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A CHARACTERIZATION OF FINITE p-SOLUBLE GROUPS OF p-LENGTH ONE BY COMMUTATOR IDENTITIES

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Abstract

A classical result of M. Zorn states that a finite group is nilpotent if and only if it satisfies an Engel condition. If this is the case, it satisfies almost all Engel conditions. We shall give a similar description of the class of p-soluble groups of p-length one by a sequence of commutator identities.

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1. Introduction

The classical result of Zorn (1936) characterizes the class of finite nilpotent groups by commutator identities. Indeed, a finite group is nilpotent if and only if it satisfies for almost all positive integers k the kth Engel condition, that is for all elements x, y of the group we have $[x, {}_k y] = 1$. The purpose of this note is to characterize the class of p-soluble groups of p-length one in a similar way. All groups considered in this paper are finite, all unexplained notation is standard and can be found in Huppert (1967) or Gorenstein (1968). This paper is part of the author's Ph. D. Thesis (Dissertation zur Erlangung des naturwissenschaftlichen Doktorgrades) written under supervision of Professor H. Heineken.

2. Preliminaries

In order to describe p'-groups by identities, we firstly introduce a sequence of positive integers such that any p'-number divides almost all of these numbers.

DEFINITION. Let p be a prime and let $\{p_1, p_2, \ldots\}$ be the set of all primes different from p. For each positive integer k we define $m_{k,p} = (p_1 \ldots p_k)^k$. If it is clear, for which prime p the sequence is defined, we shall write m_k instead of $m_{k,p}$.

We are now able to give a sequence of laws in two variables which turn out to hold identically in groups having p-length one.

DEFINITION. Let x, y be variables and p be a prime. For any positive integer k let

$$\lambda_k(x, y) = \left[\left[y^{m_k}, x \right]^{p^k}, {}_2x \right]^{m_k}.$$

We have the following easy observation:

LEMMA 1. *Let G be a p-soluble group having p-length one. Then $\lambda_k(x, y) = 1$ is a law in G for almost all k.*

PROOF. Let M and N be normal subgroups of G such that N and G/M are p'-groups and M/N is a p-group.

Select k such that $\exp(N)$ and $\exp(G/M)$ both divide m_k. Then for all $y \in G$ we get $y^{m_k} \in M$, hence $[y^{m_k}, x] \in M$ as M is a normal subgroup of G.

If we increase k such that $\exp(M/N)$ divides p^k then we get $[y^m, x]^{p^k} \in N$. The result now follows readily.

The purpose of this paper is the proof of the converse of Lemma 1. To do this, we shall firstly examine the structure of minimal non p-length one groups.

LEMMA 2. *Let H be a p-soluble group which does not have p-length one, but all of whose proper subgroups and factor groups have p-length one. Then:*

(a) *H has a unique minimal normal subgroup N which has a complement Q,*

(b) *H has p-length two, indeed $H = O_{p,p',p}(H)$,*

(c) *$Q = AB$, A is a normal q-subgroup of Q and B acts irreducibly on $A/\Phi(A)$. Moreover $|B| = p$ and A is elementary abelian or extraspecial.*

PROOF. (a) and (b) follow immediately from Huppert (1967), p. 693. (c) If $|B| \neq p$, we select a maximal subgroup X of H with $O_{p,p'}(H) < X < H$. So X is normal in H and $l_p(X) = 1$ by minimality. Moreover $O_{p'}(X) \leq O_{p'}(H) = 1$, so X is p-closed. But $O_p(X) \leq O_p(H)$ and we arrive at the contradiction $|O_p(H)| < |X|_p = |O_p(X)|_p$.

By (a) and (b) the group Q is a semidirect product of $O_{p'}(Q)$ with $B \in \mathrm{Syl}_p(Q)$. B operates nontrivially on $O_{p'}(Q)$ because otherwise $B < O_p(H)$ and H would be p-closed.

Let $A < O_{p'}(Q)$ be a group of least possible order which is normalized but not centralized by B. By Gorenstein (1968), Theorem 6.2.2, A is a q-group for some prime $q \neq p$. By Huppert (1967), p. 351, A is special.

We claim $H = NAB$. Let $X = NAB$. Then we have $[O_{p'}(X), N] = 1$. So $O_{p'}(X) < C_H(N) = N$ which implies that $O_{p'}(X) = 1$. Suppose that X is a proper subgroup of H. Then $l_p(X) = 1$ and so X is p-closed. But then the subgroup AB of X is p-closed which implies $[B, A] = 1$ contradicting the choice of A. So we have $H = X = NAB$.

Assume that A is not elementary abelian. Then, by minimality, we have $[Z(A), B] = 1$ and so $Z(A) \leqslant Z(Q)$. By (a) the group Q acts faithfully and irreducibly on N, so $Z(Q)$ is cyclic. In particular, $|Z(A)| = q$ and the result follows.

3. The main result

This section is devoted to give a proof of the following result

THEOREM. *Let G be a finite p-soluble group. Then the following conditions are equivalent*:

(i) *G has p-length one*,

(ii) *$\lambda_k(x, y) = 1$ is a law in G for almost all positive integers k.*

We have shown in Lemma 1 that condition (i) implies (ii). To prove the converse, let H be a counterexample of least possible order. Then every proper subgroup and factor group of H has p-length one. These groups have been dealt with in Lemma 2. We divide the proof of the Theorem into two parts according to whether A is elementary abelian or A is extraspecial.

LEMMA 3. *Let H be as in Lemma 2. Let furthermore A be elementary abelian. Then $\lambda_k(H) \neq 1$ for infinitely many k.*

PROOF. We choose k such that $p^k \equiv 1$ (mod(exp(A))). Furthermore, select $b_0 \in B = \langle b \rangle$ such that $b_0^{m_k} = b$. Finally, let $1 \neq n \in C_N(b)$. Such elements exist because N is a p-group and b is a p-element.

For an element $1 \neq a \in A$ we calculate $\lambda_k(na, b_0)$. We have

$$\begin{aligned}\lambda_k(na, b_0) &= \left[\left[b_0^{m_k}, na\right]^{p^k}, {}_2na\right]^{m_k}\\ &= \left[\left[b, a\right]^{p^k}, na, na\right]^{m_k}\\ &= \left[b, a, na, na\right]^{m_k},\end{aligned}$$

since $[n, b] = 1$ and $p^k \equiv 1 \pmod{o([b, a])}$.

Suppose $\lambda_k(H) = 1$. As $[b, a, na, na] \in N$ and N is a p-group, we conclude $[b, a, na, na] = 1$. Hence

$$\begin{aligned}1 &= \left[b, a, na, na\right]\\ &= \left[\left[b, a, a\right]\left[b, a, n\right]^a, na\right]\\ &= \left[\left[b, a, n\right]^a, a\right]\left[\left[b, a, n\right]^a, n\right]^a\\ &= \left[b, a, n, a\right]^a.\end{aligned}$$

This means

(1) $$\left[n, \left[b, a\right], a\right] = 1.$$

Replacing b by b^i for a positive integer i yields the following

(2) $$\left[n, \left[b^i, a\right], a\right] = 1$$

We claim

(3) $$\left[n, \left[b, a\right], a^{b^{-i}}\right] = 1 \quad \text{for all } i.$$

Indeed, for $i = 0$ this is equation (1). Proceeding by induction on i, assume (3) to be true for some $i - 1$.

Then (2) implies

$$\begin{aligned}1 &= \left[n, \left[b^{i+1}, a\right], a\right]\\ &= \left[n, \left[b, a\right]^{b^i}\left[b^i, a\right], a\right]\\ &= \left[n, \left[b^i, a\right], a\right]\left[\left[n, \left[b, a\right]^{b^i}\right]^{[b^i, a]}, a\right]\\ &= \left[n, \left[b, a\right]^{b^i}, a\right]^{[b^i, a]}.\end{aligned}$$

This implies $1 = [n, [b, a], a^{b-i}]$ as n commutes with b.

Application of (3) now gives immediately

$$\left[n, \left[b, a\right]\right] \in C_N(\langle a^{b^i} | i = 1, 2, \cdots \rangle) = C_N(A).$$

As Q acts faithfully and irreducibly on N this gives $[n, [b, a]] = 1$. So

$$1 \neq n \in C_N(\langle [b, a], b\rangle) = C_N(Q).$$

This contradicts Lemma 2.

The second part of the proof deals with the case when A is extraspecial. We have

LEMMA 4. *Let H be as in Lemma 2. Furthermore let A be extraspecial. Then $\lambda_k(H) \neq 1$ for infinitely many k.*

PROOF. As in the proof of Lemma 3, let k be such that

$$p^k \equiv 1 \ (\mathrm{mod}(\exp A)).$$

Furthermore let $a \in A \setminus Z(A)$ and $1 \neq n \in C_N(B)$. By Lemma 2(c) we can select $b \in B$ such that $[a, a^b] \neq 1$. As p and m_k are coprime, we can find $b_0 \in B = \langle b \rangle$ with $b_0^{m_k} = b$.

Then

$$\begin{aligned}\lambda_k(na, b_0) &= \left[\left[b_0^{m_k}, na\right]^{p^k}, na, na\right]^{m_k} \\ &= \left[\left[b, na\right]^{p^k}, na, na\right]^{m_k} \\ &= \left[\left[b, a\right]^{p^k}, na, na\right]^{m_k} \\ &= \left[b, a, na, na\right]^{m_k},\end{aligned}$$

as $[n, b] = 1$ and $p^k \equiv 1 \ (\mathrm{mod}\ o([b, a]))$.

Suppose $\lambda_k(H) = 1$. As $\lambda_k(H) \leqslant N$ and N is a p-group, we can conclude $[b, a, na, na] = 1$, because p and m_k are coprime. Hence

$$\begin{aligned}1 &= \left[b, a, na, na\right] \\ &= \left[\left[b, a, a\right]\left[b, a, n\right]^a, na\right] \\ &= \left[z\left[b, a, n\right]^a, na\right] \quad (\text{setting } z = \left[b, a, a\right] \in Z(A)) \\ &= \left[zn_1, na\right] \quad \left(\text{setting } n_1 = \left[b, a, n\right]^a\right) \\ &= \left[zn_1, a\right]\left[zn_1, n\right]^a \\ &= \left[n_1, a\right]\left[z, n\right]^a \\ &= \left[b, a, n, a\right]^a\left[z, n\right]^a.\end{aligned}$$

This implies

$$\left[n, z\right] = \left[b, a, n, a\right]. \tag{4}$$

Now we evaluate $\lambda_k(naz, b_0)$. We have

$$\begin{aligned}\left[b_0^{m_k}, naz\right] &= \left[b, naz\right] \\ &= \left[b, z\right]\left[b, na\right]^z \\ &= \left[b, a\right]^z \\ &= \left[b, a\right],\end{aligned}$$

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as $z \in Z(A)$ and B centralizes $Z(A)$. So

$$\lambda_k(naz, b_0) = \left[\left[b_0^{m_k}, naz\right]^{p^k}, {}_2naz\right]^{m_k}$$
$$= \left[\left[b, a\right]^{p^k}, {}_2naz\right]^{m_k}$$
$$= \left[b, a, naz, naz\right]^{m_k}, \quad \text{as } p^k \equiv 1 \ (\mathrm{mod}\ o([b, a])).$$

This implies

$$\lambda_k(naz, b_0) = \left[\left[b, a, z\right]\left[b, a, na\right]^z, naz\right]^{m_k}$$
$$= \left[\left[b, a, a\right]^z\left[b, a, n\right]^{az}, naz\right]^{m_k}$$
$$= \left[z^z n_2, naz\right]^{m_k},$$

where we have defined $n_2 = [b, a, n]^{az}$,

Finally

$$\lambda_k(naz, b_0) = \left(\left[zn_2, az\right]\left[zn_2, n\right]^{az}\right)^{m_k}$$
$$= \left(\left[n_2, az\right]\left[z, n\right]^{az}\right)^{m_k}$$
$$= \left(\left[b, a, n, az\right]^{az}\left[z, n\right]^{az}\right)^{m_k}.$$

As above, this implies

(5) $$[n, z] = [b, a, n, az].$$

Comparing (4) with (5) yields

$$[b, a, n, a] = [b, a, n, az]$$
$$= [b, a, n, za]$$
$$= [b, a, n, a][b, a, n, z]^a, \quad \text{as } z \in Z(A).$$

So $[b, a, n, z] = 1$. But $z = [b, a, a] \neq 1$ and so z acts fixedpointfreely on N. So we have $[n, [b, a]] = 1$, i.e. $n \in C_N([b, a])$. As n and b commute, we arrive at

$$1 \neq n \in C_N(\langle b, [b, a]\rangle) = C_N(Q).$$

This clearly contradicts Lemma 2.

The proof of the Theorem is now complete. As a corollary we obtain the (well-known) result

COROLLARY. *Let G be a p-soluble group. If every subgroup of G which can be generated by two elements has p-length one then G has p-length one.*

PROOF. This follows easily, as in the sequence of laws there are only two variables.

4. An example

One might ask why we have taken the particular sequence λ_k. Should we not rather look for a simpler sequence such as for example

$$\alpha_k(x,y) = [x, {}_k y^{m_k}]^{m_k}?$$

Again, a result similar to Lemma 1 holds. The proof of this is virtually the same. But, unfortunately, there are p-soluble groups of p-length two which satisfy almost all laws $\alpha_k(x,y) = 1$. For $p = 2$ the symmetric group of degree four is an example. We shall now construct a series of examples for each prime p.

EXAMPLE. Let q be a prime with $q \equiv 1 \pmod p$ and let X be the nonabelian group of order pq. Then X has a faithful and irreducible representation over $GF(p)$. Let G be the splitting extension of this module M by X. By construction, $C_M(O_q(X)) = 1$. Hence $O_{p,q}(G)$ is a Frobenius group and all elements different from 1 in $O_{p,q}(G)$ have order p or q.

Now let $x, y \in G$ be given. If $1 \neq yM$ is a p-element in G/M then all $y^{m_k}M \neq 1$ and so all elements $[x, {}_k y^{m_k}] \notin M$ or $[x, y^{m_k}] \in M$. In the first case the above remark shows that the order of $[x, {}_k y^{m_k}]$ is q. So $\alpha_k(x,y) = 1$ for all k. In the second case $[x, y^{m_k}]$ and y^{m_k} both lie in a p-Sylow subgroup and for large enough k we have $[x, {}_k y^{m_k}] = 1$.

If $1 \neq yM$ is a q-element in G/M we get $y^{m_k} \in M$ for almost all k. Then we have $[x, y^{m_k}] \in M$ and finally $[x, {}_2 y^{m_k}] = 1$. This shows that G satisfies almost all laws $\alpha_k(x,y) = 1$. But $l_p(G) = 2$.

References

D. Gorenstein (1968), *Finite groups* (Harper & Row, New York).
B. Huppert (1967), *Endliche Gruppen I* (Springer Verlag, Berlin).
M. Zorn (1936), 'Nilpotency of finite groups', *Bull. Amer. Math. Soc.* **42**, 485.

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