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# Abelian Gradings on Upper Block Triangular Matrices

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*Abstract.* Let $G$ be an arbitrary finite abelian group. We describe all possible $G$-gradings on upper block triangular matrix algebras over an algebraically closed field of characteristic zero.

## 1 Introduction

Let $G$ be an arbitrary group and $R$ an associative algebra over a field $F$. A $G$-grading on $R$ is a vector space decomposition of $R$ into the direct sum of subspaces $R = \bigoplus_{g\in G} R_g$ such that $R_g R_h \subseteq R_{gh}$ for any $g, h \in G$. The elements of the $R_g$-component are called *homogeneous* of degree $g$. If $e$ is the identity element of $G$, then $R_e$ is called the *neutral component*. The support of a graded algebra is defined as

$$\operatorname{Supp} R = \{g \in G \mid R_g \neq 0\}.$$

Similarly, one can define the support of any homogeneous subspace of $R$.

Gradings arise in a natural way in many classes of rings and algebras. A special place in the theory of graded ring and algebras is occupied by the problem of describing all possible gradings on most important structures. For example, one of the well-known results in Lie theory is the description of $\mathbb{Z}$-gradings on finite-dimensional complex Lie algebras [9]. Finite $\mathbb{Z}$-gradings of infinite-dimensional simple Lie algebras were classified in [13]. Also, gradings on some finite dimensional simple Lie algebras of Cartan type were classified in [3].

The description of the gradings on matrix algebras has an important role in PI-theory (see for instance [2, 11]) and in the theory of Lie superalgebras and colour Lie superalgebras [1]. The gradings on a matrix algebra by a finite group were described in [2, 4] provided the field $F$ is algebraically closed. Recently all possible gradings on an upper triangular matrix algebra were described (see [14, 15]). Moreover, in [6] the elementary gradings on upper triangular matrix algebras were described as well as the corresponding graded identities.

In this paper we deal with finite-dimensional graded algebras over an algebraically closed field $F$. The main object of our interest is the so-called upper block triangular matrix algebras $UT(d_1, \dots, d_m)$. These algebras are a generalization of upper triangular matrix algebras and play an exceptional role in PI-theory, especially in the study of the asymptotic behavior of the sequence of codimensions.

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Recall that

$$UT(d_1,\dots,d_m)=\begin{pmatrix} M_{d_1}(F) & B_{12} & \cdots & B_{1m}\\ 0 & M_{d_2}(F) & \cdots & B_{2m}\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & M_{d_m}(F)\end{pmatrix},$$

where $M_{d_i}(F)$ is the algebra of $d_i \times d_i$ matrices over $F$ and the $B_{ij}$ are rectangular matrices over $F$ of corresponding size. Then

$$UT(d_1,\dots,d_m) \cong M_{d_1}(F) \oplus \cdots \oplus M_{d_m}(F) + J,$$

where $\bigoplus_{i,j} B_{ij} \cong J$ is the Jacobson radical of $UT(d_1,\dots,d_m)$.

## 2 Abelian Gradings on Matrix Algebras

In this section we recall the main results about abelian gradings on finite-dimensional simple algebras over an algebraically closed field $F$.

A grading $R=\bigoplus_{g\in G} R_g$ on the matrix algebra $R=M_n(F)$ is called *elementary* if there exists an $n$-tuple $(g_1,\dots,g_n)\in G^n$ such that the matrix units $E_{ij}$, $1\le i,j\le n$ are homogeneous and $E_{ij}\in R_g \iff g=g_i^{-1}g_j$.

A grading is called *fine* if $\dim R_g = 1$ for any $g\in \operatorname{Supp} R$. In this case $T=\operatorname{Supp} R$ is always a subgroup of $G$ [2].

A special case of a fine grading is the so-called $\varepsilon$-grading, where $\varepsilon$ is an $n$-th primitive root of 1. Let $G=\langle a\rangle_n \times \langle b\rangle_n$ be the direct product of two cyclic groups of order $n$.

We set

$$X_a=\begin{pmatrix} \varepsilon^{n-1} & 0 & \cdots & 0\\ 0 & \varepsilon^{n-2} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix},\quad Y_b=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ 1 & 0 & 0 & \cdots & 0\end{pmatrix}.$$

Then

$$X_aY_bX_a^{-1}=\varepsilon Y_b,\quad X_a^n=Y_b^n=E, \tag{2.1}$$

where $E$ is the identity matrix and all $X_a^iY_b^j$, $1\le i,j\le n$, are linearly independent. Clearly, the elements $X_a^iY_b^j$, $i,j=1,\dots,n$, form a basis of $R$ and the products of these elements are uniquely defined by (2.1).

Now for any $g\in G$, $g=a^ib^j$, we denote by $R_g$ the one-dimensional subspace

$$R_g=\langle X_a^iY_b^j\rangle. \tag{2.2}$$

Then from (2.1) it follows that $R=\bigoplus_{g\in G} R_g$ is a $G$-grading on $M_n(F)$.

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The grading on $M_n(F)$ given by (2.1) and (2.2) is called an $\varepsilon$-*grading*.

One of the ways for constructing new gradings is through the tensor products. Let $G$ be an abelian group and $S, T$ two subgroups of $G$. If $A = \bigoplus_{s\in S} A_s$ and $B = \bigoplus_{t\in T} B_t$ are an $S$-grading and a $T$-grading on $A$ and $B$, respectively, then $C = A \otimes B$ is a $G$-graded algebra with $C_g = \bigoplus_{st=g} A_s B_t$ and $\operatorname{Supp} C$ is a subgroup of $ST$. In particular, one can equip $C = A \otimes B$ with a $G = S \times T$-grading if $A$ is $S$-graded and $B$ is $T$-graded.

The next result (see [2]) shows how to construct any grading on a matrix algebra starting from these examples.

***Theorem 2.1*** *Let $G$ be an abelian group and $M_n(F) = R = \bigoplus_{g\in G} R_g$ a matrix algebra over an algebraically closed field $F$ with a $G$-grading. Then there exist a decomposition $n = tq$, a subgroup $H \subseteq G$, and a $q$-tuple $(g_1, \dots, g_q) \in G^q$ such that $M_n(F)$ is isomorphic to $M_t(F) \otimes M_q(F)$ as a $G$-graded algebra where $M_t(F)$ is an $H$-graded algebra with a "fine" $H$-grading and $M_q(F)$ has an elementary grading defined by $(g_1, \dots, g_q)$.*

Recall that $R = \bigoplus_{g\in G} R_g$ is called a graded division algebra if any nonzero homogeneous element is invertible.

***Theorem 2.2*** *Let $F$ be an algebraically closed field of characteristic zero and $M_n(F) = R = \bigoplus_{g\in G} R_g$, a grading on a matrix algebra over $F$ by an abelian group $G$ such that $\dim R_g \leq 1$ for any $g \in G$. Then $H = \operatorname{Supp} R$ is a subgroup of $G$, $H = H_1 \times \cdots \times H_k$, $H_i \simeq \mathbf{Z}_{n_i} \times \mathbf{Z}_{n_i}$, $i = 1, \dots, k$, and $R$ is isomorphic to $M_{n_1}(F) \otimes \cdots \otimes M_{n_k}(F)$ as an $H$-graded algebra, where $M_{n_i}(F)$ is an $H_i$-graded algebra with some $\varepsilon_i$-grading. In particular, $M_n(F)$ is a graded division algebra.*

The algebra of upper block triangular matrices also admits an elementary grading. Indeed, if we embed such an algebra into a full matrix algebra with any elementary grading, then it will be a graded subalgebra. On the other hand it is not difficult to see that the tensor product $UT(d_1, \dots, d_m) \otimes M_k(F)$ is isomorphic to $UT(kd_1, \dots, kd_m)$ and any grading defined on $UT(d_1, \dots, d_m)$ and on $M_k(F)$ induces a grading on $UT(kd_1, \dots, kd_m)$.

## 3 Gradings on Block Triangular Matrix Algebras

In what follows we shall use the decomposition given in the next lemma. The proof of this result in case of rings can be found in [10, Lemma 3.11] or in [8, Ch.4, Sect.4]. We remark that the same arguments can be applied also in the case of algebras.

***Lemma 3.1*** *Let $R$ be an algebra over $F$ with identity element $E$ and let $C$ be a subalgebra of $R$ isomorphic to $M_n(F)$. If $E_{ij}, i, j = 1, \dots, n$, are the matrix units of $C$ and $E = E_{11} + \cdots + E_{nn}$ then $R = CD \simeq C \otimes D \simeq M_n(D)$ where $D$ is the centralizer of $C$ in $R$.*

The main result of the paper is the following.

***Theorem 3.2*** *Let $G$ be a finite abelian group and let $R = UT(d_1, \dots, d_m)$ be an upper block triangular matrix algebra over an algebraically closed field $F$ of characteristic* *zero with a G-grading. Then there exist a decomposition* $d_1 = tp_1, \dots, d_m = tp_m$, *a subgroup* $H \subseteq G$, *and an* $n-$*tuple* $(g_1, \dots, g_n) \in G^n$, *where* $n = p_1 + \cdots + p_m$ *such that* $UT(d_1, \dots, d_m)$ *is isomorphic to* $M_t(F) \otimes UT(p_1, \dots, p_m)$ *as a G-graded algebra where* $M_t(F)$ *is an H-graded algebra with a "fine" H-grading and* $UT(p_1, \dots, p_m)$ *has an elementary grading defined by* $(g_1, \dots, g_n)$.

**Proof** First of all recall the duality between $G$-grading and $\hat{G}$-action. Given a $G$-graded algebra $R = \bigoplus_{g\in G} R_g$, the dual group $\hat{G}$ of irreducible $G$-characters acts on $R$ by automorphisms. If $\chi \in \hat{G}$ and $\sum_{g\in G} a_g \in R$, then

$$\chi * \Big(\sum_{g\in G} a_g\Big) = \sum_{g\in G} \chi(g) a_g.$$

A subspace $V$ of $R$ is a graded subspace if and only if it is $\hat{G}$-stable. By [5, Lemma 2.2] the Jacobson radical of $UT(d_1, \dots, d_m)$ is graded and there exists a maximal semisimple subalgebra $B$ of $R$ homogeneous in this grading. Moreover, any maximal semisimple subalgebra of $UT(d_1, \dots, d_m)$ is isomorphic to $B_1 \oplus \cdots \oplus B_m$ where $B_i \simeq M_{d_i}(F)$. From the relations

$$B_1 J B_2 \dots J B_m \neq 0, \quad B_{\sigma(1)} J B_{\sigma(2)} \dots J B_{\sigma(m)} = 0$$

for any nonidentical permutation $\sigma \in S_n$, it follows that $B_1, \dots, B_m$ are stable under the $\hat{G}$-action. This means that $B_1, \dots, B_m$ are graded subalgebras. Using Theorem 2.1, we decompose $B_1, \dots, B_m$ into the tensor product of elementary and fine components. Hence for $i = 1, \dots, m$, let $B_i = M_{p_i}(F) \otimes M_{t_i}(F)$ where $M_{p_i}(F)$ has an elementary grading and $M_{t_i}(F)$ has a fine grading. Our goal now is to prove that all $M_{t_i}(F)$ are isomorphic.

Denote for shortness $C^{(1)} = M_{t_1}(F), \dots, C^{(m)} = M_{t_m}(F)$. We claim that if $M \subseteq R$ is a non-trivial homogeneous left (right) $C^{(i)}$-submodule of $R$, then $\dim M \leq \dim C^{(i)}$. In fact, if $u \in M$ is a homogeneous element, $C^{(i)} u \neq 0$ implies that $x_g u \neq 0$ for all $x_g \in C^{(i)}_g, x_g \neq 0$, as it follows from Theorem 2.2. On the other hand, the elements $x_g u$ belong to distinct homogeneous components for distinct $g \in G$. This proves the claim.

For our purpose it is more convenient to write the above decomposition of the $B_i$'s in the form $B_i = A^{(i)} C^{(i)}$ where $A^{(i)} \simeq M_{p_i}(F)$ and $C^{(i)} \simeq M_{t_i}(F)$. Let us now fix the two algebras $C^{(1)}$ and $C^{(2)}$. If $e_1 \in A^{(1)}, e_2 \in A^{(2)}$ are two minimal idempotents of $A^{(1)}$ and $A^{(2)}$, respectively, *e.g.*, diagonal matrix units, then $\operatorname{rank} e_1 = t_1, \operatorname{rank} e_2 = t_2$ in $R$ and we have $\dim e_1 R e_2 = \operatorname{rank} e_1 \operatorname{rank} e_2 = t_1 t_2$. On the other hand, since $e_1$ centralize $C^{(1)}$, we have

$$\dim C^{(1)} e_1 R e_2 = \dim e_1 C^{(1)} R e_2 \leq \dim e_1 R e_2 = t_1 t_2.$$

By the above claim, the dimension of the left-hand side cannot be less than $\dim C^{(1)} = t_1^2$. Hence $t_1 t_2 \geq t_1^2$. Similarly $t_1 t_2 \geq t_2^2$ and then $t_1 = t_2$. Thus,

$$\dim e_1 R e_2 = t_1^2 = t_2^2. \tag{3.1}$$

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Note that $e_1Re_2$ is a graded subspace of $R$. Hence from (3.1) it follows that for any nonzero homogeneous $X_{12} \in e_1Re_2$

$$T = C^{(1)}X_{12} = X_{12}C^{(2)}. \tag{3.2}$$

Denote by $H_1, H_2$ the supports of $C^{(1)}, C^{(2)}$, respectively. Then from (3.2) it follows that Supp $T = gH_1 = gH_2$, where $g = \deg X_{12}$. Hence $H_1 = H_2$. On the other hand, for any homogeneous $a \in C_h^{(1)}$ there exists $b \in C^{(2)}$ such that

$$aX_{12} = X_{12}b \tag{3.3}$$

as it follows from (3.2). This $b$ is homogeneous, $b \in C_h^{(2)}$, and it is uniquely defined. It is easy to check that the relation (3.3) defines an isomorphism $\varphi\colon C^{(1)} \to C^{(2)}$ of $G$-graded algebras.

Similarly we choose $X_{23}, \dots, X_{m-1m}$ and prove that $H_1 = \cdots = H_m$ and all $C^{(1)}, \dots, C^{(m)}$ are isomorphic as $H_1$-graded algebras and also $t_1 = \cdots = t_m = t$. Moreover, we can take $X_{12}, X_{23}, \dots, X_{m-1m}$ such that

$$X_{12}X_{23}\dots X_{m-1m} \neq 0. \tag{3.4}$$

Denote by $\varphi_i$, $i = 2, \dots, m$, these isomorphisms $\varphi_i\colon C^{(1)} \to C^{(i)}$ and consider the subalgebra $C$ in $R$ of the form

$$C = \{x + \varphi_2(x) + \cdots + \varphi_m(x) \mid x \in C^{(1)}\}.$$

Then $C$ is a simple homogeneous subalgebra of $R$.

Finally we take the centralizer $D$ of the subalgebra $C$ in $R$. Then by Lemma 3.1, $R = CD \simeq C \otimes D$ where $D$ is a graded subalgebra. We only need to prove that $D \simeq UT(p_1, \dots, p_m)$ and that the grading on $D$ is elementary. First note that the semisimple component of $D$ is

$$A_1 \oplus \cdots \oplus A_m \simeq M_{p_1} \oplus \cdots \oplus M_{p_m}.$$

Denote by $I$ the radical of $D$. By the choice of $X_{12}, \dots, X_{m-1m}$ and $\varphi_2, \dots, \varphi_m$ it follows that $X_{12}, \dots, X_{m-1m} \in I$

Then from (3.4) it follows that $I^{m-1} \neq 0$. Moreover, since all $A_1, \dots, A_m$ are unitary algebras and the sum of their units is the identity matrix of $R$, we have

$$A_1IA_2 \cdots IA_m \neq 0.$$

By [7, Theorem 8.2.1] $D$ contains a subalgebra isomorphic to $UT(p_1, \dots, p_m)$. But $\dim D = \dim UT(p_1, \dots, p_m)$; hence we have an isomorphism.

By construction, all matrix algebras $A_1, \dots, A_m$ have an elementary grading. In particular, all diagonal matrix units $E_{11}, \dots, E_{nn}$ of $D$ where $n = p_1 + \cdots + p_m$ are homogeneous. Then by [16, Lemma 1] the grading on $UT(p_1, \dots, p_m)$ is elementary and the proof is complete. ■

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