ASYMPTOTIC EXPANSIONS

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1. Introduction. Let $a_1\, a_2, \ldots, a_m$ be a set of real non-negative numbers and let

1.1 $$P(x) = a_1x + a_2x^2 + \ldots + a_mx^m \qquad (a_m \neq 0).$$

Many combinatorial problems can be reduced to the study of numbers B_n generated by

1.2 $$\sum_{n=0}^{\infty} B_n x^n/n! = e^{P(x)}.$$

Some problems of this type were treated by Touchard **(7)**, Jacobsthal **(3)**, Chowla, Herstein, Moore and Scott **(1**; **2)**, and the present authors **(4)**. In **(2)**, the problem of finding asymptotic formulae for B_n in terms of $P(x)$ was proposed. Essentially the same problem was solved earlier by Pólya **(5)**, as a by-product of an investigation of the zeros of the derivatives of certain functions. The object of the paper is to give a different and more explicit solution to this problem. Furthermore our method yields complete asymptotic expansions, while that of Pólya gave only the first term.

2. Preliminary notions. Since some of the coefficients a_k in (1.1) may be zero, $P(x)$ will in general have the form

2.1 $$P(x) = b_1x^r + b_2x^s + \ldots + a_mx^m,$$

where the coefficients $b_1, b_2, \ldots, a_m$ are positive. In what follows we shall assume

2.2 $$(r, s, \ldots, m) = 1.$$

This involves no essential loss of generality since one can always reduce the problem to this case by a substitution of the form $y = x^q$.

LEMMA 1. *If* $0 \leqslant \theta \leqslant \pi$ *and* $\cos r\theta = 1, \cos s\theta = 1, \ldots, \cos m\theta = 1$ *then* $\theta = 0$.

Proof. If $0 < \theta \leqslant \pi$ then $\cos r\theta = 1$ implies the existence of positive integers a and b, $(a, b) = 1$, $b > 1$, such that $\theta = \pi\, a/b$. Now $r\theta = r\pi\, a/b$, $s\theta = s\pi\, a/b, \ldots, m\theta = m\pi\, a/b$ are each integral multiples of 2π. Hence b divides $r, s, \ldots, m$, which contradicts (2.2).

Corresponding to $P(x)$ as defined in (1.1) we define a trigonometric polynomial $S(R, \theta)$ by

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2.3 $$S(R, \theta) = \tfrac{1}{2}[P(Re^{i\theta}) + P(Re^{-i\theta})] = \sum_{k=1}^{m} a_k R^k \cos k\,\theta.$$

Further we define ϵ by

2.4 $$\epsilon = R^{(1-4m)/8},$$

and prove

LEMMA 2. *For $\epsilon \leqslant \theta \leqslant \pi$ and R sufficiently large, $S(R, \theta) \leqslant S(R, \epsilon)$.*

Proof. Since $a_k \geqslant 0$ for $k = 1, 2, \ldots, m$. $S(R, \theta)$ assumes its greatest value at $\theta = 0$. Also by (1.1), (2.3) and (2.4) we have

2.5 $$S(R, 0) - S(R, \epsilon) = \sum_{k=1}^{m} a_k R^k (1 - \cos k\epsilon) = O(R^{\frac{1}{4}}).$$

On the other hand, there exists by (2.2) and Lemma 1, a positive integer $t < m$ such that $\cos(t\theta) \neq 1$ and $a_t \neq 0$. Hence for fixed θ,

2.6 $$S(R, 0) - S(R, \theta) = \sum_{k=1}^{m} a_k R^k (1 - \cos k\,\theta) \geqslant C_1 R^t,$$

where C_1 is a fixed positive constant. Comparing (2.5) and (2.6) gives the required result.

3. Asymptotic formulae. By (1.2) and Cauchy's theorem

3.1 $$B_n = \frac{n!}{2\pi i} \int_c \frac{e^{P(z)}}{z^{n+1}}\, dz,$$

where c denotes the circle $z = Re^{i\theta}$. We note that R, the radius of the circle, is arbitrary. From (3.1) we obtain

3.2 $$B_n = A \int_{-\pi}^{\pi} e^{F(R,\theta)} d\theta,$$

where

3.3 $$A = n!\, e^{P(R)}/2\pi R^n,$$

and

3.4 $$F(R, \theta) = P(Re^{i\theta}) - P(R) - in\theta.$$

Let I be defined by

3.5 $$I = \int_{\epsilon}^{\pi} e^{F(R,\theta)} d\theta,$$

where ϵ is given by (2.4).

LEMMA 3. $|I| = O(\exp(-R^{\frac{1}{4}}))$.

Proof. Clearly

$$|I| \leqslant \int_{\epsilon}^{\pi} e^{S(R,\theta) - S(R,0)} d\theta,$$

and the required result follows from (2.5) and Lemma 2.

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Since we will show that the integral in (3.2) can be expanded in powers of $1/R$, we may neglect integrals of type (3.5) and write

3.6 $$B_n \sim A \int_{-\epsilon}^{\epsilon} e^{F(R,\theta)} d\theta.$$

Our next step is to expand $F(R, \theta)$ in a Maclaurin series of the form

3.7 $$F(R, \theta) = \sum_{j=1}^{\infty} C_j(R)(i\theta)^j/j!,$$

where

3.8 $$C_1(R) = \sum_{k=1}^{m} k\, a_k R^k - n,$$

and

3.9 $$C_j(R) = \sum_{k=1}^{m} k^j a_k R^k \qquad (j > 1).$$

At this stage we choose R so that

3.10 $$C_1(R) = 0.$$

For large n, (3.10) will have a unique solution which may be calculated by iteration starting with

3.11 $$R \sim (n/m\, a_m)^{1/m}.$$

When (3.10) holds, (3.7) can be written in the form

3.12 $$F(R, \theta) = -\tfrac{1}{2} C_2(R)\theta^2 + \sum_{j=3}^{\infty} C_j(R)(i\theta)^j/j!\,.$$

In order to simplify some of the expressions which occur, we introduce the following notations:

3.13 $$R = z^{-2}, \quad z = R^{-\frac{1}{2}},$$

3.14 $$\bar{C}_j(z) = z^{2m} C_j(z^{-2}) = \sum_{k=1}^{m} k^j a_k z^{2m-2k}, \qquad (j > 1),$$

3.15 $$f_j(z) = z^{m(j-2)}\, \bar{C}_j(z)\{2/\bar{C}_2(z)\}^{\frac{1}{2}j},$$

3.16 $$\lambda = \epsilon(C_2(R)/2)^{\frac{1}{2}},$$

3.17 $$\phi = \theta(C_2(R)/2)^{\frac{1}{2}},$$

3.18 $$H = A(2/C_2(R))^{\frac{1}{2}},$$

3.19 $$\psi(z, \phi) = \sum_{j=3}^{\infty} f_j(z)(i\phi)^j/j!\,.$$

If we now make the substitution (3.17) in (3.6) and use (3.12) and (3.19) we obtain

3.20 $$B_n \sim H \int_{-\lambda}^{\lambda} e^{-\phi^2 + \psi(z,\phi)} d\phi.$$

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From (2.4), (3.9) and (3.16) we see that for R large,

3.21 $$K_2 R^{1/8} > \lambda > K_3 R^{1/8},$$

where K_2 and K_3 are fixed positive constants. Further, for R sufficiently large it is not difficult to show that there exists an interval $-\sigma < z < \sigma$ for which $\psi(z, \phi)$ and $e^{\psi(z,\phi)}$ have Maclaurin expansions in z of the form

3.22 $$\psi(z, \phi) = \sum_{k=1}^{\infty} \psi_k(\phi)\, z^k$$

and

3.23 $$e^{\psi(z,\phi)} = \sum_{r=0}^{\infty} \Psi_r(\phi)\, z^r, \quad \Psi_0(\phi) = 1,$$

which are uniformly convergent for $|\phi| \leqslant \lambda$. It is further easy to justify the fact that the $\psi_k(\phi)$ are given by means of (3.19) to be

3.24 $$\psi_k(\phi) = \sum_{j=3}^{\infty} \frac{1}{k!}\left[\frac{d^k f_j(z)}{dz^k}\right]_{z=0} \frac{(i\phi)^j}{j!}.$$

From (3.19) we see that $\psi_k(\phi)$ are polynomials in ϕ and hence $\Psi_k(\phi)$ are also polynomials in ϕ. In fact $\Psi_{2n}(\phi)$ contains only even powers of ϕ while $\Psi_{2n+1}(\phi)$ only contains odd powers of ϕ.

Using (3.24) in (3.20) we have

3.25 $$B_n \sim H\left[\sum_{k=0}^{s-1}\left(\int_{-\lambda}^{\lambda} \Psi_k(\phi)\, e^{-\phi^2} d\phi\right) z^k + R_s\right],$$

where

3.26 $$R_s = \int_{-\lambda}^{\lambda} e^{-\phi^2} \sum_{k=s}^{\infty} \Psi_k(\phi)\, z^k d\phi\,.$$

Using (3.21) and the fact that the $\Psi_k(\phi)$ are polynomials in ϕ, (3.25) yields

3.27 $$B_n \sim H\left[\sum_{k=0}^{s-1}\left(\int_{-\infty}^{\infty} \Psi_k(\phi)\, e^{-\phi^2} d\phi\right) z^k + R_s\right],$$

where R_s is still given by (3.26). In order to complete our proof it remains to show that for fixed s, $R_s = O(z^s)$. If this is so our complete asymptotic formula becomes

3.28 $$B_n \sim H\left[\sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \Psi_k(\phi)\, e^{-\phi^2} d\phi / R^{\frac{1}{2}k}\right].$$

Finally, in view of the remarks following (3.24), the integrals in (3.28) will vanish for odd k and (3.28) can be put in the form

3.29 $$B_n \sim H\left[\sum_{k=0}^{\infty} \left(\int_{-\infty}^{\infty} \Psi_{2k}(\phi)\, e^{-\phi^2} d\phi\right) / R^k\right].$$

We shall now consider $\psi(z, \phi)$ as given by (3.19) to be a function of a complex variable z and a real variable ϕ. We restrict z to be in a neighborhood $|z| < \sigma < 1$ and ϕ to be bounded. Under these restrictions (3.14) yields

3.30 $$|\tfrac{1}{2}\bar{C}_2(z)| \geqslant \left| m^2 a_m - \sum_{k=1}^{m-1} k^2 a_k \sigma^{2m-2k} \right| .$$

Clearly, by taking σ small enough we may say that

3.31 $$|\tfrac{1}{2}\bar{C}_2(z)| \geqslant \alpha^2,$$

where α is a positive constant.

Similarly from (3.19) and the fact that $|z| < 1$ we can say that

3.32 $$|\bar{C}_j(z)| \leqslant \sum_{k=1}^{m} k^j |a_k| < a(m^{k+1}),$$

where $a = \max(|a_1|, |a_2|, \ldots, |a_m|)$. Hence from (3.15), (3.31) and (3.32) we obtain

3.33 $$|f_j(z)| \leqslant a(m^{j+1})/\alpha^j,$$

where a, m, and α are independent of j and z. From (3.3) and Cauchy's theorem on derivatives we obtain

3.34 $$\left| \frac{d^k f_j(z)}{dz^k} \right|_{z=0} \leqslant a(m^{j+1})\, k!/\alpha^j \sigma^k.$$

By introducing the notation $am = M$, $m/\alpha = K$, $\sigma = 1/S$, (3.34) may be written

3.35 $$\left| \frac{d^k f_j(z)}{dz^k} \right|_{z=0} \leqslant M K^j k!\, S^k.$$

From (3.15) the derivatives in (3.35) vanish for $m(j-2) > k$. Since m is a positive integer, $k + 2 > (k/m) + 2$. Hence (3.24) can be written

3.36 $$\psi_k(\phi) = \sum_{j=3}^{k+2} \frac{1}{k!} \left[\frac{d^k f_j(z)}{dz^k} \right]_{z=0} \frac{(i\phi)^j}{j!}.$$

From (3.35) we now obtain

3.37 $$|\psi_k(\phi)| \leqslant MS^k \sum_{j=3}^{k+2} (K|\phi|)^j/j!,$$

and by induction on k we easily deduce

3.38 $$|\psi_k(\phi)| \leqslant MS^k\, [K|\phi|]^2\, [1 + K|\phi|]^k.$$

By a lemma proved in **(4)**, (3.38) implies

3.39 $$|\Psi_j(\phi)| \leqslant M[K|\phi|]^2[1 + M(K\phi)^2]^{j-1} S^j (1 + K|\phi|)^j.$$

From (3.39) we obtain

3.40 $$\left| \sum_{j=s}^{\infty} \Psi_j(\phi)\, z^j \right| \leqslant \frac{1}{T} M[K|\phi|]^2[1 + M(K\phi)^2]^{s-1} S^s (1 + K|\phi|)^s |z|^s$$

where T is given by

3.41 $$T = 1 - [1 + M(K|\phi|)^2][1 + K|\phi|]|z|.$$

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We now revert to real values of z. Recalling that $z = R^{-\frac{1}{2}}$ and $|\phi| \leqslant \lambda$ we have, from (3.21), that $\lambda < K_2 R^{1/8}$,

3.42 $$|\phi|^3|z| = O(R^{-1/8}).$$

Hence for R sufficiently large we have $T > \frac{1}{2}$. Thus (3.40) yields

3.43 $$\left|\sum_{j=s}^{\infty} \Psi_j(\phi)\, z^j\right| \leqslant Q_s(|\phi|)\, z^s,$$

where $Q_s(|\phi|)$ is a polynomial in $|\phi|$. From (3.26) we now obtain

3.44 $$|R_s| \leqslant \int_{-\lambda}^{\lambda} e^{-\phi^2} Q_s(|\phi|)\, d\phi\, z^s \leqslant z^s \int_{-\infty}^{\infty} e^{-\phi^2} Q_s(|\phi|)\, d\phi.$$

Since $Q_s(|\phi|)$ is a polynomial, the last integral of (3.44) exists and hence

3.45 $$R_s = O(z^s).$$

This completes the proof of the main result (3.29) which can be written in the form

3.46 $$B_n \sim \frac{n!\, e^{P(R)}}{2\pi R^n}\left[\frac{2}{C_2(R)}\right]^{\frac{1}{2}} \sum_{k=0}^{\infty} \left(\int_{-\infty}^{\infty} \Psi_k(\phi)\, e^{-\phi^2} d\phi / R^k \right),$$

where R is determined by

3.47 $$\sum_{k=1}^{m} k\, a_k R^k - n = 0.$$

In concluding this section we might point out that $\Psi_0(\phi) = 1$. Hence the first term of the asymptotic expansion is easily calculated. If we introduce the operator Θ by

3.48 $$\Theta = R \frac{d}{dR},$$

then the first term of the expansion is given by

3.49 $$B_n \sim \frac{n!\, e^{P(R)}}{R^n}\left[\frac{1}{2\pi \Theta^2 P(R)}\right]^{\frac{1}{2}},$$

and R as a function of n is given by

3.50 $$\Theta P(R) = n.$$

4. Applications. To illustrate applications of the method we consider three special cases.

Example 1. $P(x) = x$.

In this case the numbers B_n are all 1. However, since the asymptotic formula obtained by our method involves the factor $n!$, it will lead in this case to Stirling's expansion for $n!$. Equations (3.8), (3.9) and (3.10) yield $R = n$ and $C_j(R) = R = n$ $(j > 1)$. Applying (3.46) we obtain

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4.1 $$1 \sim \frac{n!\, e^n}{(2\pi)^{\frac{1}{2}} n^{n+\frac{1}{2}}}\left(1 - \frac{1}{12n} + \dots\right)$$

or

4.2 $$n! \sim \left(\frac{n}{e}\right)^n (2\pi\, n)^{\frac{1}{2}}\left(1 + \frac{1}{12n} + \dots\right)$$

as required.

Example 2. $P(x) = x + (x^p/p)$.

In this case it is known **(3)** that if p is a prime then $B_n = B_{n,p}$ is the number of solutions of $x^p = 1$ in the symmetric group of degree n. The case $p = 2$ was treated in **(1)** and **(4)** and the result for $p > 2$ was announced in **(4)**. In this case we have

4.3 $$P(R) = R + (R^p/p),$$

4.4 $$C_1(R) = R + R^p - n = 0$$

and

4.5 $$C_2(R) = R + pR^p.$$

From these and (3.49) the first term of the asymptotic expansion is given by

4.6 $$B_{n,p} \sim \frac{n!\,\exp(R + R^p/p)}{R^n[2\pi(R + pR^p)]^{\frac{1}{2}}}.$$

Now using (4.2) and (4.4) we obtain,

4.7 $$n! \sim n^n\, e^{-n}(2\pi n)^{\frac{1}{2}},$$

4.8 $$\exp\left(\frac{R^p}{p}\right) = \exp\left(\frac{n}{p} - \frac{R}{p}\right).$$

Also

4.9 $$R^n = (n - R)^{n/p} = n^{n/p}\exp\left(\frac{n}{p}\log\left(1 - \frac{R}{n}\right)\right).$$

Expanding $\log(1 - R/n)$ yields

4.10 $$R^n \sim n^{n/p}\exp\left(-\frac{R}{p} - \frac{R^2}{2pn}\right)$$

Finally

4.11 $$(R + pR^p)^{\frac{1}{2}} \sim (pn)^{\frac{1}{2}}.$$

Using (4.7) to (4.11) in (4.6) yields

4.12 $$B_{n,p} \sim \left(\frac{n}{e}\right)^{n(1-1/p)} p^{-\frac{1}{2}}\exp\left(R + \frac{R^2}{2pn}\right).$$

We now consider two cases:

Case 1. $p = 2$. Here $e^{R+(R^2/2np)} \sim e^{n-\frac{1}{2}+\frac{1}{4}} = \exp(n^{\frac{1}{2}} - \frac{1}{4})$.

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Case 2. $p > 2$. Here $e^{R+(R^2/2np)} \sim \exp(n^{1/p})$.
Thus we obtain

4.13 $$B_{n,2} \sim \left(\frac{n}{e}\right)^{\frac{1}{2}n} \exp(n^{\frac{1}{2}})\, 2^{-\frac{1}{2}} e^{-\frac{1}{4}}$$

and

4.14 $$B_{n,p} \sim \left(\frac{n}{e}\right)^{n(1-1/p)} p^{-\frac{1}{2}} \exp(n^{1/p}) \qquad (p > 2).$$

Example 3. $P(x) = 2tx + x^2 \qquad (t > 0)$.

Here $B_n = B_n(t)$ are polynomials in t. In this case we have

4.15 $$P(R) = 2tR + R^2,$$

4.16 $$\Theta P(R) = 2tR + 2R^2 = n,$$

4.17 $$R = \tfrac{1}{2}[-t + (2n + t^2)^{\frac{1}{2}}].$$

From these and (3.49) we obtain

4.18 $$B_n(t) \sim \frac{n!\exp(2Rt + R^2)}{R^n} \frac{1}{[2\pi(2Rt + 4R^2)]^{\frac{1}{2}}},$$

where R is given by (4.17).

By computing the first two terms of the asymptotic expansion, $B_n(t)$ can be put in the form

4.19 $$B_n(t) \sim \left(\frac{n}{e}\right)^{\frac{1}{2}n} 2^{\frac{1}{2}(n-1)} \exp((2n)^{\frac{1}{2}}t - \tfrac{1}{2}t^2)\left\{1 + \frac{t^3 + 2t}{6(2n)^{\frac{1}{2}}}\right\}.$$

Our method restricts t to be positive. However, the above result is valid also for $t < 0$. $B_n(t)$ is of course related to the Hermite polynomials and (4.19) can be checked by means of the known expansion formula for these polynomials given in (**6**, p. 194).

5. Conclusion. We have given here a method of finding asymptotic expansions for numbers or functions whose generating function is of the form $e^{P(x)}$. In this paper we have restricted $P(x)$ to be a polynomial in x with non-negative coefficients. If this severe restriction on $P(x)$ is relaxed (3.49) may no longer be valid. We hope, in a subsequent paper, to show how the method may be modified to cope with the case of less restricted functions $P(x)$.

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