < \infty$. We call $\mathcal{W}$ an *F-Poincaré decomposition* of G, if there are constants c_2, c_3, and N with the following properties:

(i) $G = \bigcup_{D\in\mathcal{W}} D$,

(ii) $\sum_{D\in\mathcal{W}} \chi_D(x) \leq N\chi_G(x)$ for all $x \in R^n$, and

(iii) there is a domain $D_0 \in \mathcal{W}$ such that for each $D \in \mathcal{W}$ there is a chain $C(D) = (D_0, D_1, \ldots, D_k)$ of domains in $\mathcal{W}$ with

$$\max\{|D_i|, |D_{i+1}|\} \leq c_2|D_i \cap D_{i+1}| \tag{2.3}$$

for $i = 0, 1, \ldots, k-1$; $D = D_k$; and

$$\sum_{A\in C(D)} \kappa_1(A) \leq c_3 b_0^{-1} c_2^{-1} \operatorname{dia}(G). \tag{2.4}$$

For each $D \in \mathcal{W}$ we fix a chain $C(D)$ satisfying (2.3) and (2.4) and call this chain the F-Poincaré chain from D_0 to D. For a fixed set $A \in \mathcal{W}$ we write

$$A(\mathcal{W}) = \{D \in \mathcal{W} \mid A \in C(D)\}.$$

If D in R^n is an (α, β)-John domain and W is a Whitney decomposition of D into Whitney cubes Q, [6, VI], then $\{\operatorname{int} \frac{9}{8}Q \mid Q \in W\}$ forms an F-Poincaré decomposition of D, see the proof for Theorem 4.1.

3. A sufficient condition for a domain to be an F-Poincaré domain.

Our main result is the following theorem which gives a sufficient condition for a domain to be an F-Poincaré domain.

THEOREM 3.1. *Let $G \subset R^n$ be a bounded domain and let $\mathcal{W}$ be an F-Poincaré decomposition of G. Suppose that there are constants $b_1 < \infty$ and $\varepsilon \in [0, 1]$ such that*

$$\sum_{D\in A(\mathcal{W})} |D| \leq b_1\, \kappa_1(A)^{-\varepsilon} |A| \tag{3.2}$$

for all $A \in \mathcal{W}$. Then $G \in \mathcal{P}(F)$.

PROOF FOR THEOREM 3.1. Since F is an increasing, convex, continuous function,

$$F\big(|u(x) - u_G|\big) \leq \frac{1}{2}\Big(F\big(2|u(x) - c|\big) + F(2|c - u_G|)\Big),$$

where by Jensen's inequality

$$F(2|u_G - c|) \leq F\Big(2 \fint_G |u(y) - c|\, dy\Big) \leq \fint_G F\big(2|u(y) - c|\big)\, dy;$$

here, $c \in R$. Hence

$$\int_G F\big(|u(x) - u_G|\big)\, dx \leq \int_G F\big(2\,|u(x) - c|\big)\, dx \tag{3.3}$$

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for each $c \in R$. Thus we need to estimate $F(2|u(y) - c|)$ for some constant $c \in R$.

We apply a similar argument as in [3, Theorem 4.4]. Since $\mathcal{W}$ is an F-Poincaré decomposition of G, there is a domain $D_0 \in \mathcal{W}$ such that for each $D \in \mathcal{W}$ we can fix a chain satisfying (2.3) and (2.4). We will estimate

$$\begin{aligned}F\Big(\frac{b_0}{4c_3\operatorname{dia}(G)}|u(x) - u_{D_0}|\Big) \le \frac{1}{2}\Big(&F\Big(\frac{b_0}{2c_3\operatorname{dia}(G)}|u(x) - u_D|\Big)\\ &+F\Big(\frac{b_0}{2c_3\operatorname{dia}(G)}|u_D - u_{D_0}|\Big)\Big).\end{aligned} \tag{3.4}$$

Recall $D \in \mathcal{P}(F)$ with $K(D) \le c_0 < \infty$ and especially $D \in \mathcal{P}(1)$ with $\kappa_1(D) \le c_1 < \infty$. Inequality (2.3) and the fact $D_j \in \mathcal{P}(1)$ yield

$$\begin{aligned}|u_{D_k} - u_{D_0}| &\le \sum_{j=0}^{k-1} |u_{D_j} - u_{D_{j+1}}|\\ &= \sum_{j=0}^{k-1} \fint_{D_j\cap D_{j+1}} |u_{D_j} - u_{D_{j+1}}|\,dx\\ &\le 2c_2 \sum_{j=0}^{k} \fint_{D_j} |u_{D_j} - u(x)|\,dx\\ &\le 2c_2 \sum_{j=0}^{k} \kappa_1(D_j) \fint_{D_j} |\nabla u(x)|\,dx.\end{aligned}$$

Thus using (2.4), convexity, and Jensen's inequality we obtain

$$\begin{aligned}F\Big(\frac{b_0c_2}{2c_3\operatorname{dia}(G)}|u_D - u_{D_0}|\Big) &\le F\Big(\frac{b_0c_2}{c_3\operatorname{dia}(G)} \sum_{A\in C(D)} \kappa_1(A) \fint_A |\nabla u(y)|\,dy\Big)\\ &\le F\Big(\sum_{A\in C(D)} \frac{\kappa_1(A)}{\sum_{B\in C(D)}\kappa_1(B)} \fint_A |\nabla u(y)|\,dy\Big)\\ &\le \sum_{A\in C(D)} \frac{\kappa_1(A)}{\sum_{B\in C(D)}\kappa_1(B)} F\Big(\fint_A |\nabla u(y)|\,dy\Big)\\ &\le \sum_{A\in C(D)} \frac{\kappa_1(A)}{\kappa_1(D_0)} \fint_A F\big(|\nabla u(y)|\big)\,dy.\end{aligned} \tag{3.5}$$

Inequalities (3.3)–(3.5) imply

$$\begin{aligned}&\int_G F\Big(\frac{b_0}{8c_3\operatorname{dia}(G)}|u(x) - u_G|\Big)\,dx\\ &\quad\le \frac{1}{2}\sum_{D\in\mathcal{W}} \int_D F\Big(\frac{b(D)}{2c_3\operatorname{dia}(D)}|u(x) - u_D|\Big)\,dx\\ &\quad\quad+ \frac{1}{2}\sum_{D\in\mathcal{W}} |D| \sum_{A\in C(D)} \frac{\kappa_1(A)}{\kappa_1(D_0)} \fint_A F\big(|\nabla u(y)|\big)\,dy.\end{aligned} \tag{3.6}$$

We may assume that $1 \leq c_3$. Since $D \in \mathcal{P}(F)$, inequality (1.1) yields

$$(3.7)\qquad \begin{aligned} \sum_{D\in\mathcal{W}} \int_D F\Big(\frac{b(D)}{c_3\,\mathrm{dia}(D)}|u(x)-u_D|\Big)\,dx &\leq c_0 \sum_{D\in\mathcal{W}} \int_D F\big(|\nabla u(x)|\big)\,dx \\ &\leq Nc_0 \int_G F\big(|\nabla u(x)|\big)\,dx. \end{aligned}$$

Rearranging the double sum in (3.6), and using (3.2) and the inequality $\kappa_1(A) \leq c_1$ we obtain

$$(3.8)\qquad \begin{aligned} &\sum_{D\in\mathcal{W}} \sum_{A\in\mathcal{C}(D)} |D|\kappa_1(A) \fint_A F\big(|\nabla u(y)|\big)\,dy \\ &= \sum_{A\in\mathcal{W}} \sum_{D\in A(\mathcal{W})} |D|\kappa_1(A) \fint_A F\big(|\nabla u(y)|\big)\,dy \\ &\leq b_1 \sum_{A\in\mathcal{W}} \kappa_1(A)^{1-\varepsilon} \int_A F\big(|\nabla u(y)|\big)\,dy \\ &\leq b_1 c_1^{1-\varepsilon} N \int_G F\big(|\nabla u(x)|\big)\,dx. \end{aligned}$$

Substituting (3.7) and (3.8) into (3.6) implies the desired inequality

$$\int_G F\Big(\frac{b_0}{8c_3\,\mathrm{dia}(G)}|u(x)-u_G|\Big)\,dx \leq c(b_1,c_0,c_1,\varepsilon,N) \int_G F\big(|\nabla u(x)|\big)\,dx$$

and Theorem 3.1 is proved.

4. **John domains.** Applying Theorem 3.1 and its proof to a John domain yields

THEOREM 4.1. *An (α,β)-John domain D in R^n is an F-Poincaré domain with $b = c(n)\frac{\alpha}{\beta}$ and $K = c(n)(\frac{\beta}{\alpha})^{n+1}$.*

PROOF. Let $\mathcal{W}$ be a Whitney decomposition of D into cubes Q. Theorem 1.2 yields that $K(Q) = c(n) := c_0$ and $\kappa_1(Q) = c(n)\,\mathrm{dia}(Q) := c_1$. Fix $Q_0 \in \mathcal{W}$ with $x_0 \in Q_0$. The John property in a domain means that for each $Q \in \mathcal{W}$ there is a chain $C(\mathrm{int}\,\frac{9}{8}Q)$ of cubes $\mathrm{int}\,\frac{9}{8}Q_j$ $Q_j \in \mathcal{W}, j = 0,1,\ldots,k$, $Q = Q_k$, such that

$$\sum_{j=i}^{k} \mathrm{dia}(Q_j) \leq c(n)\frac{\beta}{\alpha}\,\mathrm{dia}(Q_i) \leq c(n)\frac{\beta}{\alpha}\,\mathrm{dia}(D)$$

for all $i = 0,1,\ldots,k$, see [3, proofs for Proposition 6.1 and Lemma 8.3]. Hence there are constants $c_2 = c_2(n)$ and $c_3 = c_3(n)\frac{\beta}{\alpha}$ such that (2.3) and (2.4) are true.

The above result combined to the fact that there are not too many Whitney cubes of the same size in a John domain yields that for all $A \in \mathcal{W}' = \{\mathrm{int}\,\frac{9}{8}Q \mid Q \in \mathcal{W}\}$

$$\begin{aligned} \sum_{\mathrm{int}\frac{9}{8}Q\in A(\mathcal{W}')} |Q| &\leq \sum_{j=1}^{\infty} \sum_{Q_j\in S} |Q_j| \\ &\leq \sum_{j=1}^{\infty} c(n)\Big(\frac{\beta}{\alpha}\Big)^n \Big(\frac{\alpha}{c(n)\beta}\Big)^{\delta} 2^{-j\delta}|A| \\ &\leq c(n)\Big(\frac{\beta}{\alpha}\Big)^n |A|, \end{aligned}$$

where $S = \left\{\operatorname{int}\frac{9}{8}Q : \frac{\alpha\,\mathrm{dia}(A)}{2^j\beta c(n)} \leq \mathrm{dia}(\frac{9}{8}Q) \leq 2\frac{\alpha\,\mathrm{dia}(A)}{2^j\beta c(n)}\right\}$ and $\delta = \delta(n, \frac{\alpha}{\beta})$, see [3, Lemma 8.4]. Now $\{\operatorname{int}\frac{9}{8}Q \mid Q \in \mathcal{W}\}$ is an F-Poincaré decomposition of D and (3.2) is satisfied, when $\varepsilon = 0$. Thus $D \in \mathcal{P}(F)$ by Theorem 3.1.

The proof for the Theorem 3.1 gives that $b = c(n)\frac{\alpha}{\beta}$ and

$$K = c(n)\left(1 + \left(\frac{\beta}{\alpha}\right)^n \frac{\mathrm{dia}(D)}{\mathrm{dia}(Q_0)}\right) \leq \left(\frac{\beta}{\alpha}\right)^{n+1},$$

since $\alpha \leq \mathrm{dia}(Q_0) \leq \mathrm{dia}(D) \leq \beta$.

5. **Starshaped domains.**

THEOREM 5.1. *If D in R^n is a domain which is starshaped with respect to a point x_0, then $D \in \mathcal{P}(F)$. Here, $b = \frac{1}{18}$ and $K = K\big(n, d(x_0, \partial D), \max_{x\in\partial D} d(x, x_0)\big)$.*

We need the following trace lemma for the proof of Theorem 5.1.

LEMMA 5.2. *Let D be a domain in R^n and let $B^n(2\ell) \subset D$. If $F\colon [0,\infty) \longrightarrow [0,\infty)$ is a convex, continuous function with $F(0) = 0$, then*

$$\int_{S^{n-1}(\xi)} F\Big(\frac{1}{2\ell}|u(z)|\Big)\,dm_{n-1}(z) \leq \frac{1}{\ell}\int_{B^n(\ell)} F\Big(\frac{1}{\ell}|u(x)|\Big)\,dx + \frac{2^{n-1}}{\ell}\int_{B^n(\ell)} F\big(|\nabla u(x)|\big)\,dx,$$

for each $\xi \in [\ell/2, \ell]$ whenever $u \in C^1(D)$.

PROOF. By the mean value theorem for integrals we have

$$\int_{r=\ell/2}^{\ell}\int_{S^{n-1}(1)} F\big(|u(\theta, r)|\big)\,r^{n-1}\,dm_{n-1}(\theta)\,dr = (\ell - \ell/2)\int_{S^{n-1}(1)} F\big(|u(\theta,\sigma)|\big)\,\sigma^{n-1}\,dm_{n-1}(\theta) \tag{5.3}$$

for some $\sigma \in [\ell/2, \ell]$.

On the other hand for ξ with $\ell/2 \leq \xi \leq \sigma \leq \ell$,

$$\begin{aligned} F\Big(\frac{1}{2\ell}|u(\theta,\xi)|\Big) &\leq F\Big(\frac{1}{2\ell}|u(\theta,\sigma)| + \frac{1}{2\ell}\int_\xi^\sigma |D_r u(\theta, t)|\,dt\Big)\\ &\leq \frac{1}{2}F\Big(\frac{1}{\ell}|u(\theta,\sigma)|\Big) + \frac{1}{2}\frac{1}{\ell/2}\int_{\ell/2}^{\ell} F\big(|D_r u(\theta,t)|\big)\,dt. \end{aligned}$$

Hence

$$\begin{aligned} &F\Big(\frac{1}{2\ell}|u(\theta,\xi)|\Big)\xi^{n-1}\\ &\quad\leq \frac{1}{2}F\Big(\frac{1}{\ell}|u(\theta,\sigma)|\Big)\sigma^{n-1} + \frac{1}{\ell}\Big(\frac{2}{\ell}\Big)^{n-1}\ell^{n-1}\int_{\ell/2}^{\ell} F\big(|D_r u(\theta,t)|\big)t^{n-1}\,dt, \end{aligned}$$

where we have used $\xi^{n-1} \leq \sigma^{n-1}$ and $1 \leq t^{n-1}/(\ell/2)^{n-1}$.

Combining the estimates and (5.3) we obtain

$$\begin{aligned}
\int_{S^{n-1}(\xi)} F\Big(\frac{1}{2\ell}|u(z)|\Big)\, dm_{n-1}(z)
&= \int_{S^{n-1}(1)} F\Big(\frac{1}{2\ell}|u(\theta,\xi)|\Big)\xi^{n-1}\, dm_{n-1}(\theta)\\
&\le \frac{1}{2}\int_{S^{n-1}(1)} F\Big(\frac{1}{\ell}|u(\theta,\sigma)|\Big)\sigma^{n-1}\, dm_{n-1}(\theta)\\
&\quad + \frac{2^{n-1}}{\ell}\int_{t=\ell/2}^{\ell}\int_{S^{n-1}(1)} F\big(|D_r u(\theta,t)|\big)t^{n-1}\, dm_{n-1}(\theta)\, dt\\
&\le \frac{1}{\ell}\int_{r=\ell/2}^{\ell}\int_{S^{n-1}(1)} F\Big(\frac{1}{\ell}|u(\theta,r)|\Big)r^{n-1}\, dm_{n-1}(\theta)\, dr\\
&\quad + \frac{2^{n-1}}{\ell}\int_{B^n(\ell)\setminus B^n(\ell/2)} F\big(|\nabla u(x)|\big)\, dx\\
&= \frac{1}{\ell}\int_{B^n(\ell)\setminus B^n(\ell/2)} F\Big(\frac{1}{\ell}|u(x)|\Big)\, dx\\
&\quad + \frac{2^{n-1}}{\ell}\int_{B^n(\ell)\setminus B^n(\ell/2)} F\big(|\nabla u(x)|\big)\, dx.
\end{aligned}$$

This yields the desired inequality and the proof is complete.

PROOF FOR THEOREM 5.1. Write $d(x_0,\partial D) = 2\ell$, $\max_{x\in\partial D} d(x,x_0) = L$, and $B^n(x_0,\frac{1}{2}\ell) = B$. It suffices to consider functions $u \in W^1_p(D)\cap C^\infty(D)$, *cf.* [5, Theorem 1.1.6/1]. We assume, for convenience, that $x_0 = 0$. By (3.3) it is enough to estimate the term $\int_D F(|u(x)-u_B|)\, dx$.

First we note that

$$\begin{aligned}
\int_D F\big(|u(x)-u_B|\big)\, dx &\le \int_D F\Big(\fint_B |u(x)-u(y)|\, dy\Big)\, dx\\
&\le \int_B \fint_B F\big(|u(x)-u(y)|\big)\, dy\, dx\\
&\quad + \int_{D\setminus B}\fint_B F\big(|u(x)-u(y)|\big)\, dy\, dx.
\end{aligned}\tag{5.4}$$

The function F is increasing and by Theorem 1.2 a ball B is an F-Poincaré domain and hence

$$\begin{aligned}
\int_B \fint_B F\Big(\frac{1}{9\,\mathrm{dia}(D)}|u(x)-u(y)|\Big)\, dy\, dx
&\le \int_B \fint_B F\Big(\frac{1}{\mathrm{dia}(B)}|u(x)-u_B|\Big)\, dx\, dy\\
&\le c(n)\int_B F\big(|\nabla u(x)|\big)\, dx.
\end{aligned}\tag{5.5}$$

We estimate the last double integral in (5.4) in three parts:

$$
\begin{aligned}
&\int_{D\setminus B} \fint_B F\Big(\frac{1}{9\,\mathrm{dia}(D)}|u(x)-u(y)|\Big)\,dy\,dx \\
&\qquad \le \frac{1}{3}\int_{D\setminus B}\fint_B F\Big(\frac{1}{3\,\mathrm{dia}(D)}\Big|u(x)-u\Big(\frac{\ell}{2|x|}x\Big)\Big|\Big)\,dy\,dx \\
&\qquad\quad + \frac{1}{3}\int_{D\setminus B}\fint_B F\Big(\frac{1}{3\,\mathrm{dia}(D)}\Big|u\Big(\frac{\ell}{2|x|}x\Big)-u_B\Big|\Big)\,dy\,dx \\
&\qquad\quad + \frac{1}{3}\int_{D\setminus B}\fint_B F\Big(\frac{1}{3\,\mathrm{dia}(D)}|u(y)-u_B|\Big)\,dy\,dx,
\end{aligned} \tag{5.6}
$$

where

$$
\begin{aligned}
&\int_{D\setminus B}\fint_B F\Big(\frac{1}{\mathrm{dia}(D)}|u(y)-u_B|\Big)\,dy\,dx \\
&\qquad \le c(n)\frac{|D\setminus B|}{|B|}\int_B F\big(|\nabla u(x)|\big)\,dx
\end{aligned} \tag{5.7}
$$

by Theorem 1.2.

Since D is starshaped the first integral of the right hand side in (5.6) can be estimated by using spherical coordinates: for $\theta \in S^{n-1}(1)$ write $R(\theta) = |z|$ where z is the unique common point of ∂D and the ray $t\theta$, $t \ge 0$. Thus

$$
\begin{aligned}
&\int_{D\setminus B}\fint_B F\Big(\frac{1}{\mathrm{dia}(D)}\Big|u(x)-u\Big(\frac{\ell}{2|x|}x\Big)\Big|\Big)\,dy\,dx \\
&\qquad = \int_{S^{n-1}(1)}\int_{\ell/2}^{R(\theta)} F\Big(\frac{1}{\mathrm{dia}(D)}\Big|u(r,\theta)-u\Big(\frac{\ell}{2},\theta\Big)\Big|\Big)\,r^{n-1}\,dr\,dm_{n-1}(\theta).
\end{aligned}
$$

Applying the inequalities $\ell/2 \le r \le R(\theta) \le L$ and $\ell/2 = \mathrm{dia}(B) \le \alpha \le r$ we obtain

$$
\begin{aligned}
&\int_{\ell/2}^{R(\theta)} F\Big(\frac{1}{\mathrm{dia}(D)}\Big|u(r,\theta)-u\Big(\frac{\ell}{2},\theta\Big)\Big|\Big)\,r^{n-1}\,dr \\
&\qquad \le \int_{\ell/2}^{R(\theta)} F\Big(\frac{1}{|R(\theta)-\ell/2|}\int_{\ell/2}^{r}|u_\alpha(\alpha,\theta)|\,d\alpha\Big)\,r^{n-1}\,dr \\
&\qquad \le \int_{\ell/2}^{R(\theta)} F\Big(\frac{1}{|R(\theta)-\ell/2|}\int_{\ell/2}^{R(\theta)}|u_\alpha(\alpha,\theta)|\,d\alpha\Big)\,r^{n-1}\,dr \\
&\qquad \le \int_{\ell/2}^{R(\theta)} \frac{1}{|R(\theta)-\ell/2|}\int_{\ell/2}^{R(\theta)} F\big(|u_\alpha(\alpha,\theta)|\big)\,d\alpha r^{n-1}\,dr \\
&\qquad \le \int_{\ell/2}^{R(\theta)} \frac{1}{\ell/2}\int_{\ell/2}^{L} F\big(|u_\alpha(\alpha,\theta)|\big)\,d\alpha r^{n-1}\,dr \\
&\qquad \le \frac{2}{n}\frac{L^n}{\ell}\int_{\ell/2}^{R(\theta)} F\big(|\nabla u(\alpha,\theta)|\big)\frac{\alpha^{n-1}}{\alpha^{n-1}}\,d\alpha \\
&\qquad \le \frac{1}{n}\Big(\frac{2L}{\ell}\Big)^n\int_{\ell/2}^{R(\theta)} F\big(|\nabla u(\alpha,\theta)|\big)\alpha^{n-1}\,d\alpha.
\end{aligned}
$$

Hence

$$\int_{D\setminus B} \fint_B F\Big(\frac{1}{\operatorname{dia}(D)}\Big|u(x) - u\Big(\frac{\ell}{2|x|}x\Big)\Big|\Big)\,dy\,dx \le c(n)\Big(\frac{L}{\ell}\Big)^n \int_{D\setminus B} F\big(|\nabla u(x)|\big)\,dx. \tag{5.8}$$

In order to estimate the second integral of the right hand side of (5.6) we need Lemma 5.2. Changing the variables and using Lemma 5.2 and Theorem 1.2 we obtain

$$\begin{aligned}
&\int_{D\setminus B} \fint_B F\Big(\frac{1}{\operatorname{dia}(D)}\Big|u\Big(\frac{\ell}{2|x|}x\Big) - u_B\Big|\Big)\,dy\,dx \\
&\le \int_{\ell/2}^{L} \int_{S^{n-1}(r)} F\Big(\frac{1}{2\ell}\Big|u\Big(\frac{\ell}{2|x|}x\Big) - u_B\Big|\Big)\,dm_{n-1}(x)\,dr \\
&= \int_{\ell/2}^{L} \int_{S^{n-1}(\ell/2)} F\Big(\frac{1}{2\ell}|u(z) - u_B|\Big)\frac{r^{n-1}}{(\ell/2)^{n-1}}\,dm_{n-1}(z)\,dr \\
&\le c(n)\Big(\frac{L}{\ell}\Big)^{n-1} L \int_{S^{n-1}(\ell/2)} F\Big(\frac{1}{2\ell}|u(z) - u_B|\Big)\,dm_{n-1}(z) \\
&\le c(n)\Big(\frac{L}{\ell}\Big)^n \int_D F\big(|\nabla u(x)|\big)\,dx.
\end{aligned} \tag{5.9}$$

Estimates (5.4)–(5.9) and (3.3), where $G = D$, together yield the inequality (1.1) with $b = \frac{1}{18}$.

6. **Further remarks.** We need an additional assumption of F to get $b = 1$ in inequality (1.1) for more general domains than convex domains. In this case, a variation of inequality (1.1) is the following one which was studied by Bhattacharya and Leonetti

$$\int_D F\Big(\frac{|u(x) - u_D|}{\operatorname{dia}(D)}\Big)\,dx \le K_F \int_D F\big(|\nabla u(x)|\big)\,dx, \tag{6.1}$$

where D is a bounded domain in R^n, u is a function from an appropriate Sobolev space, $F\colon [0,\infty) \longrightarrow [0,\infty)$ is a convex, continuous function satisfying the Δ_2-condition, and $F(0) = 0$. Here constant K_F depends at most on F and D. By the Δ_2-condition we mean that there is a constant τ_F such that $F(2x) \le \tau_F F(x)$ for all $x > 0$.

Then Theorems 4.1 and 5.1 read as

THEOREM 6.2. *An (α,β)-John domain in R^n satisfies the inequality (6.1) with $K_F = c(n)(\frac{\beta}{\alpha})^{n+1}\tau_F^{\eta}$; here $\eta = \eta(\frac{\alpha}{\beta}) < 0$.*

THEOREM 6.3. *A starshaped domain in R^n satisfies inequality (6.1) with a constant $K_F = \tau_F^{-5}K\big(n, d(x_0,\partial D), \max_{x\in\partial D} d(x,x_0)\big)$; here K is a constant from Theorem 5.1.*

The proofs for Theorems 6.2 and 6.3 are essentially the same as the proofs for Theorems 4.1 and 5.1.

6.4. REMARK. Let $F_i\colon [0,\infty) \longrightarrow [0,\infty)$, $i = 1,2$, be continuous functions with constants c_1 and c_2 such that the inequalities $c_1F_1(x) \leq F_2(x) \leq c_2F_1(x)$ hold for all $x \in [0,\infty)$. If F_1 is a convex function and $F_1(0) = 0$, then D is an F_2-Poincaré domain whenever D is an F_1-Poincaré domain in the sense of (1.1). Further if F_1 satisfies the Δ_2-condition and D satisfies (6.1) with F_1, then D is an F_2-Poincaré domain in the sense of (6.1).

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