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Homogeneity of the Pure State Space of a Separable C^*-Algebra

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Abstract. We prove that the pure state space is homogeneous under the action of the automorphism group (or the subgroup of asymptotically inner automorphisms) for all the separable simple C^*-algebras. The first result of this kind was shown by Powers for the UHF algbras some 30 years ago.

1 Introduction

If A is a C^*-algebra, an automorphism α of A is *asymptotically inner* if there is a continuous family $(u_t)_{t\in[0,\infty)}$ in the group $\mathcal{U}(A)$ of unitaries in A (or $A + \mathbf{C}1$ if A is non-unital) such that $\alpha = \lim_{t\to\infty} \operatorname{Ad} u_t$; we denote by $\operatorname{AInn}(A)$ the group of asymptotically inner automorphisms of A, which is a normal subgroup of the group of approximately inner automorphisms. Note that each $\alpha \in \operatorname{AInn}(A)$ leaves each (closed two-sided) ideal of A invariant. It is shown, in [11], [1], [3], for a large class of separable C^*-algebras that if ω_1 and ω_2 are pure states of A such that the GNS representations associated with ω_1 and ω_2 have the same kernel, then there is an $\alpha \in \operatorname{AInn}(A)$ such that $\omega_1 = \omega_2\alpha$. We shall show in this paper that this is the case for *all* the separable C^*-algebras; formally, denoting by π_ω the GNS representation associated with a state ω, we state:

Theorem 1.1 *Let A be a separable C^*-algebra. If ω_1 and ω_2 are pure states of A such that* $\ker \pi_{\omega_1} = \ker \pi_{\omega_2}$, *then there is an* $\alpha \in \operatorname{AInn}(A)$ *such that* $\omega_1\alpha = \omega_2$.

In particular the pure state space of a separable simple C^*-algebra A is homogeneous under the action of $\operatorname{AInn}(A)$. We need the separability for this statement to be true even if we replace $\operatorname{AInn}(A)$ by the full automorphism group $\operatorname{Aut}(A)$ (see 2.3). But if we instead assume that A is nuclear, the situation is unclear, *i.e.*, we do not know if the pure state space of a non-separable simple nuclear C^*-algebra is homogeneous under the action of $\operatorname{Aut}(A)$ or not. See [2] for some problems on this.

We note here that $\operatorname{AInn}(A)$ can be considered as a *core* of $\operatorname{Aut}(A)$ whose inner structure is beyond algebraic grasp; $\operatorname{AInn}(A)$ is characterized as the subgroup of automorphisms which have the same KK class with the identity automorphism for the class of purely infinite simple separable C^*-algebras classified by Kirchberg and Phillips [7] (see [9] for a similar result for a class of AT algebras).

The proof of the above theorem comprises three observations taken from [3] and [5]. By combining these, the theorem will follow immediately.

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The first observation from [3] is that the following property for a C^*-algebra A will imply the above theorem.

Property 1.2 For any finite subset $\mathcal{F}$ of A, any pure state ω of A with $\pi_\omega(A) \cap \mathcal{K}(\mathcal{H}_\omega) = (0)$, and $\epsilon > 0$, there exist a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: If φ is a pure state of A such that π_φ is quasi-equivalent to π_ω, and

$$|\varphi(x) - \omega(x)| < \delta, \quad x \in \mathcal{G},$$

then there is a continuous path $(u_t)_{t\in[0,1]}$ in $\mathcal{U}(A)$ such that $u_0 = 1$, $\varphi = \omega \operatorname{Ad} u_1$, and

$$\| \operatorname{Ad} u_t(x) - x\| < \epsilon, \quad x \in \mathcal{F},\ t \in [0,1].$$

In the above statement, $\mathcal{K}(\mathcal{H}_\omega)$ is the C^*-algebra of compact operators on $\mathcal{H}_\omega$, the Hilbert space for π_ω.

Another observation from [3] is that the following property of A, a kind of weak amenability, implies the above property:

Property 1.3 Let $\mathcal{F}$ be a finite subset of A, π an irreducible representation of A on a Hilbert space $\mathcal{H}$, E a finite-dimensional projection on $\mathcal{H}$, and $\epsilon > 0$. Then there exists an $x = (x_1, x_2, \dots, x_n) \in M_{1n}(A)$ for some n such that $\|xx^*\| \le 1$, $\pi(xx^*)E = E$, and $\| \operatorname{ad} a \operatorname{Ad} x\| < \epsilon$ for all $a \in \mathcal{F}$, where $\operatorname{Ad} x$ and $\operatorname{ad} a$ denote the linear maps on A defined by $b \mapsto xbx^* = \sum x_i b x_i^*$ and $b \mapsto [a, b]$ respectively.

Here $M_{mn}(A)$ denotes the $m \times n$ matrices over A.

The final observation, from Haagerup [5], is that this property holds for *all* C^*-algebras, which is shown by repeating, almost verbatim, the proof of 3.1 of [5] employed for verifying the statement that all nuclear C^*-algebras are amenable.

Although those observations are mostly immediate from the cited references if once properly formulated as above, we shall outline the proofs for the reader's convenience: 1.2 implies 1.1 in Section 2, 1.3 implies 1.2 in Section 3, and Property 1.3 is universal in Section 4.

The present method is further exploited in connection with one-parameter automorphism groups [8] and for type III representations [4].

2 Homogeneity

We denote by $\operatorname{AInn}_0(A)$ the set of $\alpha \in \operatorname{AInn}(A)$ which has a continuous family $(u_t)_{t\in[0,\infty)}$ in $\mathcal{U}(A)$ with $u_0 = 1$ and $\alpha = \lim_{t\to\infty} \operatorname{Ad} u_t$.

Theorem 2.1 *Let A be a separable C^*-algebra satisfying Property 1.2. If ω_1 and ω_2 are pure states of A such that $\ker \pi_{\omega_1} = \ker \pi_{\omega_2}$, then there is an $\alpha \in \operatorname{AInn}_0(A)$ such that $\omega_1 = \omega_2 \alpha$.*

The following gives a slightly weaker version of Property 1.2.

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Lemma 2.2 *Let A be a C^*-algebra with Property 1.2. Then for any finite subset $\mathcal{F}$ of A, any pure state ω of A with $\pi_\omega(A) \cap \mathcal{K}(\mathcal{H}_\omega) = (0)$, and $\epsilon > 0$, there exist a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: If φ is a pure state of A such that $\ker \pi_\varphi = \ker \pi_\omega$, and*

$$|\varphi(x) - \omega(x)| < \delta, \quad x \in \mathcal{G},$$

then for any finite subset $\mathcal{F}'$ of A and $\epsilon' > 0$ there is a continuous path $(u_t)_{t\in[0,1]}$ in $\mathcal{U}(A)$ such that $u_0 = 1$, and

$$|\varphi(x) - \omega \operatorname{Ad} u_1(x)| < \epsilon', \quad x \in \mathcal{F}',$$

$$\| \operatorname{Ad} u_t(x) - x\| < \epsilon, \quad x \in \mathcal{F}.$$

Proof Given $(\mathcal{F}, \omega, \epsilon)$, choose $(\mathcal{G}, \delta)$ as in Property 1.2. Let φ be a pure state of A such that $\ker \pi_\varphi = \ker \pi_\omega$ and

$$|\varphi(x) - \omega(x)| < \delta/2, \quad x \in \mathcal{G}.$$

Let $\mathcal{F}'$ be a finite subset of A and $\epsilon' > 0$ with $\epsilon' < \delta/2$. We can mimic φ as a vector state through π_ω; by Kadison's transitivity there is a $v \in \mathcal{U}(A)$ such that

$$|\varphi(x) - \omega \operatorname{Ad} v(x)| < \epsilon', \quad x \in \mathcal{F}' \cup \mathcal{G},$$

(see 2.3 of [3]). Since $|\omega \operatorname{Ad} v(x) - \omega(x)| < \delta$, $x \in \mathcal{G}$, we have, by applying Property 1.2 to the pair ω and $\omega \operatorname{Ad} v$, a continuous path (u_t) in $\mathcal{U}(A)$ such that $u_0 = 1$, and

$$\omega \operatorname{Ad} v = \omega \operatorname{Ad} u_1,$$

$$\| \operatorname{Ad} u_t(x) - x\| < \epsilon, \quad x \in \mathcal{F}.$$

Since $|\varphi(x) - \omega \operatorname{Ad} u_1(x)| < \epsilon'$, $x \in \mathcal{F}'$, this completes the proof. ∎

We shall now turn to the proof of Theorem 2.1.

Once we have Lemma 2.2, we can prove this in the same way as 2.5 of [3]. We shall only give an outline here.

Let ω_1 and ω_2 be pure states of A such that $\ker \pi_{\omega_1} = \ker \pi_{\omega_2}$.

If $\pi_{\omega_1}(A) \cap \mathcal{K}(\mathcal{H}_{\omega_1}) \neq (0)$, then $\pi_{\omega_1}(A) \supset \mathcal{K}(\mathcal{H}_{\omega_1})$ and π_{ω_1} is equivalent to π_{ω_2}. Then by Kadison's transitivity (see, *e.g.*, 1.21.16 of [12]), there is a continuous path (u_t) in $\mathcal{U}(A)$ such that $u_0 = 1$ and $\omega_1 = \omega_2 \operatorname{Ad} u_1$.

Suppose that $\pi_{\omega_1}(A) \cap \mathcal{K}(\mathcal{H}_{\omega_1}) = (0)$, which also implies that $\pi_{\omega_2}(A) \cap \mathcal{K}(\mathcal{H}_{\omega_2}) = (0)$.

Let (x_n) be a dense sequence in A.

Let $\mathcal{F}_1 = \{x_1\}$ and $\epsilon > 0$ (or $\epsilon = 1$). Let $(\mathcal{G}_1, \delta_1)$ be the $(\mathcal{G}, \delta)$ for $(\mathcal{F}_1, \omega_1, \epsilon/2)$ as in Lemma 2.2 such that $\mathcal{G}_1 \supset \mathcal{F}_1$. For this $(\mathcal{G}_1, \delta_1)$ we choose a continuous path (u_{1t}) in $\mathcal{U}(A)$ such that $u_{1,0} = 1$ and

$$|\omega_1(x) - \omega_2 \operatorname{Ad} u_{1,1}(x)| < \delta_1, \quad x \in \mathcal{G}_1.$$

Let $\mathcal{F}_2 = \{x_i, \operatorname{Ad} u_{1,1}^*(x_i) \mid i = 1, 2\}$ and let $(\mathcal{G}_2, \delta_2)$ be the $(\mathcal{G}, \delta)$ for $(\mathcal{F}_2, \omega_2 \operatorname{Ad} u_{1,1}, 2^{-2}\epsilon)$ as in Lemma 2.2 such that $\mathcal{G}_2 \supset \mathcal{G}_1 \cup \mathcal{F}_2$ and $\delta_2 < \delta_1/2$. By 2.2 there is a continuous path (u_{2t}) in $\mathcal{U}(A)$ such that $u_{2,0} = 1$ and

$$\|\operatorname{Ad} u_{2t}(x) - x\| < 2^{-1}\epsilon, \quad x \in \mathcal{F}_1,$$

$$|\omega_2 \operatorname{Ad} u_{1,1}(x) - \omega_1 \operatorname{Ad} u_{2,1}(x)| < \delta_2, \quad x \in \mathcal{G}_2.$$

Let $\mathcal{F}_3 = \{x_i, \operatorname{Ad} u_{2,1}^*(x_i) \mid i = 1, 2, 3\}$ and let $(\mathcal{G}_3, \delta_3)$ be the $(\mathcal{G}, \delta)$ for $(\mathcal{F}_3, \omega_1 \operatorname{Ad} u_{2,1}, 2^{-3}\epsilon)$ as in 2.2 such that $\mathcal{G}_3 \supset \mathcal{G}_2 \cup \mathcal{F}_3$ and $\delta_3 < \delta_2/2$. By 2.2 there is a continuous path (u_{3t}) in $\mathcal{U}(A)$ such that $u_{3,0} = 1$ and

$$\|\operatorname{Ad} u_{3t}(x) - x\| < 2^{-2}\epsilon, \quad x \in \mathcal{F}_2,$$

$$|\omega_1 \operatorname{Ad} u_{2,1}(x) - \omega_2 \operatorname{Ad}(u_{1,1}u_{3,1})(x)| < \delta_3, \quad x \in \mathcal{G}_3.$$

We shall repeat this process.

Assume that we have constructed $\mathcal{F}_n$, $\mathcal{G}_n$, δ_n, and $(u_{n,t})$ inductively. In particular if n is even, $\mathcal{F}_n$ is given as

$$\{x_i, \operatorname{Ad}(u_{n-1,1}^* u_{n-3,1}^* \cdots u_{1,1}^*)(x_i) \mid i = 1, 2, \dots, n\}$$

and (G_n, δ_n) is the $(\mathcal{G}, \delta)$ for $\bigl(\mathcal{F}_n, \omega_2 \operatorname{Ad}(u_{1,1}u_{3,1} \cdots u_{n-1,1}), 2^{-n}\epsilon\bigr)$ as in 2.2 such that $\mathcal{G}_n \supset \mathcal{G}_{n-1} \cup \mathcal{F}_n$ and $\delta_n < \delta_{n-1}/2$. And $(u_{n,t})$ is given by 2.2 for $\bigl(\mathcal{F}_{n-1}, \omega_1 \operatorname{Ad}(u_{2,1} \cdots u_{n-2,1}), 2^{-n+1}\epsilon\bigr)$ and for $\mathcal{F}' = \mathcal{G}_n$ and $\epsilon' = \delta_n$ and it satisfies

$$\|\operatorname{Ad} u_{nt}(x) - x\| < 2^{-n+1}\epsilon, \quad x \in \mathcal{F}_{n-1},$$

$$|\omega_1 \operatorname{Ad}(u_{2,1}u_{4,1} \cdots u_{n,1})(x) - \omega_2 \operatorname{Ad}(u_{1,1}u_{3,1} \cdots u_{n-1,1})(x)| < \delta_n, \quad x \in \mathcal{G}_n.$$

We define continuous paths (v_t) and (w_t) in $\mathcal{U}(A)$ with $t \in [0, \infty)$ by: For $t \in [n, n+1]$

$$v_t = u_{1,1}u_{3,1} \cdots u_{2n-1,1}u_{2n+1,t-n},$$

$$w_t = u_{2,1}u_{4,1} \cdots u_{2n-2,1}u_{2n+2,t-n}.$$

Then, since $\|\operatorname{Ad} u_{nt}(x) - x\| < 2^{-n+1}\epsilon$, $x \in \mathcal{F}_{n-1}$ and $\delta_n \to 0$, we can show that $\operatorname{Ad} v_t$ (resp. $\operatorname{Ad} w_t$) converges to an automorphism α (resp. β) as $t \to \infty$ and that $\omega_1\beta = \omega_2\alpha$. Since $\alpha, \beta \in \operatorname{AInn}_0(A)$ and $\operatorname{AInn}_0(A)$ is a group, this will complete the proof. See the proofs of 2.5 and 2.8 of [3] for details.

Remark 2.3 Let A be a factor of type II_1 or type III with separable predual A_*, which is a unital simple non-separable non-nuclear C^*-algebra. Then the pure state space of A is not homogeneous under the action of the automorphism group $\operatorname{Aut}(A)$ of A.

This is shown as follows. Since A contains a C^*-subalgebra isomorphic to $C_b(\mathbf{N}) \equiv C(\beta\mathbf{N})$ and $\beta\mathbf{N}$ has cardinality 2^c, the pure state space of A has cardinality (at least) 2^c, where c denotes the cardinality of the continuum. (We owe this argument to J. Anderson.) On the other hand any $\alpha \in \operatorname{Aut}(A)$ corresponds to an isometry on the predual A_*, a separable Banach space. Thus, since the set of bounded operators on a separable Banach space has cardinality c, $\operatorname{Aut}(A)$ has cardinality (at most) c. Hence the pure state space of A cannot be homogeneous under the action of $\operatorname{Aut}(A)$.

3 1.3 implies 1.2

Theorem 3.1 *Any C^*-algebra with Property 1.3 has Property 1.2.*

Proof Let $\mathcal{F}$ be a finite subset of A, ω a pure state of A with $\pi_\omega(A) \cap \mathcal{K}(\mathcal{H}_\omega) = (0)$, and $\epsilon > 0$. For $\pi = \pi_\omega$ and the projection E onto the subspace $\mathbf{C}\Omega_\omega$, we choose an $x \in M_{1n}(A)$ for some n as in Property 1.3, *i.e.*, $\|x\| \le 1$, $\pi(xx^*)\Omega_\omega = \Omega_\omega$ with $\Omega = \Omega_\omega$, and $\|\operatorname{ad} a \operatorname{Ad} x\| < \epsilon$ for all $a \in \mathcal{F}$.

Let

$$\mathcal{G} = \{x_i x_j^* \mid i, j = 1, 2, \dots, n\},$$

which will be the subset $\mathcal{G}$ required in Property 1.2. We will choose $\delta > 0$ sufficiently small later. Suppose that we are given a unit vector $\eta \in \mathcal{H}_\omega$ satisfying

$$|\langle \pi(x_i^*)\eta, \pi(x_j^*)\eta\rangle - \langle \pi(x_i^*)\Omega, \pi(x_j^*)\Omega\rangle| < \delta$$

for any $i, j = 1, 2, \dots, n$, where $\Omega = \Omega_\omega$. Note that

$$\sum_{j=1}^{n} \|\pi(x_j^*)\Omega\|^2 = \langle \pi(xx^*)\Omega, \Omega\rangle = 1,$$

which implies, in particular, that $|\langle \pi(xx^*)\eta, \eta\rangle - 1| < n\delta$. Thus the two finite sets of vectors $S_\Omega = \{\pi(x_i^*)\Omega \mid i = 1, \dots, n\}$ and $S_\eta = \{\pi(x_i^*)\eta \mid i = 1, \dots, n\}$ have similar geometric properties in $\mathcal{H}_\omega$ if δ is sufficiently small. Hence we are in a situation where we can apply 3.3 of [3].

Let us describe how we proceed from here in a simplified case. Suppose that the linear span $\mathcal{L}_\Omega$ of S_Ω is orthogonal to the linear span $\mathcal{L}_\eta$ of S_η and that the map $\pi(x_i^*)\Omega \mapsto \pi(x_i^*)\eta$ and $\pi(x_i^*)\eta \mapsto \pi(x_i^*)\Omega$ extends to a unitary U on $\mathcal{L}_\Omega + \mathcal{L}_\eta$; in particular we have assumed that $\langle \pi(x_i^*)\eta, \pi(x_j^*)\eta\rangle = \langle \pi(x_i^*)\Omega, \pi(x_j^*)\Omega\rangle$ for all i, j. Since U is a self-adjoint unitary, $F \equiv (1 - U)/2$ is a projection and satisfies that $e^{i\pi F} = U$ on the finite-dimensional subspace $\mathcal{L}_\Omega + \mathcal{L}_\eta$. By Kadison's transitivity we choose an $h \in A$ such that $0 \le h \le 1$ and $\pi(h)|\mathcal{L}_\Omega + \mathcal{L}_\eta = F$. We set $\overline{h} = \operatorname{Ad} x(h)$, which entails that $\|[a, \overline{h}]\| < \epsilon$, $a \in \mathcal{F}$. Then we have that

$$\begin{aligned}
\pi(\overline{h})(\Omega - \eta) &= \pi(xhx^*)(\Omega - \eta) \\
&= \sum \pi(x_i) F \pi(x_i^*)(\Omega - \eta), \\
&= \sum \pi(x_i)\pi(x_i^*)(\Omega - \eta) \\
&= \Omega - \eta
\end{aligned}$$

and that $\pi(\overline{h})(\Omega + \eta) = 0$. Hence it follows that

$$\pi(e^{i\pi\overline{h}})\Omega = \pi(e^{i\pi\overline{h}})(\Omega - \eta)/2 + \pi(e^{i\pi\overline{h}})(\Omega + \eta)/2 = -(\Omega - \eta)/2 + (\Omega + \eta)/2 = \eta.$$

Thus the path $(e^{it\pi\overline{h}})_{t \in [0,1]}$ is what is desired.

Whenever $\mathcal{L}_\Omega$ is orthogonal to $\mathcal{L}_\eta$, this argument can be made rigorous if $\delta > 0$ is sufficiently small. See [3] for details.

If $\mathcal{L}_\eta$ is not orthogonal to $\mathcal{L}_\Omega$, we still find a unit vector $\zeta \in \mathcal{H}_\omega$ such that

$$|\langle \pi(x_i^*)\zeta, \pi(x_j^*)\zeta\rangle - \langle \pi(x_i^*)\Omega, \pi(x_j^*)\Omega\rangle| < \delta$$

and such that $\mathcal{L}_\zeta$ is orthogonal to both $\mathcal{L}_\Omega$ and $\mathcal{L}_\eta$. Here we use the assumption that $\pi_\omega(A) \cap \mathcal{K}(\mathcal{H}_\omega) = (0)$. Then we combine the path of unitaries sending η to ζ and then the path sending ζ to Ω to obtain the desired path. ■

4 Property 1.3 is Universal

Let $\mathrm{Bil}(A)$ denote the bounded bilinear forms on a C^*-algebra A. We have a canonical isometric identification of $\mathrm{Bil}(A)$ with $(A \widehat{\otimes} A)^*$, which is given by

$$\langle V, a \otimes b\rangle = V(a, b).$$

Here $A \widehat{\otimes} A$ is the completion of the algebraic tensor product $A \otimes A$ equipped with the projective tensor norm:

$$\|S\|_\wedge = \inf\Big\{ \sum_{i=1}^{n} \|x_i\|\,\|y_i\| \Big\},$$

where the infimum is taken all over the possible representations $S = \sum_{i=1}^n x_i \otimes y_i$. For $a \in A$ the bounded linear maps L_a and R_a on $A \widehat{\otimes} A$ are defined by

$$L_a(x \otimes y) = ax \otimes y \quad \text{and} \quad R_a(x \otimes y) = x \otimes ya$$

and the bounded linear map $p \colon A \widehat{\otimes} A \to A$ is defined by

$$p(x \otimes y) = xy.$$

If $\mathcal{M}$ is a von Neumann algebra, $\mathrm{Bil}_\sigma(\mathcal{M})$ denotes the subspace of $\mathrm{Bil}(\mathcal{M})$ consisting of separately σ-weakly continuous forms on $\mathcal{M}$. For $a \in \mathcal{M}$, the dual maps $(L_a)^*$ and $(R_a)^*$ leave $\mathrm{Bil}_\sigma(\mathcal{M})$ invariant. We define a contraction $\varphi\colon \mathrm{Bil}(\mathcal{M}) \to \ell^\infty(\mathcal{M}_1)$ by $\varphi(V)(a) = V(a^*, a)$, where $\mathcal{M}_1$ is the unit ball of $\mathcal{M}$.

We rely on the following result [5]:

Theorem 4.1 **(Haagerup)** *Let $\mathcal{M}$ be an injective von Neumann algebra. Then there exists a mean m on the (discrete) semigroup $I(\mathcal{M})$ of isometries in $\mathcal{M}$ which is invariant in the sense that*

$$m\big(\varphi(L_a^*V)|I(\mathcal{M})\big) = m\big(\varphi(R_a^*V)|I(\mathcal{M})\big)$$

for all $V \in \mathrm{Bil}_\sigma(\mathcal{M})$ and all $a \in \mathcal{M}$.

By using the above result and the proof of 3.1 of [5] we prove:

Lemma 4.2 *Let $\pi\colon A \to \mathcal{B}(\mathcal{H})$ be a non-degenerate representation of a C^*-algebra A. If $\pi(A)''$ is injective, then there exists a net $\{T_\lambda\}_\lambda$ in $A \otimes A$ such that*

1. *the net $\{T_\lambda\}$ is in the convex hull of $\{x \otimes x^* \mid x \in A, \|x\| \le 1\}$,*
2. $\lim_\lambda \|L_a T_\lambda - R_a T_\lambda\|_\wedge = 0$ *for any $a \in A$,*
3. $\pi\big(p(T_\lambda)\big) \to 1$ *σ-weakly in $\mathcal{B}(\mathcal{H})$.*

Proof What is shown as Theorem 3.1 in [5] is the above statement (or more precisely the statement on ω below) for a nuclear C^*-algebra A and its universal representation π. But the proof there depends only on the fact that $\mathcal{M} = \pi(A)''$ is injective. We shall just give an outline of the proof here.

Let e denote the central projection in A^{**} corresponding to π; we shall identify $\mathcal{M}$ with $A^{**}e$.

By using the fact that $V \in \mathrm{Bil}(A)$ uniquely extends to $\tilde{V} \in \mathrm{Bil}_\sigma(A^{**})$ [10], we define an $\omega \in (A \mathbin{\widehat{\otimes}} A)^{**} \cong \mathrm{Bil}(A)^*$ by

$$\omega(V) = m\big(\varphi(\tilde{V})|I(\mathcal{M})\big),$$

where m is an *invariant* mean on $I(\mathcal{M})$ as in the above theorem. We then assert that

1. ω is in the weak*-closed convex hull of $\{x \otimes x^* \mid x \in A, \|x\| \le 1\}$,
2. $L_a^{**}\omega = R_a^{**}\omega$ for any $a \in A$,
3. $p^{**}(\omega) = e$ in A^{**}.

Property 1 follows by the Hahn-Banach separation argument using the crucial fact that $\tilde{V}$ is jointly σ-strong* continuous [6]. Property 2 reflects the invariance of m in the above theorem: $(L_a^{**}\omega)(V) = \omega(L_a^* V) = m\big(\varphi(L_a^*\tilde{V})|I(\mathcal{M})\big) = m\Big(\varphi\big(L_{ae}^*(\tilde{V}|\mathcal{M})|I(\mathcal{M})\big)\Big) = m\Big(\varphi\big(R_{ae}^*(\tilde{V}|\mathcal{M})|I(\mathcal{M})\big)\Big)$, which is equal to $(R_a^{**}\omega)(V)$, for all $V \in \mathrm{Bil}(A)$ and $a \in A$, where $\tilde{V}|\mathcal{M} \in \mathrm{Bil}_\sigma(\mathcal{M})$ is the restriction of $\tilde{V}$. Since $(p^* f)\tilde{} = p^* f$ and $\varphi(p^* f)(a) = f(a^* a)$ for $f \in A^*$, Property 3 follows from: $p^{**}(\omega)(f) = \omega(p^* f) = f(e)$.

Now, we may find a net $\{T_\lambda\}$ in the convex hull of $\{x \otimes x^* \mid x \in A, \|x\| \le 1\}$ such that T_λ weak*-converges to ω in $(A \mathbin{\widehat{\otimes}} A)^{**}$. It follows that $p(T_\lambda)$ weak*-converges to e in A^{**}. Since for any $a \in A$, $L_a T_\lambda - R_a T_\lambda$ converges weakly to 0 in $A \mathbin{\widehat{\otimes}} A$, we may assume

$$\lim_\lambda \|L_a T_\lambda - R_a T_\lambda\|_\wedge = 0$$

by convexity. ∎

By applying the above lemma to an irreducible representation π on $\mathcal{H}$, a finite-dimensional projection E on $\mathcal{H}$, and $\epsilon > 0$, we obtain a sequence $(x_1, x_2, \dots, x_n)$ in A such that $\sum_{i=1}^n \|x_i\|^2 \le 1$, and

$$\Big\| \sum_i a x_i \otimes x_i^* - \sum_i x_i \otimes x_i^* a \Big\|_\wedge < \epsilon, \quad a \in \mathcal{F},$$

$$\Big\| \pi\Big(\sum_i x_i x_i^*\Big) E - E \Big\| < \epsilon.$$

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By using Kadison's transitivity, we find a $b \in A$ (or $A + \mathbf{C}1$) such that $b \approx 1$, $\|yy^*\| \leq 1$, and $\pi(yy^*)E = E$, where $y = (bx_1, bx_2, \dots, bx_n) \in M_{1n}(A)$. Since there is a contraction ψ of $A \widehat{\otimes} A$ into $\mathcal{B}(A)$, which is defined by $\psi(a \otimes b)(x) = axb$, we obtain:

Theorem 4.3 *Any C^*-algebra has Property 1.3.*

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