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DISTANCE MATRICES AND RIDGE FUNCTION INTERPOLATION

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ABSTRACT A geometric characterization is given for a collection of points in $\mathbb{R}^d$ to produce a singular l_1-distance matrix Some quantitative results are established in terms of "characteristic matrices" The results in this paper generalize those of Dyn, Light and Cheney and have application to ridge function interpolation

1. **Introduction.** In a series of papers concerning the imbeddings of metric spaces in Hilbert space, Schoenberg ([S1], [S2], [S3]) systematically studied the properties of the matrices $(\|x_i - x_j\|_p)$ where $x_1, \ldots, x_n \in \mathbb{R}^d$ and $\|\cdot\|_p$ is the l_p-norm. These matrices are called *l_p-distance matrices*. Among other things, he proved that for $1 \leq p \leq 2$ the $n \times n$ matrix $(\|x_i - x_j\|_p)$ has at most 1 positive eigenvalue. Schoenberg also proved that the matrix $(\|x_i - x_j\|_\infty)$ has at most 1 positive eigenvalue if the points are restricted to be in $\mathbb{R}^2$. Herz [H] showed that in $\mathbb{R}^3$, the matrix $(\|x_i - x_j\|_\infty)$ no longer enjoys this property. On the other hand, Herz proved the following general result: if $(X, \|\cdot\|)$ is a two-dimensional real normed linear space, and $x_1, \ldots, x_n \in X$, then the matrix $(\|x_i - x_j\|)$ has at most 1 positive eigenvalue.

Schoenberg's embedding argument implies that for $1 < p \leq 2$ the matrix $(\|x_i - x_j\|_p)$ has exactly 1 positive eigenvalue and $(n-1)$ negative eigenvalues if the points $x_1, \ldots, x_n$ are distinct, which implies that the matrix is nonsingular. The case $p = 1$ is more interesting: singular l_1-distance matrices exist and one wishes to give a geometric characterization of those configurations of points that give rise to singular matrices. Let $\mathcal{N} := \{x_1, \ldots, x_n\}$ be a set of n distinct points in $\mathbb{R}^d$, $d \geq 2$. Dyn, Light and Cheney [DLC] showed that if $\mathcal{N} \subset \mathbb{R}^2$ then the matrix $(\|x_i - x_j\|_1)$ is singular if and only if $\mathcal{N}$ contains a *closed path*; see [DLC] for the definition of closed path in $\mathbb{R}^2$. The following pictures are examples of closed paths formed by four points, six points and ten points respectively.

In the plane, there is an analogous "closed path" characterization for the l_∞-distance matrices due to the similarity between the l_1 and l_∞ norm in $\mathbb{R}^2$; see [DLC].

The concept of "closed path" appeared in a different context in the influential paper by Diliberto and Straus [DS]. In that paper, they also introduced "multidimensional closed path" in $\mathbb{R}^d$, $d \geq 3$. However, as pointed out by Light [L], there are a number of ambiguities and faults in their formulation. Light [L] re-defined multidimensional closed

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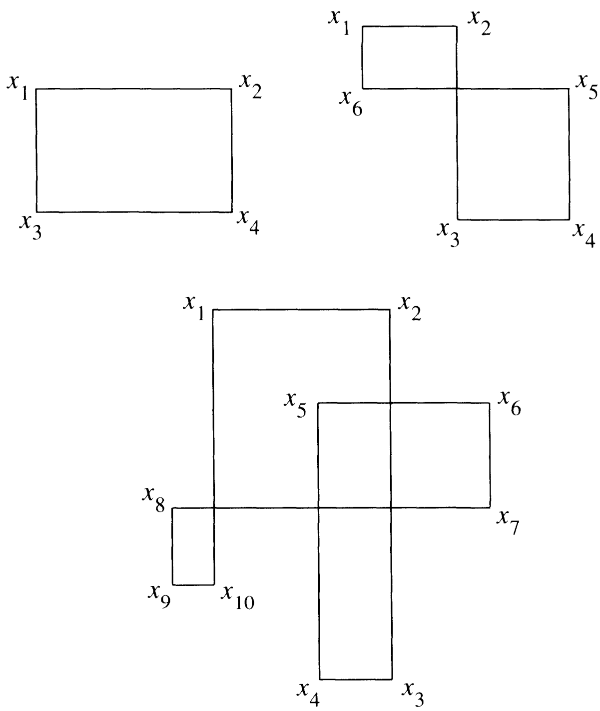

FIGURE 1

path in $\mathbb{R}^d$, $d \geq 3$, using the structure of a tree. He proved that if the set $\mathcal{N}$ contains a multidimensional closed path under the new definition, then the matrix $(\|x_i - x_j\|_1)$ is singular. The converse is however not true; see [SX].

In Section 2, we give a necessary and sufficient geometric condition for a configuration of points in $\mathbb{R}^d$, $d \geq 2$, to yield a singular l_1-distance matrix. Inspired by the work of Braess and Pinkus [BP], we define what it means for a configuration of points to be a *sum of rectangles* and show that the associated l_1-distance matrix is singular if and only if there is a nonempty subset which is a sum of rectangles. This generalizes the results of Dyn, Light and Cheney, since in $\mathbb{R}^2$ a nonempty sum of rectangles is a closed path.

In Section 3, we obtain some quantitative results. We associate to $\mathcal{N}$ two matrices: (i) the l_1-distance matrix, (ii) the characteristic matrix which is nonnegative definite and has nonnegative integer entries less than or equal to d. There exist interesting relationships between the matrix $(\|x_i - x_j\|_1)$ and the characteristic matrix. We show that the matrix $(\|x_i - x_j\|_1)$ is singular if and only if the characteristic matrix is singular. Furthermore, the characteristic matrix allows us to get an asymptotically best possible estimate for the lower bound of the eigenvalues of the matrix $(\|x_i - x_j\|_1)$ which is desired in application.

Finally in Section 4, we discuss the application of our results to the problem of interpolating scattered data using ridge functions.

2. **Sum of rectangles.** In what follows, the discussion depends in part on a semi-norm defined in [SX] whose definition we now recall. We write a generic point in $\mathbb{R}^d$ in the form $x = (\xi_1, \ldots, \xi_d)$. Using the coordinate functional $p_\nu : p_\nu(x) = \xi_\nu, \nu = 1, \ldots, d$, we define the semi-norm $|\cdot|_{\mathcal{N}}$ on $\mathbb{R}^n$ associated with $\mathcal{N}$ as follows. If $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$, then

$$(1) \qquad |c|_{\mathcal{N}} := \Big[\sum_{\nu=1}^{d} \sum_{y \in p_\nu(\mathcal{N})} \Big(\sum\{c_j : p_\nu x_j = y\}\Big)^2\Big]^{\frac{1}{2}}.$$

In [SX], it was proved that $|\cdot|_{\mathcal{N}}$ being a norm is equivalent to the nonsingularity of certain interpolation matrices. In particular, Theorem 10 in [SX] implies that the matrix $(\|x_i - x_j\|_1)$ is nonsingular if and only if the semi-norm $|\cdot|_{\mathcal{N}}$ is a norm. Note that this is still true if the semi-norm in (1) is replaced by some equivalent semi-norm, for example,

$$\Big[\sum_{\nu=1}^{d} \sum_{y \in p_\nu(\mathcal{N})} \Big(\sum\{c_j : p_\nu x_j = y\}\Big)^p\Big]^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

We use our definition to simplify the calculations in Section 3.

Recall that the free abelian group on a set X is the set of all functions $f : X \longrightarrow \mathbf{Z}$ such that $f(a) = 0$ for all but finitely many $a \in X$. The group operation is given by addition of functions. Let G denote the free abelian group on $\mathbb{R}^d$. We have the following theorem.

THEOREM 1. *Let $\mathcal{N} := \{x_1, \ldots, x_n\}$ be n points in $\mathbb{R}^d$. In order that the matrix $(\|x_i - x_j\|_1)$ be singular it is necessary and sufficient that there exist an element f of G supported on a nonempty subset of $\mathcal{N}$ such that for each ν, $1 \le \nu \le d$, and all $y \in p_\nu(\mathcal{N})$,*

$$(2) \qquad \sum\{f(x_j) : p_\nu(x_j) = y\} = 0.$$

PROOF. To prove the sufficiency, assume that $f \in G$ is supported on a nonempty subset of $\mathcal{N}$ and satisfies Equation (2). Then the semi-norm $|\cdot|_{\mathcal{N}}$ is not a norm. Hence the matrix $(\|x_i - x_j\|_1)$ is singular. To prove the necessity, assume that the matrix is singular. Then the semi-norm $|\cdot|_{\mathcal{N}}$ is not a norm. Thus the following system of linear equations with unknowns $c_1, \ldots, c_n$

$$\sum\{c_j : p_\nu(x_j) = y\} = 0, \quad y \in p_\nu(\mathcal{N}), \quad \nu = 1, \ldots, d.$$

has nontrivial solutions. Let $(c_1, \ldots, c_n)$ be such a solution with integer components (this is possible because the entries of the coefficient matrix of the linear system are either 1 or 0). Define an element f of G by letting

$$f(x) = \begin{cases} c_j, & \text{if } x = x_j \\ 0, & \text{otherwise.} \end{cases}$$

It is then easy to see that f satisfies Equation (2). ∎

We need to introduce some notations and definition. The function $g \in G$ which is supported on four points x_1, x_2, x_3, x_4 and satisfies $g(x_1) = g(x_4) = 1$ and $g(x_2) = g(x_3) = -1$ will be denoted by $g(x_1, x_2, x_3, x_4)$. If x_i are of the form

$$\begin{aligned} x_1 &= (\xi_1, \dots, \xi_d), \\ x_2 &= (\xi_1, \dots, \xi_{i-1}, \zeta_i, \xi_{i+1}, \dots, \xi_d), \\ x_3 &= (\xi_1, \dots, \xi_{j-1}, \zeta_j, \xi_{j+1}, \dots, \xi_d), \\ x_4 &= (\xi_1, \dots, \xi_{i-1}, \zeta_i, \xi_{i+1}, \dots, \xi_{j-1}, \zeta_j, \xi_{j+1}, \dots, \xi_d), \end{aligned}$$

where $i < j$, $\xi_i \neq \zeta_i$, $\xi_j \neq \zeta_j$, then we say that g is a *signed rectangle* which we will denote by $r(x_1, x_2, x_3, x_4)$.

DEFINITION 1. A nonempty finite set in $\mathbb{R}^d$, $d \geq 2$, is said to be a *sum of rectangles* if it is the support of a function of the form $\sum n_i r_i$ in which each r_i is a signed rectangle, the coefficients n_i are integers, and the sum is finite.

This is similar to the definition of a sum of "bricks" given in [BP]. This formalism may seem somewhat daunting, but the geometric picture is quite simple. One may consider a signed rectangle to be a rectangle in $\mathbb{R}^d$ whose edges are parallel to the coordinate axes and which has opposite signs associated to adjacent vertices. Given a collection of signed rectangles with integers associated to each, a point is in the corresponding sum of rectangles if and only if the sum of associated (signed) integers is nonzero.

EXAMPLE 1. In $\mathbb{R}^3$, consider the points $x_1 = (0,0,0)$, $x_2 = (0,0,1)$, $x_3 = (0,1,0)$, $x_4 = (0,1,1)$, $x_5 = (1,0,0)$, $x_6 = (1,0,1)$, $x_7 = (1,1,1)$. Let r_1 be the signed rectangle $r(x_1, x_2, x_3, x_4)$, r_2 be $r(x_1, x_2, x_5, x_6)$, and r_3 be $r(x_2, x_4, x_6, x_7)$. In this case $\operatorname{supp}(r_1 + r_2 + r_3) = \{x_1, x_2, x_3, x_6, x_7\}$. Thus $\{x_1, x_2, x_3, x_6, x_7\}$ is a sum of rectangles. (See Figure 2 for the geometric picture.)

This is also an example of a collection of points whose l_1-distance matrix is singular, but which is not a multidimensional closed path under Light's definition [L]; see [SX, Example 4]. This is no coincidence as we have the following theorem.

THEOREM 2. *The l_1-distance matrix of a finite set of distinct points $\mathcal{N} = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$, $d \geq 2$, is singular if and only if it contains a sum of rectangles.*

PROOF. To prove the sufficiency, let $f = \sum n_i r_i$ be the associated sum of signed rectangles. It is easy to see that for any fixed $y \in \mathbb{R}^d$ and $1 \leq \nu \leq d$, we have

$$\sum \{ r_i(x) : p_\nu(x) = y \} = 0.$$

Since f is supported on a subset of $\mathcal{N}$, we can write

$$\begin{aligned} \sum \{ f(x_j) : p_\nu(x_j) = y \} &= \sum \{ f(x) : p_\nu(x) = y \} \\ &= \sum_i n_i \sum \{ r_i(x) : p_\nu(x) = y \} = 0. \end{aligned}$$

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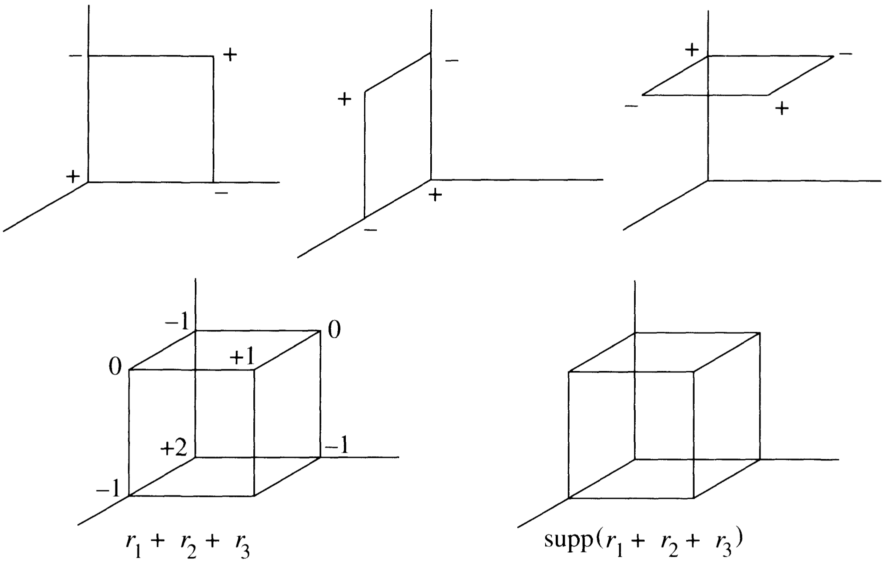

FIGURE 2

Hence f satisfies Equation (2) and by Theorem 1, the associated l_1-distance matrix is singular.

To prove the necessity, we borrow the techniques used in the proof of [BP, Theorem 4.2]. We define $f \in G$ as in Theorem 1. It suffices to show that f is a sum of signed rectangles. We will induct on $M(f) := \sum_{i=1}^{n} |f(x_i)|$. In order for Equation (2) to hold for all ν and y, supp(f) must contain at least four points, thus, $M \geq 4$. We break the proof into four cases:

CASE (i). $d = 2, M = 4$. In this case f itself must be a signed rectangles and we are done.

CASE (ii). $d = 2, M > 4$. By renumbering the points of $\mathcal{N}$ if necessary, we may assume that $f(x_1) > 0, x_1 = (\xi_1, \xi_2)$. In order for Equation (2) to hold for $\nu = 1, y = \xi_1$ and $\nu = 2, y = \xi_2$, there must be $x_i, x_j \in \mathcal{N}$ such that $x_i = (\xi_1, \zeta_2)$, $x_j = (\zeta_1, \xi_2)$ and $f(x_i) < 0, f(x_j) < 0$. Let $z = (\zeta_1, \zeta_2)$, r be the signed rectangle $r(x_1, x_i, x_j, z)$ and let $\tilde{f} = f - r$. Now $|\tilde{f}(x_1)| = |f(x_1)| - 1$, $|\tilde{f}(x_i)| = |f(x_i)| - 1$, $|\tilde{f}(x_j)| = |f(x_j)| - 1$ $|\tilde{f}(z)| \leq |f(z)| + 1$ and $|\tilde{f}(x)| = |f(x)|$ for the other $x \in \mathcal{N}$. Since (as was seen in the first part of the proof) signed rectangles contribute nothing to the left hand side of Equation (2), $\tilde{f}$ also satisfies Equation (2) and $M(\tilde{f}) < M(f)$, so by the inductive hypothesis, $\tilde{f}$ is a sum of signed rectangles and consequently, so is f.

CASE (iii). $d \geq 3, M = 4$. As above, supp(f) must consist of four points. Here, however, the situation is more complicated. Without loss of generality, we may renumber the coordinates so that $f = g(x_1, x_2, x_3, x_4)$ where

$$\begin{aligned} x_1 &= (\xi_1, \ldots, \xi_k, \alpha_{k+1}, \ldots, \alpha_l, \beta_{l+1}, \ldots, \beta_d), \\ x_2 &= (\xi_1, \ldots, \xi_k, \alpha_{k+1}, \ldots, \alpha_l, \delta_{l+1}, \ldots, \delta_d), \\ x_3 &= (\xi_1, \ldots, \xi_k, \gamma_{k+1}, \ldots, \gamma_l, \beta_{l+1}, \ldots, \beta_d), \\ x_2 &= (\xi_1, \ldots, \xi_k, \gamma_{k+1}, \ldots, \gamma_l, \delta_{l+1}, \ldots, \delta_d), \end{aligned}$$

where $\alpha_\iota \neq \gamma_\iota$, $\beta_\iota \neq \delta_\iota$. Note that in order for the points to be distinct, there must be at least one "α coordinate" and one "β coordinate". Unlike the case $d = 2$, this f will not, in general, be a signed rectangles. However, it can be written as a sum of signed rectangles. We need the following lemma to verify this claim.

SUBLEMMA. *If*

$$\begin{aligned} Q_1 &= (a_1, \ldots, a_m, b_{m+1}, \ldots, b_d), \\ Q_2 &= (u_1, \ldots, u_m, b_{m+1}, \ldots, b_d), \\ Q_3 &= (a_1, \ldots, a_m, v_{m+1}, \ldots, v_d), \\ Q_2 &= (u_1, \ldots, u_m, v_{m+1}, \ldots, v_d), \end{aligned}$$

are four distinct points, then $g(Q_1, Q_2, Q_3, Q_4)$ is a sum of signed rectangles.

PROOF OF SUBLEMMA. Let

$$\begin{aligned} z_0 &= (a_1, \ldots, a_m, b_{m+1}, \ldots, b_d), \\ z_r &= (u_1, \ldots, u_r, a_{r+1}, \ldots, a_m, b_{m+1}, \ldots, b_d), \\ w_0 &= (a_1, \ldots, a_m, v_{m+1}, \ldots, v_d), \\ w_r &= (u_1, \ldots, u_r, a_{r+1}, \ldots, a_m, v_{m+1}, \ldots, v_d). \end{aligned}$$

Note that $z_0 = Q_1, z_m = Q_2, w_0 = Q_3, w_m = Q_4$ and we have

$$g(Q_1, Q_2, Q_3, Q_4) = \sum_{\iota=0}^{m-1} g(z_\iota, z_{\iota+1}, w_\iota, w_{\iota+1}). \tag{3}$$

Let

$$\begin{aligned} h_{r,0} &= z_r, \\ h_{0,s} &= (a_1, \ldots, a_m, v_{m+1}, \ldots, v_{m+s}, b_{m+s+1}, \ldots, b_d), \\ h_{r,s} &= (u_1, \ldots, u_r, a_{r+1}, \ldots, a_m, v_{m+1}, \ldots, v_{m+s}, b_{m+s+1}, \ldots, b_d). \end{aligned}$$

We now have

$$g(z_\iota, z_{\iota+1}, w_\iota, w_{\iota+1}) = \sum_{j=0}^{d-m-1} r(h_{\iota,j}, h_{\iota+1,j}, h_{\iota,j+1}, h_{\iota+1,j+1})$$

which combined with Equation (3) shows that $g(Q_1, Q_2, Q_3, Q_4)$ is a sum of signed rectangles. ∎

Returning to the proof of Theorem 2, we apply the sublemma with $m = l$ and

$$(a_1, \dots, a_m) = (\xi_1, \dots, \xi_k, \alpha_{k+1}, \dots, \alpha_l),$$
$$(b_{m+1}, \dots, b_d) = (\beta_{m+1}, \dots, \beta_d),$$
$$(u_1, \dots, u_m) = (\xi_1, \dots, \xi_k, \gamma_{k+1}, \dots, \gamma_l),$$
$$(v_{m+1}, \dots, v_d) = (\delta_{m+1}, \dots, \delta_d)$$

to see that $g(x_1, x_2, x_3, x_4)$ is a sum of signed rectangles.

CASE (iv). $d \geq 3$, $M > 4$. We write $x_1 = (\xi_1, \dots, \xi_d)$ and assume, without loss of generality, that $f(x_1) > 0$. In order for Equation (2) to hold for $\nu = 1, \dots, d, y = \xi_\nu$, there must be $z_\nu = (\zeta_1^\nu, \dots, \zeta_{\nu_1}^\nu, \xi_\nu, \zeta_{\nu+1}^\nu, \dots, \zeta_d^\nu)$ with $f(z_\nu) < 0$, $\nu = 1, \dots, d$. We now proceed by a series of steps. If $\zeta_1^2 \neq \xi_1$, let $s_2 = (\xi_1, \xi_2, \zeta_3^2, \dots, \zeta_d^2)$ and $t_2 = (\zeta_1^2, \zeta_2^1, \zeta_3^1, \dots, \zeta_d^1)$. We see from the sublemma above that $g(z_1, s_2, t_2, z_2)$ is a sum of signed rectangles and let $f_2 = f + g(z_1, s_2, t_2, z_2)$. If $f(s_2) > 0$, then $M(f_2) < M(f)$ and the inductive step is complete. If not, we still have $M(f_2) \leq M(f)$ and $f_2(s_2) < 0$. If $\zeta_1^2 = \xi_1$ let $s_2 = z_2$ and let $f_2 = f$.

In the second step if $(\zeta_1^3, \zeta_2^3) \neq (\xi_1, \xi_2)$, let $s_3 = (\xi_1, \xi_2, \xi_3, \zeta_4^3, \dots, \zeta_d^3)$ and $t_3 = (\zeta_1^3, \zeta_2^3, \zeta_3^2, \dots, \zeta_d^2)$. The sublemma shows that $g(s_2, s_3, t_3, z_3)$ is a sum of signed rectangles. Let $f_3 = f_2 + g(s_2, s_3, t_3, z_3)$. If $f_2(s_3) > 0$, then $M(f_3) < M(f_2)$ and the inductive step is complete. If not, we still have $M(f_3) \leq M(f_2)$ and $f_3(s_3) < 0$. If $(\zeta_1^3, \zeta_2^3) = (\xi_1, \xi_2)$ let $s_3 = z_3$ and let $f_3 = f_2$.

This process must eventually terminate with $M(f_i) < M(f_{i-1})$ completing the inductive step, since if we do not terminate at an earlier stage, we must end with $s_{d-1} = (\xi_1, \dots, \xi_{d-1}, \zeta_d), f(s_{d-1}) < 0$ and $z_d = (\zeta_1^d, \dots, \zeta_{d-1}^d, \xi_d)$ yielding $s_d = (\xi_1, \dots, \xi_{d-1}, \xi_d) = x_1$. In this case, we have $f_d(s_d) = f_d(x_1) = \cdots = f_2(x_1) = f(x_1) > 0$ (since x_1 is not in the support of any of the g_i) and thus $M(f_d) < M(f_{d-1}) \leq \cdots \leq M(f)$. ∎

REMARK. Although Theorem 2 gives a geometric criterion for the singularity of the l_1-distance matrices, for $d \geq 3$ we still do not have an effective geometric algorithm for ascertaining whether a given collection of points satisfies that criterion.

3. Characteristic matrices.

Given the set $\mathcal{N} = \{x_1, \dots, x_n\}$, we define the matrix $A = (a_{ij})$ by letting

$$a_{ij} = \#\{p_\nu : p_\nu(x_i) = p_\nu(x_j)\}.$$

It is clear from the definition that A is symmetric and that $a_{ii} = d$. If the points $x_1, \dots, x_n$ are distinct, then we also have $a_{ij} \leq d - 1$ when $i \neq j$. We call A the characteristic matrix of $\mathcal{N}$.

EXAMPLE 2. The characteristic matrices of the three closed paths in $\mathbb{R}^2$ discussed in Section 1 (Figure 1) are the circulant matrices with top rows being $(2, 1, 0, 1)$, $(2, 1, 0, 0, 0, 1)$ and $(2, 1, 0, 0, 0, 0, 0, 0, 0, 1)$ respectively.

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EXAMPLE 3. The characteristic matrix of the five points in Example 4 of [SX] is

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 1 \\ 2 & 3 & 2 & 0 & 2 \\ 1 & 2 & 3 & 1 & 1 \\ 1 & 0 & 1 & 3 & 1 \\ 1 & 2 & 1 & 1 & 3 \end{pmatrix}$$

THEOREM 3. *The characteristic matrix A of $\mathcal{N}$ is nonnegative definite. And A is positive definite if and only if the matrix* $(\|x_i - x_j\|_1)$ *is nonsingular.*

PROOF. Let $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$. We have

$$cAc^T = \sum_{i,j=1}^{n} a_{ij}c_ic_j = \sum_{\nu=1}^{d} \sum_{y \in p_i(\mathcal{N})} \Big(\sum \{c_j : p_\nu(x_j) = y\} \Big)^2 = |c|^2_{\mathcal{N}} \geq 0.$$

This shows that A is nonnegative definite. To see the second result of the theorem, we notice that $cAc^T > 0$ for all $c \neq 0$ if and only if $|c|_{\mathcal{N}}$ is a norm, which is equivalent to the nonsingularity of the matrix $(\|x_i - x_j\|_1)$. ∎

The characteristic matrix is in general much simpler than the original l_1- distance matrix. Thus Theorem 3 provides another practical way to determine the singularity of the matrix $(\|x_i - x_j\|_1)$. The following theorem shows that the characteristic matrix can also be used to estimate the lower bound of the eigenvalues of the matrix $(\|x_i - x_j\|_1)$.

THEOREM 4. *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix $(\|x_i - x_j\|_1)$. Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of the characteristic matrix A in ascending order. Assume A is positive definite. Then the following estimate holds:*

$$\min_{1 \leq j \leq n} |\lambda_j| \geq \frac{1}{2}\mu_1\delta,$$

where

$$\delta = \min_{1 \leq \nu \leq d} \{ |p_\nu(x_i - x_j)| : p_\nu(x_i) \neq p_\nu(x_j) \}.$$

Some elementary Fourier transform techniques will be used in the proof of Theorem 4. Here we define the Fourier transform $\hat{B}$ of the function $B \in L^1(\mathbb{R})$ using the formula

$$\hat{B}(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ist} B(s)\, ds.$$

It is well-known that if both B and $\hat{B}$ belong to $L^1(\mathbb{R})$, then the following Fourier inversion formula holds:

$$B(s) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{its} \hat{B}(t)\, dt.$$

Let B be the function defined by

$$B(s) = \begin{cases} 0, & \text{if } s < -1 \\ s + 1, & \text{if } -1 \leq s < 0 \\ -s + 1, & \text{if } 0 \leq s \leq 1 \\ 0, & \text{if } s > 1. \end{cases}$$

It is elementary to see that

$$\hat{B}(t) = (2\pi)^{-1/2}\frac{\sin^2(t/2)}{(t/2)^2},$$

and that both B and $\hat{B}$ are elements of $L^1(\mathbb{R})$. We are now ready to prove Theorem 4.

PROOF OF THEOREM 4. Without loss of generality, we may assume that $\delta = 1$. A basic calculation shows that

$$|t| = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos(ts)}{s^2}\,ds.$$

Let $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ satisfying $\sum_{j=1}^n c_j = 0$. We have

$$\begin{aligned}
&\sum_{i,j=1}^n c_i c_j |p_\nu(x_i) - p_\nu(x_j)| \\
&= -\frac{1}{\pi}\int_{-\infty}^{\infty} s^{-2}\sum_{i,j=1}^n c_i c_j \cos\Big[\big(p_\nu(x_i) - p_\nu(x_j)\big)s\Big]\,ds \\
&= -\frac{1}{\pi}\int_{-\infty}^{\infty} s^{-2}\Big|\sum_{j=1}^n c_j e^{ip_\nu(x_j)s}\Big|^2\,ds \\
&\le -\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big|\sum_{j=1}^n c_j e^{ip_\nu(x_j)s}\Big|^2\hat{B}(s)\,ds \\
&= -\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big|\sum_{y\in p_\nu(\mathcal{N})}\sum\{c_j : p_\nu(x_j) = y\}e^{iys}\Big|^2\hat{B}(s)\,ds. \\
&= -\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{y\in p_\nu(\mathcal{N})}\sum_{z\in p_\nu(\mathcal{N})}\sum\{c_j : p_\nu(x_j) = y\}\sum\{c_j : p_\nu(x_j) = z\}e^{i(y-z)s}\hat{B}(s)\,ds.
\end{aligned}$$

Using the Fourier inversion formula and the facts that B is supported on $[-1, 1]$ and that $B(0) = 1$, we have

$$\begin{aligned}
&\sum_{i,j=1}^n c_i c_j |p_\nu(x_i) - p_\nu(x_j)| \\
&\qquad \le -\frac{1}{2}\sum_{y\in p_\nu(\mathcal{N})}\sum_{z\in p_\nu(\mathcal{N})}\sum\{c_j : p_\nu(x_j) = y\}\sum\{c_j : p_\nu(x_j) = z\}B(y-z) \\
&\qquad = -\frac{1}{2}\sum_{y\in p_\nu(\mathcal{N})}\Big(\sum\{c_j : p_\nu(x_j) = y\}\Big)^2.
\end{aligned}$$

Here we use the assumption that $\delta = 1$ which implies that $B(y - z) = 0$ for $y, z \in p_\nu(\mathcal{N})$ with $y \neq z$.

Hence, we have

$$\begin{aligned}
\sum_{i,j=1}^n c_i c_j \|x_i - x_j\|_1 &\le -\frac{1}{2}\sum_{\nu=1}^d\sum_{y\in p_\nu(\mathcal{N})}\Big(\sum\{c_j : p_\nu(x_j) = y\}\Big)^2. \\
&= -\frac{1}{2}cAc^T \le -\frac{1}{2}\mu_1\sum_{j=1}^n c_j^2.
\end{aligned}$$

By the Courant-Fischer min-max Theorem (see Bellman [B, p. 113]), we know that the matrix $(\|x_i - x_j\|_1)$ has at least $(n-1)$ negative eigenvalues which are in absolute value no less that $\frac{1}{2}\mu_1$. Since the trace of the matrix $(\|x_i - x_j\|_1)$ is 0, we have $\lambda_1 + \cdots + \lambda_n = 0$. Therefore the matrix $(\|x_i - x_j\|_1)$ has exactly $(n-1)$ negative eigenvalues and 1 positive eigenvalue, and it follows that all the eigenvalues of the matrix $(\|x_i - x_j\|_1)$ have absolute value no less than $\frac{1}{2}\mu_1$. ∎

It is worthwhile to point out that the estimate given by Theorem 4 is independent of n, the number of points, and it is asymptotically (with respect to n) best possible. To see this, let n be even and consider the n points $x_j = (j, \ldots, j) \in \mathbb{R}^d$, $j = 1, \ldots, n$. We have $\|x_i - x_j\|_1 = d|i-j|$. Let $c = \frac{1}{\sqrt{4n-6}}(-1, 2, -2, \ldots, 2, -2, 1)$ and $X_j = (\|x_j - x_1\|_1, \ldots, \|x_j - x_n\|_1)$. We calculate that

$$X_j c^T = \frac{(-1)^j d}{\sqrt{4n-6}}.$$

Let L denote the matrix $(\|x_i - x_j\|_1)$. We have

$$cLc^T = \frac{-2dn}{4n-6} \sim -\frac{d}{2}.$$

The characteristic matrix A of these points is simply dI, where I denotes the unit matrix. Thus $\mu_1 = d$. It is obvious that $\delta = 1$, so our claim is verified.

4. **Application to ridge function interpolation.** A function $f\colon \mathbb{R}^d \longrightarrow \mathbb{R}$ is said to be a ridge function, if there is a function $\phi\colon \mathbb{R} \longrightarrow \mathbb{R}$ and a fixed vector $a \in \mathbb{R}^d \setminus \{0\}$ such that $f(x) = \phi(ax)$ for all $x \in \mathbb{R}^d$, where ax denotes the usual inner product of a and x. The vector a is called the *direction* of f. Let $\mathcal{R}_a$ denote the group of all ridge functions with direction a. Let $a_1, \ldots, a_k \in \mathbb{R}^d$ be pairwise linearly independent, and let $\mathcal{R}_{a_1} + \cdots + \mathcal{R}_{a_k}$ denote the sum of $\mathcal{R}_{a_1}, \ldots, \mathcal{R}_{a_k}$. The terminology "ridge function interpolation" in this paper refers to the following problem:

Given n distinct points $x_1, \ldots, x_n \in \mathbb{R}^d$, and an arbitrary set of data $d_1, \ldots, d_n$, find a function $g \in \mathcal{R}_{a_1} + \cdots + \mathcal{R}_{a_k}$ such that $g(x_j) = d_j, j = 1, \ldots, n$.

A simple and efficient realization of the interpolation is to choose a suitable function $h \in \mathcal{R}_{a_1} + \cdots + \mathcal{R}_{a_k}$ and to interpolate the given data by a function in the linear space generated by the n functions $h(x - x_1), \ldots, h(x - x_n)$. When the interpolation conditions are imposed, the result is a system of n linear equations in the unknown coefficients $c_1, \ldots, c_n$:

$$\sum_{j=1}^{n} c_j h(x_i - x_j) = d_i, \quad i = 1, \ldots, n.$$

The coefficient matrix A of the linear system has entries $A_{ij} = h(x_i - x_j)$, and is called the interpolation matrix. This interpolation scheme is well-posed if and only if the interpolation matrix is nonsingular. It is known that there exists a collection of distinct points $x_1, \ldots, x_n$ such that the interpolation matrix $h(x_i - x_j)$ is singular for every choice of $h \in \mathcal{R}_{a_1} + \cdots + \mathcal{R}_{a_k}$. One wishes to characterize the geometric configuration of these

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points. Partial results have been derived in [DLC], [LC], [L], [BP] and [SX]. Here we consider another special case. Assume that $k \leq d$ and that $a_1, \ldots, a_k$ are linearly independent. Applying a nonsingular linear transformation if necessary, we may assume that $a_\nu = e_\nu$, $\nu = 1, \ldots, k$. Set $h = \phi_1 + \cdots + \phi_k$, where $\phi_\nu(t) = |t|, \nu = 1, \ldots, k$. Then the result of Theorem 2 shows that the interpolation matrix $\big(h(x_i - x_j)\big)$ is nonsingular if and only if $\{x_1, \quad, x_n\}$ has no nonempty subset which is a sum of rectangles. In fact, Theorem 10 in [SX] and Theorem 2 yield the following stronger results:

1. Assume that $\mathcal{N}$ contains no sum of rectangles. Then the interpolation matrix $\big(h(x_i - x_j)\big)$ is nonsingular provided that $h(x) = \phi_1\big(|p_1(x)|^2\big) + \cdots + \phi_k\big(|p_k(x)|^2\big)$, where (i) $\phi_1, \ldots, \phi_k$ are completely monotonic but not constant; (ii) $\phi_1, \ldots, \phi_k$ are functions of negative type and $\phi_1', \ldots, \phi_k'$ are not constant.
2. Assume that $\mathcal{N}$ contains a sum of rectangles. Then the interpolation matrix $h(x_i - x_j)$ is singular for every choice of $h \in \mathcal{R}_{a_1} + \cdots + \mathcal{R}_{a_k}$.

We recall that a function $\phi\colon [0, \infty) \longrightarrow \mathbb{R}$ is said to be *completely monotone* if $(-1)^k \phi^{(k)}(t) \geq 0$ for all $t \in (0, \infty)$ and for all $k = 1, 2, \ldots$, and ϕ a function of negative type if $(-1)^k \phi^{(k+1)}(t) \geq 0$ for all $t \in (0, \infty)$ and for all $k = 1, 2, \ldots$.

Some interpolation matrices may be "nearly singular" by having small eigenvalues. This will lead to poor stability of the interpolation scheme and cause computational difficulties. Thus it is important to estimate the size of the eigenvalues of the interpolation matrices. Theorem 4 gives such an estimate for the matrix $(\|x_i - x_j\|_1)$ in terms of the characteristic matrices.

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