COMPACT SEMIRINGS WHICH ARE MULTIPLICATIVELY 0-SIMPLE [1]

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A *topological semiring* is a system $(S, +, \cdot)$ where $(S, +)$ and $(S, \cdot)$ are topological semigroups and the distributive laws

$$x \cdot (y+z) = (x \cdot y)+(x \cdot z),$$
$$(x+y) \cdot z = (x \cdot z)+(y \cdot z)$$

hold for all x, y, z in S; $+$ and $\cdot$ are called *addition* and *multiplication* respectively.

In this paper we suppose that $(S, \cdot)$ is a compact 0-simple semigroup and examine those additions $+$ for which $(S, +, \cdot)$ is a topological semiring. The special case where $(S, \cdot)$ is left 0-simple is dealt with in detail and we are able to give a satisfactory characterization of all possible additions. The results given when $(S, \cdot)$ is left 0-simple depend on [4] where the author has identified all additions when $(S, \cdot)$ is a group with zero (an even more special case).

Selden has found all commutative additions when $(S, \cdot)$ is left 0-simple ([6], Theorem 14 or [7], Theorem II). Although the proofs given here do not depend at all on Selden's results (which are in fact a corollary of the results in this paper), there are one or two places where the two discussions are similar in outline.

We begin by recalling some terminology. If S is a semigroup with zero 0 in which $\{0\}$ and S are the only two-sided [left, right] ideals and $S^2 \neq \{0\}$, then S is said to be *0-simple* [*left 0-simple*, *right 0-simple*]. A special case is a *group with zero*, which is a semigroup S in which 0 is a zero and $S \setminus \{0\}$ is a group. The structure of compact 0-simple semigroups is given in § 2.3 of [3], which is an extension to topological semigroups of the Rees Theorem ([1], Theorem 3.5) for algebraic semigroups.

The following lemma is implicit in the discussion of Rees matrix semigroups over a group with zero in [1], § 3.1. We sketch a proof for the sake of completeness.

[1] This paper is based on part of the author's Ph.D. thesis, written under the supervision of Dr. J. H. Michael.

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LEMMA 1. *If* $(S, \cdot)$ *is a finite semigroup which is isomorphic with a regular Rees matrix semigroup* $\mathscr{M}^0(G; I, \Lambda; P)$ *over a group with zero (see* [1], § 3.1) *and if e is any non-zero idempotent in S then*

(i) $|S| = |G|\,|I|\,|\Lambda|+1$;

(ii) $|eS| = |G|\,|\Lambda|+1$;

(iii) $|Se| = |G|\,|I|+1$;

(iv) $|eSe| = |G|+1$.

PROOF. It is easily seen that the only non-zero idempotents in $\mathscr{M}^0(G; I, \Lambda; P)$ are of the form $(p_{\mu i}^{-1}; i, \mu)$ where $i \in I$, $\mu \in \Lambda$ and $p_{\mu i} \neq 0$. Because the matrix P has a non-zero entry in each row and column ([1], Lemma 3.1), it is clear that when $p_{\mu i} \neq 0$,

$$(p_{\mu i}^{-1}; i, \mu) \cdot \mathscr{M}^0(G; I, \Lambda; P) = \{(a; i, \lambda) | a \in G, \lambda \in \Lambda\} \cup \{0\}.$$

Further, the right-hand set has $|G|\,|\Lambda|+1$ members. Hence (ii), and similarly (iii). The fourth statement follows because

$$(p_{\mu i}^{-1}; i, \mu) \cdot \mathscr{M}^0(G; I, \Lambda; P) \cdot (p_{\mu i}^{-1}; i, \mu) = \{(a; i, \mu) | a \in G\} \cup \{0\}.$$

LEMMA 2. *Let* $(S, +, \cdot)$ *be a finite semiring in which* $(S, +)$ *is a group and* $(S, \cdot)$ *is a group with zero. Then* $(S, +, \cdot)$ *is a field.*

PROOF. Because $S^2 = S$ it follows from [5], Theorem 7 that $(S, +)$ is abelian. Thus S is a finite division ring and therefore a field (Theorem 16, Chapter II, [10]).

We will use $E[+]$ to denote the set of additive idempotents in any semiring $(S, +, \cdot)$. If S is compact, $E[+]$ is non-empty ([3], Lemma 1.1.10) and is a multiplicative ideal. For if $x \in E[+]$ and $y \in S$,

$$xy+xy = x(y+y) = xy$$

and so $xy \in E[+]$; similarly $yx \in E[+]$.

THEOREM 1. *Let* $(S, +, \cdot)$ *be a compact semiring in which*

(i) $(S, \cdot)$ *has a zero* 0 *and is* 0*-simple*;

(ii) $(S, +)$ *is a group.*

Then $(S, +, \cdot)$ *is a finite field (with discrete topology).*

PROOF. Because $E[+]$ is a single point and also a multiplicative ideal, it follows that 0 is the identity of $(S, +)$. As $\{0\}$ is a maximal prcper ideal of $(S, \cdot)$, we see from Theorem 1 of [2] that $\{0\}$ is open. Hence each set $\{x\}(= x+\{0\})$ is open and S is finite. It now follows from Corollary 2.56 and Theorem 3.5 of [1] that $(S, \cdot)$ is completely 0-simple and so is isomorphic with a regular Rees matrix semigroup $\mathscr{M}^0(G; I, \Lambda; P)$ over a group with zero.

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If e is any primitive multiplicative idempotent, then, since $eS+eS \subset eS$ and $(S, +)$ is a finite group, $(eS, +)$ is a group; similarly $(Se, +)$ and $(eSe, +)$ are groups. But eSe is multiplicatively a group with zero ([1], Lemma 2.47), which means that $(eSe, +, \cdot)$ is a finite field (Lemma 2). Thus there is a prime p $(\geqq 2)$ and an integer $\nu \geqq 1$ such that $|eSe| = p^\nu$ ([10], page 104), and the order of e in $(S, +)$ is equal to p. Note that p and ν are independent of the idempotent e (Lemma 1).

Let x be any non-zero member of S. Because S is the union of its multiplicative 0-minimal left ideals (Corollary 2.49 of [1]) and each such ideal is of the form Se for some primitive idempotent e (Lemmas 2.44 and 2.46 of [1]) it follows that there is a primitive idempotent e such that $x = se$ for some s in S. Thus

$$px = p(se) = se+ \cdots +se = s(e+ \cdots +e) = s(pe) = s0 = 0,$$

and we see that x has order p in $(S, +)$. Consequently there are integers $\alpha, \beta, \mu \geqq 1$ with $|S| = p^\alpha$, $|eS| = p^\beta$ and $|Se| = p^\mu$ (Corollary to Theorem 1, Chapter IV of [10]). Now from Lemma 1,

$$\begin{aligned} p^\alpha &= |G|\,|I|\,|\Lambda|+1, \\ p^\beta &= |G|\,|\Lambda|+1, \\ p^\mu &= |G|\,|I|+1, \\ p^\nu &= |G|+1. \end{aligned}$$

Hence $|G| = p^\nu - 1$ and so

$$p^\alpha - 1 = (p^\nu - 1) \cdot \frac{p^\beta - 1}{p^\nu - 1} \cdot \frac{p^\mu - 1}{p^\nu - 1} = \frac{(p^\beta - 1)(p^\mu - 1)}{p^\nu - 1}.$$

If we multiply out and divide by p^ν, we see that

$$p^\alpha - p^{\alpha-\nu} - 1 = p^{\beta+\mu-\nu} - p^{\beta-\nu} - p^{\mu-\nu}. \tag{1}$$

Now if $\nu < \beta$ and $\nu < \mu$, it follows that $\alpha > \nu$ and so p divides the right hand side of (1) but not the left hand side. Hence either $\nu = \beta$ or $\nu = \mu$.

Suppose firstly that $\nu = \beta$; then $|\Lambda| = 1$. Let e be any primitive idempotent of $(S, \cdot)$; then $|Se| = |S|$ (Lemma 1) and so $Se = S$. Because Λ has only one member, the regularity of $\mathscr{M}^0(G; I, \Lambda; P)$ ensures that $p_{\lambda i} \neq 0$ for all $i \in I$, $\lambda \in \Lambda$ ([1], Lemma 3.1). Hence if x and y are non-zero members of S it follows from (1′) of page 88 of [1] that $xy \neq 0$. Let f be any other non-zero idempotent of $(S, \cdot)$. Because $Se = S$ it is clear that $f = se$ for some s in S and thus

$$fe = (se)e = s(ee) = se = f.$$

Hence

$$f[e+(p-1)f] = fe+(p-1)f^2 = f+(p-1)f = pf = 0,$$

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from which we see that $e+(p-1)f = 0$. Consequently,

$$e = e+0 = e+pf = e+[(p-1)f+f] = [e+(p-1)f]+f = 0+f = f.$$

Thus e is the only non-zero idempotent. But S is the union of its multiplicative 0-minimal right ideals ([1], Corollary 2.49) and each such ideal is of the form fS for a non-zero idempotent f (Lemmas 2.44 and 2.46 of [1]). Hence $S = eS$ and so

$$eSe = (eS)e = Se = S$$

from which it follows that $(S, +, \cdot)$ is a field. The result follows similarly if $\nu = \mu$.

THEOREM 2. *Let $(S, +, \cdot)$ be a compact semiring in which $(S, \cdot)$ is 0-simple. Then $S \setminus \{0\}$ is compact and one of the following holds:*

(i) $x+y = 0$ *for all x, y in S;*

(ii) $(S, +, \cdot)$ *is a finite field;*

(iii) *addition is left trivial;*

(iv) *addition is right trivial;*

(v) $(S\setminus\{0\}, +)$ *is an idempotent subsemigroup and* $x+0 = 0+x = x$ *for all x in S;*

(vi) $(S, +)$ *is idempotent and* $x+0 = 0+x = 0$ *for all x in S.*

PROOF. Because $\{0\}$ is a maximal proper multiplicative ideal it follows from Theorem 1 of [2] that $\{0\}$ is open; hence $S\setminus\{0\}$ is closed and compact. As $S+S$, $E[+]$, $S+0$ and $0+S$ are all multiplicative ideals, each is either $\{0\}$ or S.

If $S+S = \{0\}$, we have (i). Accordingly we assume that $S+S = S$.

If $E[+] = \{0\}$, it follows from Corollary 2 of [2] that $(S, +)$ is a group. Hence S is a finite field by Theorem 1. Assume now that $E[+] = S$.

If $S+0 = S$ and $0+S = \{0\}$ then, for each x in S, there is a y with $y+0 = x$; hence

$$x+0 = (y+0)+0 = y+(0+0) = y+0 = x.$$

Thus, for all x, y in S,

$$x+y = (x+0)+y = x+(0+y) = x+0 = x,$$

and we have (iii). Similarly we have (iv) if $S+0 = \{0\}$ and $0+S = S$.

If $S+0 = 0+S = S$, then, as above, $x+0 = 0+x = x$ for all x. If $x, y \in S\setminus\{0\}$, then $x+y \neq 0$, for otherwise

$$0 = x+y = (x+x)+y = x+(x+y) = x+0 = x.$$

Finally, if $S+0 = 0+S = \{0\}$, we have (vi).

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We now turn our attention to compact semigroups which are left 0-simple. If $(S, \cdot)$ is any such semigroup we are looking for a characterization of all additions $+$ for which $(S, +, \cdot)$ is a topological semiring (*Problem A*). We give what seems to be a satisfactory solution by showing how this problem can be reduced to the following more restricted problem.

Problem B. If $(T, \otimes)$ is any compact left simple semigroup, give a characterization of all additions $\oplus$ for which $(T, \oplus, \otimes)$ is a topological semiring.

That Problem B is more restricted than Problem A may be seen by considering a third problem, Problem C.

Problem C. If $(S, \cdot)$ is any compact left 0-simple semigroup, give a characterization of all additions $+$ for which $(S, +, \cdot)$ is a topological semiring in which $(S \setminus \{0\}, +, \cdot)$ is a subsemiring and $x+0 = 0+x = x$ for all x in S.

Clearly the class of semirings in Problem C is at least as restricted as that in Problem A. (In fact we shall see below that it is more restricted in the strict sense.) On the other hand, there is a $1-1$ correspondence between the semirings $(S, +, \cdot)$ in C and those $(T, \oplus, \otimes)$ in B. For given $(S, +, \cdot)$ in C, $(S \setminus \{0\}, +, \cdot)$ is one of the semirings in B (we show below that $S \setminus \{0\}$ is a compact left simple semigroup) and conversely, given $(T, \oplus, \otimes)$ in B, if we adjoin an element 0 as an isolated point to T and extend $\oplus$, $\otimes$ to $S = T \cup \{0\}$ by

$$x \oplus 0 = 0 \oplus x = x \text{ all } x \in S,$$
$$x \otimes 0 = 0 \otimes x = 0 \text{ all } x \in S,$$

then $(S, \oplus, \otimes)$ is one of the semirings considered in C. Thus B and C are essentially equivalent and each deals with a more restricted class of semirings than does A.

Unfortunately the only known results about Problem B appear to be in [5], Theorem 2, which gives but part of the information required.

Let $(S, \cdot)$ be a compact left 0-simple semigroup and let $T = S \setminus \{0\}$. Then $\{0\}$ is topologically closed and open ([2], Theorem 1) and $(T, \cdot)$ is a compact left simple semigroup ([1], Theorem 2.27). We will denote the idempotents of $(S, \cdot)$ and $(T, \cdot)$ by $E[\cdot]$ and $F[\cdot]$ respectively. If G is one of the maximal subgroups of T (say $G = f'T$ where $f' \in F[\cdot]$), then $T = F[\cdot]G$ and, in fact, T is topologically isomorphic with $F[\cdot] \times G$ ([8], Theorem 1). Also, for all x in T and $f \in F[\cdot]$, $Tx = T$ and $xf = x$ ([8]).

Example 1. Suppose $(S, \cdot)$ is as above. Let H be any normal subgroup of G which is topologically closed and open with respect to G and let $+$ be any addition of a semiring on (the compact left simple semigroup) $F[\cdot]H$ for which the normal subgroups $f'+H$ and $H+f'$ of H are also normal

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in G. (If $+$ is an addition of a semiring on $F[\cdot]H$ then is H a subsemiring ([5], Theorem 2) and it follows from [4], Theorem 1 that $f'+H$ and $H+f'$ are normal in H.) Then we can extend $+$ to the whole of S by putting

$$e\alpha+f\beta = \begin{cases} (e+f\beta\alpha^{-1})\alpha & \text{if } \beta\alpha^{-1}\in H, \\ 0 & \text{if } \beta\alpha^{-1}\notin H, \end{cases}$$
$$e\alpha+0 = 0+e\alpha = 0+0 = 0,$$

for all $e, f \in F[\cdot]$ and $\alpha, \beta \in G$.

Lemma 3. *If $+$ is defined as in Example 12 then $(S, +, \cdot)$ is a semiring.*

Proof. Because H and $G\setminus H$ are closed and open in G and the function $\varphi : T\times T \to G$, given by $\varphi(e\alpha, f\beta) = \beta\alpha^{-1}$, is continuous ([9]), we see that the sets $\varphi^{-1}(H)$ and $\varphi^{-1}(G\setminus H)$ are both closed and open. It is clear that $+$ is continuous on each of the sets $(S\times S)\setminus(T\times T)$, $\varphi^{-1}(H)$, $\varphi^{-1}(G\setminus H)$ and so, since each is closed and open and their union is $S\times S$, $+$ is continuous.

It follows from the lemma of [4] that $G \cup \{0\}$ is a semiring. For any $e, f \in F[\cdot]$ and $\alpha, \beta \in G$ we can see that there exists $h \in F[\cdot]$ with $e\alpha+f\beta = h(\alpha+\beta)$. This is trivial if $\beta\alpha^{-1}\notin H$ for then

$$\alpha+\beta = f'\alpha+f'\beta = 0$$

and any h will do. If $\beta\alpha^{-1}\in H$ then e and $f\beta\alpha^{-1}$ are members of $F[\cdot]H$ which is a semiring. Thus there is $h\in F[\cdot]$ with $e+f\beta\alpha^{-1} = h(f'+\beta\alpha^{-1})$ ([5], Theorem 2) and, since $G\cup\{0\}$ is a semiring,

$$e\alpha+f\beta = (e+f\beta\alpha^{-1})\alpha = [h(f'+\beta\alpha^{-1})]\alpha = h[(f'+\beta\alpha^{-1})\alpha] = h(\alpha+\beta).$$

The first distributive law,

$$x(y+z) = xy+xz,$$

is obviously satisfied if any of x, y, z is 0. Hence we can let $x = e\alpha$, $y = f\beta$, $z = g\gamma$ where $e, f, g \in F[\cdot]$ and $\alpha, \beta, \gamma \in G$. Then if $h\in F[\cdot]$ is such that $f\beta+g\gamma = h(\beta+\gamma)$, we see that

$$x(y+z) = e\alpha\cdot h(\beta+\gamma) = e\alpha(\beta+\gamma) = e\alpha(f'+\gamma\beta^{-1})\beta,$$
$$xy+xz = e\alpha f\beta+e\alpha g\gamma = e\alpha\beta+e\alpha\gamma.$$

If $\gamma\beta^{-1}\notin H$ then, since H is normal in G, $\alpha\gamma\beta^{-1}\alpha^{-1}\notin H$ also and so $x(y+z) = xy+xz = 0$. If $\gamma\beta^{-1}\in H$ then, because $e, f', \alpha\gamma\beta^{-1}\alpha^{-1}$ are all in $F[\cdot]H$,

$$\begin{aligned} xy+xz &= (e+e\alpha\gamma\beta^{-1}\alpha^{-1})(\alpha\beta) = [e(f'+\alpha\gamma\beta^{-1}\alpha^{-1})](\alpha\beta) \\ &= e[(f'+\alpha\gamma\beta^{-1}\alpha^{-1})(\alpha\beta)] = e(\alpha\beta+\alpha\gamma) = e[\alpha(\beta+\gamma)] = x(y+z). \end{aligned}$$

The other distributive law can be checked similarly.

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The associative law

$$(x+y)+z = x+(y+z)$$

is clearly satisfied if any of x, y, z is 0. Thus we can let $x = e\alpha$, $y = f\beta$, $z = g\gamma$ where $e, f, g \in F[\cdot]$ and $\alpha, \beta, \gamma \in G$. It is a consequence of the distributive laws that the associativity condition is equivalent to

$$[(e+f\beta\alpha^{-1})+g\gamma\alpha^{-1}]\alpha = [e+(f\beta\alpha^{-1}+g\gamma\alpha^{-1})]\alpha.$$

Thus it is sufficient to show that

$$e+(f\beta+g\gamma) = (e+f\beta)+g\gamma$$

for all $e, f, g \in F[\cdot]$ and $\beta, \gamma \in G$. Now there exist $h_1, h_2, h_3, h_4 \in F[\cdot]$ such that

$$e+(f\beta+g\gamma) = ef'+h_1(\beta+\gamma) = h_2[f'+(\beta+\gamma)],$$
$$(e+f\beta)+g\gamma = h_3(f'+\beta)+g\gamma = h_4[(f'+\beta)+\gamma].$$

Hence the result if $f'+\beta+\gamma = 0$. If $f'+\beta+\gamma \neq 0$, then $\beta, \gamma\beta^{-1} \in H$ since $f'+\beta \neq 0$ and $\beta+\gamma \neq 0$, and thus

$$e+(f\beta+g\gamma) = [e\beta^{-1}+(f+g\gamma\beta^{-1})]\beta.$$

But $e\beta^{-1}, f, g\gamma\beta^{-1} \in F[\cdot]H$ and so

$$e+(f\beta+g\gamma) = [(e\beta^{-1}+f)+g\gamma\beta^{-1}]\beta = (e+f\beta)+g\gamma.$$

THEOREM 3. *Let $(S, \cdot)$ be a compact left 0-simple semigroup and let $+$ be a binary operation on S. Then $(S, +, \cdot)$ is a topological semiring if and only if one of the following holds*:

(i) $x+y = 0$ *for all x, y in S*;

(ii) $(S, +, \cdot)$ *is a finite field*;

(iii) *addition is left trivial*;

(iv) *addition is right trivial*;

(v) $T(= S\setminus\{0\})$ *is a (compact) semiring (which is multiplicatively left simple) and $x+0 = 0+x = 0$ for all x in S*;

(vi) $+$ *is as in Example* 1.

PROOF. When one of (i)—(vi) holds it is clear that $(S, +, \cdot)$ is a semiring.

Now suppose that $(S, +, \cdot)$ is a topological semiring. It follows from Theorem 2 that either one of (i)—(v) holds or else $(S, +)$ is idempotent and $x+0 = 0+x = 0$ for all x. In this latter case, if f' is any member of $F[\cdot]$, it is clear that $f'S$ is a compact semiring (which is multiplicatively a group with zero) of the type (vi) of [4], Theorem 2. Thus if

$$H = \{\alpha | \alpha \in G = f'T \text{ and } f'+\alpha \neq 0\},$$

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it follows from [4], Theorem 2 that H is a subsemiring which is multiplicatively a normal subgroup of G, that H is topologically both open and closed with respect to G and that the normal subgroups $f'+H$ and $H+f'$ of H are also normal in G. If $e, f \in F[\cdot]$ and $\gamma \in G$ then

$$f'(e+f\gamma) = f'e+f'f\gamma = f'+f'\gamma = f'+\gamma$$

and so $e+f\gamma = 0$ if and only if $\gamma \notin H$. Thus if $\alpha, \beta \in G$ and $\beta\alpha^{-1} \notin H$,

$$e\alpha+f\beta = (e+f\beta\alpha^{-1})\alpha = 0\alpha = 0,$$

while if $\beta\alpha^{-1} \in H$,

$$e\alpha+f\beta = (e+f\beta\alpha^{-1})\alpha \neq 0.$$

If $\beta\alpha^{-1} \in H$, suppose that $e+f\beta\alpha^{-1} = g\delta$ for $g \in F[\cdot]$ and $\delta \in G$; then

$$\delta = f'\delta = f'g\delta = f'(e+f\beta\alpha^{-1}) = f'+\beta\alpha^{-1} \in H.$$

In particular, if $\alpha, \beta \in H$ then

$$e\alpha+f\beta = (e+f\beta\alpha^{-1})\alpha = g\delta\alpha \in F[\cdot]H$$

and we see that $F[\cdot]H$ is a subsemiring. Thus $+$ is as in Example 1.

Recall that a semigroup $(S, +)$ is said to be *normal* if $x+S = S+x$ for all x in S. The following lemma (which is almost certainly not original) is a consequence of this definition.

LEMMA 4. *If $(S, +)$ is a normal idempotent semigroup then it is commutative.*

PROOF. Let $x, y \in S$. Because $x+y \in x+S = S+x$, there exists z in S with $x+y = z+x$ so that

$$x+y+x = (x+y)+x = (z+x)+x = z+(x+x) = z+x = x+y.$$

Similarly, because $y+x \in S+x = x+S$, there exists w in S with $y+x = x+w$ so that

$$x+y+x = x+(y+x) = x+(x+w) = (x+x)+w = y+x.$$

We can now identify all normal additions of compact semirings which are multiplicatively left 0-simple. We need two further examples.

EXAMPLE 2. Let $(S, +)$ be any compact commutative idempotent semigroup with an isolated unit 0. If we define multiplication on S by putting $x \cdot 0 = 0 \cdot x = 0$ for all x in S and $x \cdot y = x$ for all x, y in $S \setminus \{0\}$ then it is clear that $(S, +, \cdot)$ is an additively commutative semiring in which $(S, \cdot)$ is left 0-simple.

EXAMPLE 3. Let $(F[\cdot], +)$ be a compact commutative idempotent semigroup and let $(G, \cdot)$ be any finite group. Then put $T = F[\cdot] \times G$ and

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adjoin 0 as an isolated point to T so that $S = T \cup \{0\}$. If we extend $+$ and $\cdot$ to the whole of S by putting

$$(e, \alpha)+(f, \beta) = \begin{cases} (e+f, \alpha) & \text{if} \quad \alpha = \beta, \\ 0 & \text{if} \quad \alpha \neq \beta, \end{cases}$$
$$(e, \alpha)+0 = 0+(e, \alpha) = 0+0 = 0,$$
$$(e, \alpha)\cdot(f, \beta) = (e, \alpha\cdot\beta),$$
$$(e, \alpha)\cdot 0 = 0\cdot(e, \alpha) = 0\cdot 0 = 0,$$

for all $e, f \in F[\cdot]$ and $\alpha, \beta \in G$, then $(S, +, \cdot)$ can be seen to be an additively commutative semiring in which $(S, \cdot)$ is left 0-simple.

THEOREM 4. *Let $(S, \cdot)$ be a compact semigroup which is left 0-simple and let $+$ be a binary operation on S. Then $(S, +, \cdot)$ is an additively normal topological semiring if and only if one of the following holds:*

(i) *$x+y = 0$ for all x, y in S;*

(ii) *$(S, +, \cdot)$ is a finite field;*

(iii) *$(S, +, \cdot)$ is as in Example 2;*

(iv) *$(S, +, \cdot)$ is as in Example 3.*

PROOF. When one of (i)—(iv) holds it is clear that $(S, +, \cdot)$ is an additively normal (in fact, additively commutative) semiring.

Now suppose that $(S, +, \cdot)$ is an additively normal semiring; then one of (i)—(vi) of Theorem 3 holds. Cases (i) and (ii) of Theorem 3 give (i) and (ii) of this theorem while cases (iii) and (iv) of Theorem 3 are not additively normal. In cases (v) and (vi) of Theorem 3, $E[+] = S$ and so it follows from Lemma 4 that $+$ is commutative.

In case (v) of Theorem 3, $S\diagdown\{0\}$ is a compact semiring which is multiplicatively left simple. Thus if G is any maximal multiplicative subgroup of $S\diagdown\{0\}$, then G, being an additively commutative semiring ([5], Theorem 2), is a single point (Corollary 1 to [4], Theorem 1) and so $(S, +, \cdot)$ is as in Example 2.

In case (vi) of Theorem 3, $(S, +, \cdot)$ is given by Example 1. The set H in Example 1 is a semiring which is multiplicatively a group. But because addition is commutative here, H must be a single point (Corollary 1 to [4], Theorem 1). Now H is an open subset of G so that each set $\{\alpha\}$ in G is open and G must be finite. This gives us Example 3.

The above theorem is a slight generalization of Selden's identification of all commutative additions of a compact semiring which is multiplicatively left 0-simple (see [6], Theorem 14 or [7], Theorem II). As we have seen, all normal additions of such a semiring are commutative, which is not surprising in view of Lemma 4, so that the additions in Theorem 4 are the same as those Selden found.

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