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CLOSURE OF LEAVES IN TRANSVERSELY AFFINE FOLIATIONS

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ABSTRACT. We present first examples of complete transversely affine foliations on compact manifolds with leaves whose closures are not submanifolds. Moreover, we prove that under some additional assumptions the closures of leaves form a singular foliation.

Transversely affine foliations (TAF for short) form a very interesting class of foliations, although relatively little studied, cf. [6]. They appear naturally in many situations, as for example in the study of Anosov diffeomorphisms, cf. [14]. TAF admit a transversely projectable connection and, therefore, in some respects, they are similar to Riemannian foliations. If we assume that a TAF $\mathcal{F}$ is transversely geodesically complete (i.e. for some supplementary subbundle Q geodesics of the transversely projectable connection tangent to Q are global) then $\mathcal{F}$ has the following properties which are well-known for Riemannian foliations, cf. [24,25] for TAF case:

1. leaves of $\mathcal{F}$ have the common universal covering space, cf. [17,12,22];
2. the graph of $\mathcal{F}$ is a locally trivial fibre bundle, cf. [23];
3. if the bundle Q is integrable, then the universal covering $\tilde{M}$ of M is the product $\tilde{L} \times \tilde{G}$ where $\tilde{L}$ and $\tilde{G}$ are the common universal coverings of leaves of $\mathcal{F}$ and Q, respectively, and the lifted foliation $\tilde{\mathcal{F}}$ of $\tilde{M}$ is given by the projection onto the second factor, cf. [2].

Of course this list does not exhaust similarities between TA and Riemannian foliations. In fact, these three properties are common to a much larger class of foliations admitting a foliated system of ordinary differential equations, cf. [24]. However, when we look at the closures of leaves we find the first main difference. In Riemannian foliations the closures of leaves form a singular foliation, cf. [12]. In TAF it is not the case. Examples of noncomplete TAF on compact manifolds and of complete ones on noncompact manifolds have been well-known, cf. the 1-dimensional Hopf foliation of $S^2 \times S^1$. However, even in transversely geodesically complete TAF leaves can behave very strangely. The following example is due to E. Ghys, cf. [4].

EXAMPLE 1. Take the matrix $a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. By the same letter we denote the diffeomorphism of the 2-torus induced by A. The suspension of the diffeomorphism A defines a 1-dimensional TAF $\mathcal{F}_A$ of the toral bundle T^3_A over S^1. The leaves of $\mathcal{F}_A$ correspond

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to orbits of A on T^2. Thus the leaves corresponding to points with the first coefficient rational are compact, the closures of other leaves are 2-tori. Hence the closures of leaves of $\mathcal{F}_A$, although submanifolds, they do not form a singular foliation.

Using the same suspension procedure we can construct transversely geodesically complete TAF on compact manifolds in which the closures of leaves are not necessarily submanifolds.

EXAMPLE 2. Let us consider a linear Anosov diffeomorphism A of a q-torus T^q. We would like to impose some additional conditions on A which would ensure that the foliation $\mathcal{F}_A$ obtained by suspending A has some leaves whose closures are not submanifolds. First, according to a result of M. Hirsch this cannot occur if $q = 2$, cf. [8]. Let us choose an irreducible primitive matrix $A \in SL(q, \mathbb{Z})$, $q > 2$, cf. [3]. Assume that the closure of an orbit of such a linear Anosov diffeomorphism is a submanifold. By passing to a finite covering, which corresponds to the suspension of the diffeomorphism A^k for some $k > 0$, we can assume that the closure of the orbit is a connected submanifold. Then Theorem A of [9] ensures that it must be a torus, but from the Proposition of [7] it results that our submanifold is either the torus T^q or a point. S. G. Hancock and F. Przytycki constructed very complicated invariant subsets for any linear Anosov diffeomorphism, cf. [7,16]. Thus the closure of any nonperiodic orbit contained in such an invariant subset cannot be a submanifold.

Having shown that on compact manifolds there exist transversely geodesically complete TAF with leaves whose closures are not submanifolds we would like to find out whether under some additional assumptions it is possible to demonstrate that the closures of leaves are submanifolds. First, we must reduce the study of the closures of leaves to the study of some more manageable objects.

A TAF $\mathcal{F}$ on a manifold M is developable, i.e. there exist a covering $\hat{M}$ of M, a representation $\alpha: \pi_1(M) \longrightarrow \mathrm{Aff}(\mathbf{E})$ ($\mathbf{E}$ q-dimensional affine space) and a $\pi_1(M)$-equivariant global submersion $D: \hat{M} \longrightarrow \mathbf{E}$ defining the lifted foliation $\hat{\mathcal{F}}$. If $\mathcal{F}$ is transversely geodesically complete, then $\hat{M} = \tilde{L} \times \mathbf{E}$ and D is the projection onto the second factor (i.e. $\mathcal{F}$ is complete), cf. [24]. The group $im\alpha = \Gamma \subset \mathrm{Aff}(\mathbf{E})$ is called the *affine holonomy group* and can be identified with the group of deck transformations of $\hat{M}$. Leaves of $\mathcal{F}$ correspond to orbits of Γ on $\mathbf{E}$. Let L be a leaf of $\mathcal{F}$ and Γv ($v \in \mathbf{E}$) the corresponding orbit of Γ. Then $\bar{L} = D^{-1}(\overline{\Gamma v}) \backslash \Gamma$. This equality leads to the following lemma.

LEMMA 1. *Let $\mathcal{F}$ be a complete TAF. Then the closure of a leaf L is a submanifold iff the closure of the corresponding orbit of the affine holonomy group is a submanifold.*

Thus we can concentrate our attention on the study of orbits of finitely generated subgroups of $\mathrm{Aff}(\mathbf{E})$. In [26] we have proved that for a complete TAF the algebraic closure $A(\Gamma)$ of the affine holonomy group Γ must act transitively on $\mathbf{E}$, i.e., $\mathbf{E}$ can be considered as a homogeneous space $A(\Gamma)/H$, where H is an isotropy group of the natural representation of $A(\Gamma)$ on $\mathbf{E}$. Then we have:

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LEMMA 2. *The closure of an orbit Γv is a submanifold of* $\mathbf{E}$ *iff the set* $\overline{\Gamma \cdot H}$ *is a submanifold of* $A(\Gamma)$ *where* H *is the isotropy group of* $A(\Gamma)$ *at* v.

PROOF. Consider $\mathbf{E}$ as the homogeneous space $A(\Gamma)\backslash H$. Then the orbit Γv corresponds to the orbit Γe_H where $e_H = eH \in A(\Gamma)\backslash H$. The closure of Γe_H in $A(\Gamma)\backslash H$ is equal to $\overline{\Gamma \cdot H}\backslash H$. Thus it is a submanifold iff $\overline{\Gamma \cdot H}$ is a submanifold. ∎

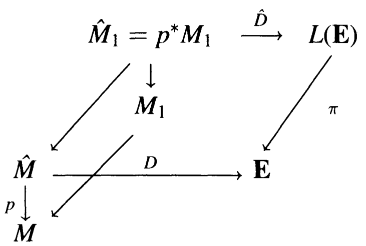
These lemmas lead to the following theorem.

THEOREM 1. *Let $\mathcal{F}$ be a complete TAF on a compact manifold M with abelian fundamental group. Then the closures of leaves of $\mathcal{F}$ form a singular foliation.*

PROOF. The affine holonomy group Γ of $\mathcal{F}$ is abelian, so is its algebraic closure $A(\Gamma)$. Since $\mathcal{F}$ is complete, the group $A(\Gamma)$ acts transitively on $\mathbf{E}$. As $A(\Gamma)$ is abelian, the isotropy groups of the representation of $A(\Gamma)$ on $\mathbf{E}$ are equal. We denote it by H. Lemma 2 ensures that the closures of orbits of Γ are the orbits of the group $\overline{H \cdot \Gamma} = H(\Gamma)$ which is a Lie subgroup of $A(\Gamma) \subset \mathrm{Aff}(\mathbf{E})$. Thus these orbits are submanifolds and, therefore, the closures of leaves are submanifolds as well. Elements of the Lie algebra of $H(\Gamma)$ define vector fields which span the tangent space to the orbits of $H(\Gamma)$. As they are Γ-invariant, these vector fields induce global foliated vector fields on M. Their foliated orbits are precisely the closures of leaves of $\mathcal{F}$. Thus, in fact, the closures of leaves form a singular foliation in the sense of Stefan, cf. [20,21]. ∎

Foliations with nilpotent affine holonomy group form another more general and very interesting class of TAF. Before formulating and proving a theorem for these foliations we must describe the commuting sheaf of a TAF.

The foliation $\mathcal{F}$ is a $\nabla - G$-foliation with ∇ being the canonical flat connection of $\mathbf{E}$. Thus we have the following commutative diagram:



where $M_1 = L(M, \mathcal{F})$ is the bundle of frames of the normal bundle of $\mathcal{F}$, i.e. the bundle of transverse frames of $(M, \mathcal{F})$. The bundle M_1 admits a canonical foliation $\mathcal{F}_1$ of the same dimension as $\mathcal{F}$ and whose leaves are coverings of leaves of $\mathcal{F}$. $\mathcal{F}_1$ is a developable foliation modelled on $L(\mathbf{E})$, the bundle of linear frames of $\mathbf{E}$; thus it is an $\mathrm{Aff}(\mathbf{E})$-Lie foliation.

In [27] we have defined the commuting sheaf of a $\nabla - G$-foliation. This definition is based on Molino's definition of the commuting sheaf of a Riemannian foliation, cf. [10,11,12]. We recall the definition and describe this sheaf in detail.

Let $\mathcal{C}_1$ be the sheaf of germs of foliated vector fields which commute with all global foliated vector fields of $(M_1, \mathcal{F}_1)$. As these vector fields must commute with the fundamental horizontal and vertical vector fields, locally, they are lifts of local infinitesimal transformations of the transversely projectable flat connection of $(M, \mathcal{F})$. Therefore $\mathcal{C}_1$ defines a sheaf $\mathcal{C}$ of germs of foliated vector fields on $(M, \mathcal{F})$. We call $\mathcal{C}$ the commuting sheaf of $\mathcal{F}$. Its stalks consist of germs of local foliated infinitesimal affine transformations of the transversely projectable flat connection and the Lie bracket endows them with the Lie algebra structure. The lift $\hat{\mathcal{C}}$ of the sheaf $\mathcal{C}$ to $\hat{M}$ consists of germs of foliated vector fields of $(\hat{M}, \hat{\mathcal{F}})$ whose lifts to $\hat{M}_1$, forming a sheaf $\hat{\mathcal{C}}_1$, commute with all Γ-invariant global foliated vector fields. This sheaf $\hat{\mathcal{C}}_1$ projects to a sheaf $\mathcal{C}_A$ on $\mathrm{Aff}(\mathbf{E})$ whose elements commute with all global (left) Γ-invariant vector fields on $\mathrm{Aff}(\mathbf{E})$, and thus with all $K = \bar{\Gamma}$-invariant vector fields. This means that the vector fields of the sheaf $\mathcal{C}_A$ must be tangent to the fibres of the K-fibre bundle $\mathrm{Aff}(\mathbf{E}) \longrightarrow K\backslash \mathrm{Aff}(\mathbf{E}) = W$. Locally, this bundle is of the form $K \times U \longrightarrow U$. Therefore the vector fields of $\mathcal{C}_A$ must commute with vector fields of the form $\sum f_i k_i$ where $k_i \in \mathbf{k}, f_i \in C^\infty(W)$ and $\mathrm{supp} f_i \subset U$. Thus, if $\mathcal{F}$ is complete, each stalk of $\mathcal{C}_A$ is isomorphic to the conjugated algebra k^-. We call this Lie algebra the structure algebra of the TAF $\mathcal{F}$. We have proved the following.

PROPOSITION 1. *Let $\mathcal{F}$ be a complete TAF with the affine holonomy group Γ. Then its commuting sheaf is a locally constant sheaf of Lie algebras whose stalk is isomorphic to the conjugated Lie algebra of Lie(K), $K = \bar{\Gamma} \subset \mathrm{Aff}(\mathbf{E})$.*

For more properties of the commuting sheaf and its relation to the closures of leaves see [27].

Now we can prove the following:

THEOREM 2. *Let $\mathcal{F}$ be a complete TAF of a compact manifold with nilpotent affine holonomy group Γ. If the group $K = \bar{\Gamma}$ has a finite number of components, then the closures of leaves form a singular foliation and they are the orbits of the commuting sheaf.*

PROOF. In [26] we have proved that the group Γ must be unipotent. Thus the connected component K_0 of K is an algebraic group, and according to [18] the orbits of K_0 are closed. Since K has a finite number of connected components, its orbits are closed, and thus equal to the closures of orbits of Γ. Therefore the closures of leaves are submanifolds. The description of the commuting sheaf ensures that these closures are the orbits of this sheaf. Hence the closures of leaves of $\mathcal{F}$ form a singular foliation. ∎

Example 1 shows that Theorem 2 is false if the group K has an infinite number of connected components, but we can still hope that the closures of leaves are submanifolds. We have seen that in the same example the space of closures of leaves has been a very irregular topological space. If we impose some separability condition on the space of orbits of the group K we can relax our other assumptions a little.

PROPOSITION 2. *Let $\mathcal{F}$ be a complete TAF on a compact manifold. If*

a) the affine holonomy group Γ is distal;

b) the group $K = \bar{\Gamma}$ *has a finite number of connected components;*
c) the space $K \backslash \mathbf{E}$ *is* T_0,

then the closures of leaves are the orbits of the commuting sheaf of $\mathcal{F}$ *and they form a singular foliation.*

PROOF. Glimm's theorem, cf. [5], ensures that orbits of K are relatively open in their closures. On the other hand these closures are minimal, cf. [13]. Therefore the orbits of K must be closed. The rest follows as in the proof of Theorem 2. ∎

To complete this short note we give an example of a TAF having a solvable affine holonomy group and with some closures of leaves not being submanifolds.

EXAMPLE 3. As in the previous examples we suspend an Anosov diffeomorphism; this time of a nontoral nilmanifold, cf. [19,1].

Let H be a 3-dimensional real Heisenberg group, and let $G = H \times H$. The group G is diffeomorphic to $\mathbb{R}^6$. G admits the following uniform subgroup Γ_0:

$$\Gamma_0 = \{(a_1, \ldots, a_6) \in \mathbb{R}^6 : a_i \in \mathbb{Z}(\sqrt{3}) \text{ and } a_{i+3} = \bar{a}_i, i = 1, 2, 3\}$$

where if $a = m + n\sqrt{3}$, $m, n \in \mathbb{Z}$, then $\bar{a} = m - n\sqrt{3}$. The space G/Γ is a compact nontoral nilmanifold. Let $\lambda = 2 + \sqrt{3}$, $\nu = (2 - \sqrt{3})^2$, $\mu = \lambda\nu = 2 - \sqrt{3}$. Then the transformation $\phi : \mathbb{R}^6 \longrightarrow \mathbb{R}^6$,

$$\phi(x_1, \ldots, x_6) = (\lambda x_1, \mu x_2, \nu x_3, \bar{\lambda} x_4, \bar{\mu} x_5, \bar{\nu} x_6)$$

preserves the lattice Γ_0 and, therefore, defines an Anosov diffeomorphism of G/Γ_0. The suspension of ϕ defines a 1-dimensional TAF $\mathcal{F}_\phi$ on the total space of $\mathbb{R} \times_\phi G/\Gamma_0$. The affine holonomy group Γ of $\mathcal{F}_\phi$ is the subgroup of Aff($\mathbb{R}^6$) generated by the group Γ_0 and ϕ. It is a solvable group. It is not difficult to verify that the closure of the Γ-orbit of the point $(x, 0, \ldots, 0)$, $x \neq 0$, is not a submanifold. Thus, indeed, the foliation $\mathcal{F}_\phi$ has the property we have been looking for.

Other examples of this kind can be constructed using the examples of H. L. Porteous, cf. [15]. Moreover, E. Ghys has informed the author that similiar examples can be constructed using the work of M. Morse on dynamics on tori, which is, of course, previous to the results of S. G. Hancock and F. Przytycki.

REFERENCES

1. T. F. Banchoff, M. I. Rosen, *Periodic points of Anosov diffeomorphisms*, Global Analysis, Proc. Symp. Pure Math. **14**(1970), 17–21.
2. R. A. Blumenthal, J. J. Hebda, *Ehresmann connections for foliations*, Indiana Un. Math. J. (4)**33**(1984), 597–611.
3. F. R. Gantmacher, *The theory of matrices, vol. 1, 2*. Chelsea Publ., New York, 1959.
4. E. Ghys, *Un feuilletage analytique dont la cohomologie basique est de dimension infinie*, Publ. IRMA Lille (1)**VII**(1985).
5. J. Glimm, *Locally compact transformation groups*, Trans. A.M.S. **101**(1961), 124–138.
6. W. Goldman, M. Hirsch, G. Levitt, *Invariant measures for affine foliations*, Proc. A.M.S. **86**(1982), 511–518.

https://doi.org/10.4153/CMB-1991-087-8 Published online by Cambridge University Press

7. S. G. Hancock, *Construction of invariant sets for Anosov diffeomorphisms*, J. London Math. Soc. **18**(1978), 339–348.

8. M. W. Hirsch, *On invariant sets of Anosov diffeomorphisms*. Essays on Topology and Related Topics, eds. A. Haefliger and R. Narasimhan, Springer, 1970, 126–135.

9. R. Mañé, *Invariant sets of Anosov diffeomorphisms*, Inventiones Math. **46**(1978), 147–152.

10. P. Molino, *Feuilletages riemanniens sur les variétés compactes; champs de Killing transverses*, C.R. Acad. Sci. Paris **289**(1979), 421–423.

11. ———, *Géométrie globale des feuilletages riemanniens*, Proc. Kon. Nederl. Acad. **85**(1982), 45–76.

12. ———, *Riemannian foliations*, Progress in Math. **73**, Birkhäuser, 1988.

13. C. C. Moore, *Distal affine transformation groups*, Amer. J. Math. **90**(1968), 733–751.

14. J. Plante, *Anosow flows, transversely affine foliations and conjecture of Verjovsky*, J. London Math. Soc. **23**(1981), 359–362.

15. H. L. Porteous, *Anosov diffeomorphisms of flat manifolds*, Topology **11**(1972), 307–315.

16. F. Przytycki, *Construction of invariant sets for Anosov diffeomorphisms and hyperbolic attractors*, Studia Math. **68**(1980), 199–213.

17. B. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. of Math. **69**(1959), 119–132.

18. M. Rosenlicht, *On quotient varieties and the affine embedding of certain homogeneous spaces*, Trans. A.M.S. **101**(1961), 211–223.

19. S. Smale, *Differentiable dynamical systems*, Bull. A.M.S. **73**(1967), 747–817.

20. P. Stefan, *Accessible sets, orbits and foliations with singularities*, Proc. London Math. Soc. **29**(1974), 699–713.

21. H. Sussmann, *Orbits of vector fields and integrability of distributions*, Trans. A.M.S. **180**(1973), 171–188.

22. Ph. Tondeur, *Foliations on Riemannian manifolds*. Springer-Verlag, 1988.

23. H. Winkelnkemper, *The graph of a foliation*, Ann. Glob. Anal. Geom. **1**(1983), 51–75.

24. R. A. Wolak, *Foliations admitting transverse systems of differential equations*, Comp. Math. **67**(1988), 89–101.

25. ———, *Le graphe d'un feuilletage admettant un système d'équations différentielles transverse*, Math. Z. **201**(1989), 177–182.

26. ———, *Transversely affine foliations compared with affine manifolds*, Quart. J. Math (Oxford) **41**(1990), 369–384.

27. ———, *Foliated G-structures and Riemannian foliations*, Manuscripta Math. **66**(1989), 45–59.

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