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SEMI-SIMPLICITY RELATIVE TO KERNEL FUNCTORS

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Introduction. Let Λ be a ring and σ a kernel functor (left exact preradical) on the category of left Λ-modules. A left Λ-module M is called *σ-semi-simple* if whenever N is a submodule of M with M/N σ-torsion, N is a direct summand of M. In Section 1 we consider alternative characterizations and properties of σ-semi-simplicity for modules. In Section 2 conditions equivalent to the σ-semi-simplicity of the ring are obtained. Section 3 is devoted to the condition, which frequently arises in Section 2, that every σ-torsion module be semi-simple.

The terminology and notation in this paper are that of Goldman [**1**], with which familiarity is assumed. In particular, $K(\Lambda)$ (respectively $I(\Lambda)$) denotes the set of kernel functors (respectively idempotent kernel functors) of the ring Λ, and when we have a module M and a submodule N of M with M/N σ-torsion we say that N is *σ-open* in M. Finally, by the term "module" we mean a left module over the ring in question.

1. σ-Semi-simplicity.

Definition. Let $\sigma \in K(\Lambda)$. A module M is called *σ-semi-simple* if every σ-open submodule of M is a direct summand of M.

Note. Throughout this section σ will stand for a fixed but arbitrary kernel functor.

We begin with some immediate consequences of the definition.

PROPOSITION 1.1. *A σ-torsion module is σ-semi-simple if and only if it is semi-simple.*

Proof. Surely any semi-simple module is σ-semi-simple. Conversely, if M is σ-torsion, every submodule is σ-open. Hence if M is σ-semi-simple as well, every submodule is a direct summand.

PROPOSITION 1.2. *If M is σ-semi-simple, and if N is any submodule of M, then M/N is σ-semi-simple.*

Proof. Let $L/N \subseteq M/N$ be σ-open. Then $M/L \approx (M/N)/(L/N)$, so L is σ-open in M. Hence $M = L \oplus T$ for some submodule T, from which it follows that $M/N = L/N \oplus (T + N)/N$.

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The following concepts are useful for obtaining alternate characterizations of σ-semi-simplicity.

Definitions. A submodule N of a module M is called *σ-dense* in M if for every σ-open submodule P of M, $P + N = M$. (Note that since σ-open submodules topologize a module, σ-dense submodules are precisely those that are dense in the topological sense.)

A submodule L of a module M is called *σ-essential* in M, or M is called a σ-essential extension of L, if L is both σ-open and essential in M (or equivalently, for every $0 \neq x \in M$, $(L : x) \in \mathscr{T}_\sigma$ and $(L : x)x \neq 0$, where $(L : x) = \{r \in \Lambda | rx \in L\}$).

The *σ-socle* of a module $M \neq 0$, denoted $\mathscr{S}_\sigma(M)$, is the intersection of all σ-essential submodules of M. If $M = 0$ we define $M = \mathscr{S}_\sigma(M)$.

THEOREM 1.3. *For any module M, the following are equivalent:*
(1) *M is σ-semi-simple;*
(2) *If L is σ-essential in M, $L = M$;*
(3) *$M = \mathscr{S}_\sigma(M)$;*
(4) *Every essential submodule of M is σ-dense in M;*
(5) *For any submodule N of M, there exists a submodule N' of M with $N \cap N' = 0$, and $N + N'$ σ-dense;*
(6) *$\mathscr{S}(M)$, the socle of M, is σ-dense in M.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) follow immediately from the definitions. (4) $\Rightarrow$ (5) follows from the definitions and the well-known existence of complements, i.e., given any submodule X of M there is a submodule Y such that $X \cap Y = 0$ and $X + Y$ is essential.

(3) $\Rightarrow$ (4) Let N be essential in M. Then for any σ-open P, $N + P$ is both σ-open and essential. Thus $M = \mathscr{S}_\sigma(M) \subseteq N + P$. So N is σ-dense.

(5) $\Rightarrow$ (1) Let L be σ-open in M, and let $N \subseteq M$ be such that $N \cap L = 0$ and $N + L$ is σ-dense. Then $N + L = (N + L) + L = M$. Thus $M = N \oplus L$.

(1) $\Rightarrow$ (6) Let L be σ-open in M and consider $\mathscr{S}(M) + L$. Suppose that $\mathscr{S}(M) + L \neq M$. Then since $\mathscr{S}(M) + L$ is σ-open, for some submodule $N \neq 0$ we have $(\mathscr{S}(M) + L) \oplus N = M$. But then N is σ-torsion, and by Proposition 1.2, N is σ-semi-simple. So by Proposition 1.1, N is semi-simple. Thus $N \subseteq \mathscr{S}(M)$, which contradicts $N \neq 0$. Therefore $\mathscr{S}(M)$ is σ-dense.

(6) $\Rightarrow$ (4) Since $\mathscr{S}(M)$ is contained in every essential submodule of M, this is immediate.

We can consider σ-semi-simplicity more closely via the following concept.

Definition. A module M is called *σ-simple* if for any σ-open submodule L of M, either $L = M$ or $L = 0$.

PROPOSITION 1.4 (1). *Every simple module is σ-simple;*
(2) *Every σ-simple module is σ-semi-simple;*
(3) *A σ-torsion module is σ-simple if and only if it is simple;*

(4) *A σ-torsion-free module is σ-simple if and only if it is σ-semi-simple;*
(5) *Any factor module of a σ-simple module is σ-simple.*

Proof. (1) and (2) follow immediately from the definitions, while (3) follows from Proposition 7.7 and (2). Now if M is σ-torsion-free and σ-semi-simple, let P be a σ-open submodule of M. Then $M = P \oplus X$, for some X. But X is σ-torsion and contained in M. Thus $X = 0$, and $P = M$; this proves (4). (5) follows from the fact that if N is a submodule of M and if L/N is σ-open in M/N, then L is σ-open in M.

Remarks. (i) $0 \in \mathrm{K}(\Lambda)$ is defined by $0(M) = 0$ for all M. Then every module is 0-semi-simple, and so by (4) above, 0-simple.

(ii) $z \in \mathrm{K}(\Lambda)$ is defined by $z(M) =$ the singular submodule of M, or equivalently $\mathscr{T}_z$ is the set of essential left ideals. If $\sigma \geqq z$, the concepts of essential and σ-essential coincide. Hence whenever $\sigma \geqq z$, a module is σ-semi-simple if and only if it is semi-simple.

THEOREM 1.5. *Let $\{M_\alpha\}$ be a family of σ-semi-simple modules. If $M = \Pi_\alpha M_\alpha$, then M is σ-semi-simple.*

Proof. Let L be a σ-open submodule of M. Then, as usual, there is a submodule P of M maximal with respect to $P \cap L = 0$. Suppose that for some β, $M_\beta \not\subseteq P + L$. Consider $N_\beta = M_\beta \cap L$. N_β is σ-open in M_β, hence $M_\beta = N_\beta \oplus X$, for some X. Then X is non-zero, σ-torsion and σ-semi-simple, so X is semi-simple by Proposition 1.1. If $X \subseteq P + L$, then we have $M_\beta = N_\beta + X \subseteq L + P$, which is not the case. Therefore $X \not\subseteq L + P$, and so there is a non-zero simple submodule S of X with $S \not\subseteq L + P$. Consider $(P + S) \cap L$. If $y \in (P + S) \cap L$, we have $y \in L$ and $y = s + p$ for some $s \in S$ and $p \in P$. But then $s \in L + P$, and so $s = 0$ (else $S \subseteq L + P$). Therefore $y \in L \cap P = 0$. Hence $(S + P) \cap L = 0$, which by the maximality of P implies that $S \subseteq P$, a contradiction. Thus for all β, $M_\beta \subseteq P + L$. Since $P \cap L = 0$, $M = P \oplus L$, and we are done.

It is clear from this theorem and Proposition 1.4 that any direct sum of σ-simple modules is σ-semi-simple. We shall later give an example to show that the converse is false. As we show now, if certain restrictions are imposed, then a converse is obtained.

THEOREM 1.6. *Let $\rho \in I(\Lambda)$, and let M be a ρ-semi-simple module. If $\rho(M)$ is finite dimensional* (i.e., $\rho(M)$ contains no infinite collection of submodules whose sum is direct), *then M has d.c.c. on ρ-open submodules, and so contains a unique minimal ρ-open submodule M_0. Furthermore M_0 is ρ-simple, and $M = M_0 \oplus X$, where X is a ρ-torsion semi-simple module. Thus M is a direct sum of ρ-simple modules.*

Proof. Since $\rho(M)$ is finite dimensional, there is an integer n such that $\rho(M)$ contains no family of more than n submodules whose sum is direct [**4**, p. 55].

Let

$$M = M_1 \supset M_2 \supset \ldots$$

be a descending chain of ρ-open submodules. Then each M/M_i is ρ-torsion and ρ-semi-simple, and thus semi-simple. Therefore for each i, there is a semi-simple $X_i \subseteq \rho(M)$ such that $M = M_i \oplus X_i$, and since $M_i \supset M_{i+1}$, $X_{i+1} \supset X_i$, which after $n + 1$ steps yields a contradiction. Thus M has d.c.c. on ρ-open submodules, and so M has minimal ρ-open submodules. But the intersection of any two ρ-open submodules is again ρ-open; hence M has a unique smallest ρ-open submodule M_0. Since M is ρ-semi-simple $M = M_0 \oplus X$, where X is semi-simple and ρ-torsion. It remains to be shown that M_0 is ρ-simple. But this is clear, since the idempotence of ρ guarantees that a ρ-open submodule of a ρ-open submodule of M is itself ρ-open in M [**1**, p. 18].

2. σ-semi-simplicity of the ring. In this section we investigate the condition that the ring Λ be σ-semi-simple with respect to some given $\sigma \in K(\Lambda)$. Some preliminaries are needed.

PROPOSITION 2.1. *σ is an exact functor if and only if for every $\mathfrak{A} \in \mathscr{T}_\sigma$, $\mathfrak{A} + \sigma(\Lambda) = \Lambda$; i.e. if and only if $\sigma(\Lambda)$ is σ-dense in Λ.*

Proof. ($\Rightarrow$) Let $\mathfrak{A} \in \mathscr{T}_\sigma$, and consider

$$0 \to \mathfrak{A} \to \Lambda \to \Lambda/\mathfrak{A} \to 0.$$

Applying σ we obtain $0 \to \sigma(\mathfrak{A}) \to \sigma(\Lambda) \to \sigma(\Lambda/\mathfrak{A}) \to 0$. Thus

$$\sigma(\Lambda/\mathfrak{A}) \approx \sigma(\Lambda)/\sigma(\Lambda) \approx \sigma(\Lambda)/(\sigma(\Lambda) \cap \mathfrak{A}) \approx (\sigma(\Lambda) + \mathfrak{A})/\mathfrak{A}.$$

But $\sigma(\Lambda/\mathfrak{A}) = \Lambda/\mathfrak{A}$, so $\sigma(\Lambda) + \mathfrak{A} = \Lambda$.

($\Leftarrow$) Let $0 \to N \to M \to M/N \to 0$ be exact, and let $x \in M$ be such that $x + N \in \sigma(M/N)$. Then for some $\mathfrak{A} \in \mathscr{T}_\sigma$, $\mathfrak{A}x \subseteq N$. Now $\mathfrak{A} + \sigma(\Lambda) = \Lambda$, so there are $a \in \mathfrak{A}$, and $s \in \sigma(\Lambda)$ such that $1 = a + s$. So $x = 1 \cdot x = ax + sx \in N + \sigma(M)$. Thus $\sigma(M/N) \subseteq (\sigma(M) + N)/N$. Since σ is a functor the reverse inclusion is true as well, and so $\sigma(M/N) = (\sigma(M) + N)/N$. But

$$(\sigma(M) + N)/N \approx \sigma(M)/(\sigma(M) \cap N) = \sigma(M)/\sigma(N).$$

Thus σ is exact.

LEMMA 2.2. *If σ is an exact functor, then σ is idempotent.*

Proof. Let M be a module and consider $0 \to \sigma(M) \to M \to M/\sigma(M) \to 0$. Applying σ we obtain $0 \to \sigma(\sigma(M)) \to \sigma(M) \to \sigma(M/\sigma(M)) \to 0$. Since $\sigma(\sigma(M)) = \sigma(M)$, $\sigma(M/\sigma(M)) = 0$, and σ is idempotent.

We can now describe the σ-semi-simplicity of Λ.

THEOREM 2.3. *For $\sigma \in K(\Lambda)$, the following are equivalent:*

(1) *Λ is σ-semi-simple;*

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(2) *every* Λ-*module is* σ-*semi-simple;*

(3) *every* Λ-*module is* σ-*injective;*

(4) *every* σ-*open left ideal is a direct summand;*

(5) *every* σ-*open left ideal contains a* σ-*open direct summand of* Λ, *and every* σ-*torsion module is semi-simple;*

(6) σ *is an exact functor, and every* σ-*torsion module is semi-simple;*

(7) $\mathscr{S}(\Lambda)$ *is* σ-*dense in* Λ.

Proof. The equivalence of (1), (4), and (7) follows from Theorem 1.3.

(1) $\Rightarrow$ (2) From Theorem 1.5, every free module is σ-semi-simple, and so by Proposition 1.2, every module is σ-semi-simple.

(2) $\Rightarrow$ (3) Since any σ-open submodule of any module is a direct summand, any homomorphism from a σ-open submodule of any module extends to the whole module.

(3) $\Rightarrow$ (4) The identity map of any σ-open left ideal splits the inclusion map into Λ.

(4) $\Rightarrow$ (5) The first part of (5) follows trivially from (4), and since (4) $\Leftrightarrow$ (1), the second part follows from (2) and Proposition 1.1.

(5) $\Rightarrow$ (6) Let $\mathfrak{A}$ be a σ-open left ideal. Then there is a σ-open left ideal $\mathfrak{B}$, with $\mathfrak{B} \subseteq \mathfrak{A}$, and $\Lambda = \mathfrak{B} \oplus Y$, for some Y. Since Y is σ-torsion we have

$$\mathfrak{A} + \sigma(\Lambda) \supseteq \mathfrak{B} + \sigma(\Lambda) \supseteq \mathfrak{B} + Y = \Lambda.$$

Hence by Proposition 2.1, σ is exact.

(6) $\Rightarrow$ (4) Let $\mathfrak{A}$ be a σ-open left ideal. Then by Proposition 2.1, $\mathfrak{A} + \sigma(\Lambda) = \Lambda$. Since $\sigma(\Lambda)$ is semi-simple, $\sigma(\Lambda) = (\sigma(\Lambda) \cap \mathfrak{A}) \oplus X$ for some X. Since $X \subseteq \sigma(\Lambda)$, we have $X \cap \mathfrak{A} = 0$. Now

$$\Lambda = \sigma(\Lambda) + \mathfrak{A} = (\sigma(\Lambda) \cap \mathfrak{A}) + X + \mathfrak{A} = X + \mathfrak{A}.$$

Thus $\Lambda = X \oplus \mathfrak{A}$.

From (6) above and Lemma 2.2 it follows that if Λ is σ-semi-simple, then σ is idempotent, and thus a ring of quotients $Q_\sigma(\Lambda)$ exists. The next theorem describes σ-semi-simplicity in terms of this ring.

LEMMA 2.4. *If* σ *is an exact functor, then* σ *has Property* (T).

Proof. As we have just noted, if σ is exact, then σ is idempotent, and so $Q_\sigma(\Lambda)$ exists. Let $i : \Lambda \rightarrow Q_\sigma(\Lambda)$ be the canonical map, and let $\mathfrak{A}$ be a σ-open left ideal of Λ. Then by Proposition 2.1, $\Lambda = \sigma(\Lambda) + \mathfrak{A}$, and there is $a \in \mathfrak{A}$ such that $i(a) = i(1) = 1$. Thus $1 \in i(\mathfrak{A})$, and so $Q_\sigma(\Lambda)i(\mathfrak{A}) = Q_\sigma(\Lambda)$. Hence by Theorem 4.3 of [**1**], σ has Property (T).

THEOREM 2.5. *For* $\sigma \in K(\Lambda)$, *the following are equivalent:*

(1) Λ *is* σ-*semi-simple;*

(2) σ *is idempotent and has Property* (T), $Q_\sigma(\Lambda) = \Lambda/\sigma(\Lambda)$, *and every* σ-*torsion module is semi-simple.*

Proof. (1) $\Rightarrow$ (2). From Theorem 2.3 (6) and Lemmas 2.2 and 2.4, σ is

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idempotent and has Property (T). That $Q_\sigma(\Lambda) = \Lambda/\sigma(\Lambda)$ is a consequence of Theorem 2.3(3).

(2) $\Rightarrow$ (1) Let $\mathfrak{A}$ be a σ-open left ideal of Λ. Since σ has Property (T), $Q_\sigma(\Lambda)i(\mathfrak{A}) = Q_\sigma(\Lambda)$, where $i : \Lambda \rightarrow Q_\sigma(\Lambda)$. Using the hypotheses, this translates to $\Lambda/\sigma(\Lambda) = \mathfrak{A} + \sigma(\Lambda)/\sigma(\Lambda)$, whence by Proposition 2.1, σ is exact. Thus, Theorem 2.3(6) holds.

Remarks. (i) If Λ is σ-semi-simple, and if $\sigma(\Lambda)$ is finite-dimensional, then Theorem 1.6 gives us some information about the structure of Λ. In particular if I is the unique minimal σ-open left ideal of Λ, it is easy to show, using the idempotence of σ, that I is two-sided, idempotent and a direct summand of Λ. Furthermore, since Λ/I is semi-simple, I is a finite intersection of maximal left ideals. Conversely, for any ring Γ, if A is a two-sided ideal of Γ with A a direct summand and Γ/A a semi-simple Γ-module, then for $\delta \in K(\Gamma)$, defined by $\mathscr{T}_\delta$ is the set of left ideals of Γ that contain A, then Γ is δ-semi-simple.

(ii) Since the left ideal $\mathfrak{A}$ satisfies $\mathfrak{A} + \mathscr{S}(\Lambda) = \Lambda$ if and only if $\Lambda/\mathfrak{A}$ is a semi-simple projective Λ-module ($\mathfrak{A} + \mathscr{S}(\Lambda) = \mathfrak{A} + ((\mathscr{S}(\Lambda) \cap \mathfrak{A}) \oplus X) = \mathfrak{A} \oplus X$, for some $X \subseteq \mathscr{S}(\Lambda)$), the set of left ideals $\mathfrak{A}$ of Λ for which $\mathfrak{A} + \mathscr{S}(\Lambda) = \Lambda$ defines an idempotent kernel functor, which, according to Theorem 2.3(7), is the unique largest kernel functor with respect to which Λ is semi-simple. In [**2**], Goldman calls this set of left ideals the *intrinsic topology* of Λ, and presents a structure theorem for rings complete in their intrinsic topologies.

We now give an example of a ring which is σ-semi-simple, but without the above finiteness conditions. This also supplies the promised example of a σ-semi-simple module which is not a direct sum of σ-simples. Let k be a field and let

$$R = \prod_\alpha k,$$

where α runs through any fixed infinite set I. Define $\sigma \in K(R)$ by $\mathscr{T}_\sigma$ as the set of ideals that contain $\prod_{\beta \in J} k$, where J is any subset of I with finite complement. Then it is easy to check that Theorem 2.3(7) holds, so that R is σ-semi-simple, and that R is not a direct sum of σ-simples.

3. Semi-simple σ-torsion modules. Since the condition that every σ-torsion module be semi-simple arises so frequently in Theorem 2.3, we consider that condition somewhat more closely in this section. Many of the results are straightforward generalizations of results about the singular torsion theory (the kernel functor z in our language) to be found in [**3**, Chapter III].

Note that if every σ-torsion module is semi-simple then every σ-open left ideal is a finite intersection of maximal left ideals, or equivalently, $\sigma \leqq \mathscr{S}$, where $\mathscr{S}$ is the kernel functor that assigns to every module its socle. The converse of this observation follows from the next proposition.

PROPOSITION 3.1. *For an idempotent kernel functor σ, the following are equivalent:*

(1) *every σ-torsion module is semi-simple;*
(2) *for every σ-open left ideal $\mathfrak{A}$, $\Lambda/\mathfrak{A}$ is semi-simple;*
(3) *every σ-torsion module is σ-injective.*

Proof. That (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Let M be a σ-torsion module, and let $f : \mathfrak{A} \to M$ be a homomorphism from a σ-open left ideal. Then $\mathfrak{A}/\mathrm{Ker} f$ is σ-torsion, and so by [**1**, Theorem 2.5], $\mathrm{Ker} f$ is σ-open. Hence

$$\Lambda/\mathrm{Ker} f = \mathfrak{A}/\mathrm{Ker} f \oplus X/\mathrm{Ker} f$$

for some left ideal X. Then f can be extended to all of Λ by being 0 on X. Thus, [**1**, Proposition 3.2] M is σ-injective.

(3) $\Rightarrow$ (1) If M is a σ-torsion module, and N is a submodule of M, then M/N is σ-torsion, so the inclusion of N in M splits, and thus N is a summand of M.

PROPOSITION 3.2. *Let σ be an idempotent kernel functor for which every σ-torsion module is semi-simple. Then for any σ-open left ideal $\mathfrak{A}$, $\mathfrak{A} = \mathfrak{A}^2$.*

Proof. Let $\mathfrak{A}$ be a σ-open left ideal. Then, since σ is idempotent, $\mathfrak{A}^2$ is also σ-open, and so $\Lambda/\mathfrak{A}^2$ is semi-simple. Thus there is a left ideal X, containing $\mathfrak{A}^2$, such that $\Lambda/\mathfrak{A}^2 = \mathfrak{A}/\mathfrak{A}^2 \oplus X/\mathfrak{A}^2$. In other words, $X + \mathfrak{A} = \Lambda$ and $X \cap \mathfrak{A} \subseteq \mathfrak{A}^2$. So there exist $a \in \mathfrak{A}$ and $x \in X$ such that $a + x = 1$. Thus for any $b \in \mathfrak{A}$, $b = b \cdot 1 = ba + bx$. But $bx = b - ba$. Hence $bx \in X \cap \mathfrak{A} \subseteq \mathfrak{A}^2$, and so $b \in \mathfrak{A}^2$.

COROLLARY 3.3. *Let R be a commutative noetherian ring, and let $\sigma \in I(R)$. Then R is σ-semi-simple if and only if every σ-torsion module is semi-simple.*

Proof. This follows from Proposition 3.2 and the fact that idempotent ideals in commutative noetherian rings are direct summands.

Remark. In [**3**], Goodearl notes that if for a non-singular ring Λ every singular module is semi-simple, then the Jacobson radical of Λ, $J(\Lambda)$, is contained in the socle of Λ, and thus $J(\Lambda)^2 = 0$. This can be generalized to arbitrary idempotent kernel functors. For $\sigma \in I(\Lambda)$ we define the *σ-radical*, $J_\sigma(M)$, of a module M to be the intersection of the kernels of homomorphisms from M into σ-simple modules. It can readily be shown that $J_\sigma(\Lambda)$ is a two-sided radical ideal of Λ. Then if every σ-torsion module is semi-simple, $J_\sigma(\Lambda)$ is contained in every σ-essential left ideal, and thus in $\mathscr{S}_\sigma(\Lambda)$, whence $J_\sigma(\Lambda)^2 = 0$.

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