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THE PERSISTENCE OF UNIVERSAL FORMULAE IN FREE ALGEBRAS

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Gilbert Baumslag, B.H. Neumann, Hanna Neumann, and Peter M. Neumann successfully exploited their concept of discrimination to obtain generating groups of product varieties via the wreath product construction. We have discovered this same underlying concept in a somewhat different context. Specifically, let V be a non-trivial variety of algebras. For each cardinal α let $F_\alpha(V)$ be a V-free algebra of rank α. Then for a fixed cardinal r one has the equivalence of the following two statements:

(1) $F_r(V)$ discriminates V. (1*) The $F_s(V)$ satisfy the same universal sentences for all $s \geq r$. Moreover, we have introduced the concept of strong discrimination in such a way that for a fixed finite cardinal r the following two statements are equivalent:

(2) $F_r(V)$ strongly discriminates V. (2*) The $F_s(V)$ satisfy the same universal formulas for all $s \geq r$ whenever elements of $F_r(V)$ are substituted for the unquantified variables. On the

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surface (2) and (2*) appear to be stronger conditions than (1) and (1*). However, we have shown that for particular varieties (of groups) (2) and (2*) are no stronger than (1) and (1*).

I. Definitions and Theorems

Following Bell and Slomson [2] we write, when A is a substructure of B,

(1) $A \; \forall \; B$ if A and B satisfy precisely the same universal sentences, and

(2) $A \subseteq_{\forall} B$ if A and B satisfy precisely the same universal formulas whenever elements of A are substituted for the unquantified variables (if any).

Let V be a non-trivial variety of algebras of some fixed but arbitrary similarity type. Here non-trivial means that V contains at least one algebra with at least two elements. We pose two questions which we (in a certain sense) quickly answer, namely:

(1) When do we have $F_r(V) \; \forall \; F_s(V)$ for all $s \geq r$? and

(2) When do we have $F_r(V) \subseteq_{\forall} F_s(V)$ for all $s \geq r$?

Here a cardinal is an ordinal not equipotent with any prior ordinal; moreover, it is to be understood that $F_r(V)$ is generated by an initial segment of cardinal r of a set of free generators of cardinal s for $F_s(V)$. It follows from a result of Vaught [4 Theorem 4 p.237] that $F_r(V)$ is elementarily embedded in $F_s(V)$ under the inclusion map whenever r is infinite. Hence, the answer to both questions (1) and (2) is "always" whenever r is infinite. We therefore focus our attention principally upon finite r.

DEFINITION: *(A)* $F_r(V)$ *(B+3N) discriminates* V provided, whenever (s_i, t_i) are finitely many pairs of terms in the language L_V appropriate for algebras of V such that none of the equations $s_i = t_i$ is a law of

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V, there are elements $b_1, b_2, \ldots \in F_r(V)$ such that $s_i(b_1, b_2, \ldots) \neq t_i(b_1, b_2, \ldots)$ for all i.

(B) Let $r \in \underset{\sim}{N}$. $F_r(V)$ *strongly discriminates* V provided it possesses an ordered set $a_1 < a_2 < \ldots < a_r$ of V-free generators such that whenever (s_i, t_i) are finitely many pairs of terms of L_V with none of the equations $s_i = t_i$ a law in V there are elements $b_{r+1}, b_{r+2}, \ldots \in F_r(V)$ such that simultaneously $s_i(a_1, \ldots, a_r, b_{r+1}, b_{r+2}, \ldots) \neq t_i(a_1, \ldots, a_r, b_{r+1}, b_{r+2}, \ldots)$ for all i.

We remark that $B + 3N$ refers to the authors of [1] in which this concept is introduced for groups.

THEOREM 1: *(1)* $F_r(V) \; \forall \; F_s(V)$ *for all* s *such that* $s \geq r$ *if and only if* $F_r(V)$ *(B + 3N) discriminates* V.

(2) Let $r \in \underset{\sim}{N}$. $F_r(V) \subseteq_V F_s(V)$ *for all* s *such that* $s \geq r$ *if and only if* $F_r(V)$ *strongly discriminates* V.

It is easy to show that the *(B+3N)* discrimination (strong discrimination) of V by its free algebra of rank r implies the *(B+3N)* discrimination (strong discrimination) of V by its free algebras of larger rank. It is also easy to see that strong discrimination implies *(B+3N)* discrimination. Thus, the persistence of universal sentences in free algebras will already imply the persistence of universal formulae precisely in the case that $F_m(V)$ strongly discriminates V where

$$m = \min \{ n \in \underset{\sim}{N} \mid F_n(V) \text{ (B+3N) discriminates } V \} .$$

Of course we are confining ourselves to V *(B+3N)* discriminated by one of its free algebras of finite rank.

THEOREM 2. *(N. Gupta and F. Levin): The concepts of (B+3N) discrimination and strong discrimination coincide for any product variety* UN_c *where* U *is any variety of groups and* N_c *is the variety of all*

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groups nilpotent of class at most c .

THEOREM 3. *The concepte of* $(B+3N)$ *discrimination and strong discrimination coincide for the varieties* $N_c \wedge A^2$ *of groups which are nilpotent of class at most* c *and metabelian.*

II. Outlines of Proofs

We first consider Theorem 1, Part (1). Note that if r is infinite there is nothing to prove since all free algebras of infinite rank discriminate. Assume r is finite. Observe that if $F_r(V) \; V \; F_s(V)$ for all $s \geq r$ then given pairs $(s_1,t_1),\ldots,(s_k,t_k)$ of terms of L_V such that $s_i(x_1,\ldots,x_m) = t_i(x_1,\ldots,x_m)$ are not laws in V and without loss of generality $m \geq r$ we may assert that if $a_1,\ldots,a_m$ V-freely generate $F_m(V)$, then simultaneously $s_i(a_i,\ldots,a_m) \neq t_i(a_1,\ldots,a_m)$ in $F_m(V)$. Hence

$$\exists x_1\ldots\exists x_m \; (\bigwedge_{i=1}^{k} s_i(x_i,\ldots,x_m) \neq t_i(x_1,\ldots,x_m))$$

holds in $F_m(V)$. Since the $F_s(V)$, $s \geq r$, satisfy precisely the same universal sentences they also satisfy the same existential sentences. Therefore,

$$\exists x_1\ldots\exists x_m \; (\bigwedge_{i=1}^{k} s_i(x_1,\ldots,x_m) \neq t_i(x_1,\ldots,x_m))$$

must also be true in $F_r(V)$. That is, there are elements $b_1,\ldots,b_m \in F_r(V)$ such that simultaneously

$$s_i(b_1,\ldots,b_m) \neq t_i(b_1,\ldots b_m) \qquad (1 < i < k) .$$

In other words, $F_r(V)$ $(B+3N)$ discriminates V .

Suppose now that $F_r(V)$ $(B+3N)$ discriminates V . By Vaught's result there is nothing to prove if r is infinite. Suppose r is finite.

The satisfaction of the same universal sentences by the $F_s(V)$, $s \geq r$, is equivalent to the satisfaction in these algebras of the same primitive sentences of Abraham Robinson. (See [2]) Such a sentence has the form

$$\exists \bar{x} \ (\bigwedge_i (p_i(\bar{x}) = P_i \ (\bar{x})) \ \wedge \bigwedge_j (q_j(\bar{x}) \neq Q_j(\bar{x})))$$

where the p_i, P_i, q_j and Q_j are terms of L_V and $\bar{x}$ is a tuple of variables. We may take care of finitely many conjuncts of the form $q_j(x_1,\ldots,x_m) \neq Q_j(x_1,\ldots,x_m)$ since $F_r(V)$ (B+3N) discriminates V. With a little finesse we can also treat the conjuncts $p_i(x_1,\ldots,x_m) = P_i(x_1,\ldots,x_m)$. (See [3])

Part (2) may be deduced by observing that the $F_s(V)$, $s \geq r$, satisfy the same universal formulas if and only if they satisfy the same primitive sentences in the extension L_V^+ of L_V formed by adjoining the elements of $F_r(V)$ as constants. A term in this extended language has the form $t(c_1,\ldots,c_n,x_1,\ldots,x_m)$ where $c_1,\ldots,c_n \in F_r(V)$. But $c_i = t_i(a_1,\ldots,a_r)$ is a term on the free generators $a_1,\ldots,a_r$ of $F_r(V)$ so ultimately $t(c_1,\ldots,c_n,x_1,\ldots,x_m)$ is a term $s(a_1,\ldots,a_r,x_1,\ldots,x_m)$ on the free generators $a_1,\ldots,a_r$ of $F_r(V)$ and variables $x_1,\ldots,x_m$. The proof of Part (1) may now be mimicked to produce the desired conclusion.

The proofs of Theorems 2 and 3 depend on the fact that strong discrimination is a kind of strong residual property. Namely, it is not difficult to prove: If $r \in \underset{\sim}{N}$ then $F_r(V)$ strongly discriminates V if and only if for every $s > r$ and every ordered V-free generating set $a_1 < a_2 < \ldots < a_s$ of $F_s(V)$ and every finite set of pairs of unequal elements $s_i \neq t_i$, $1 \leq i \leq k$, of $F_s(V)$ there is a retraction

$$F_s(V) \xrightarrow{\phi} F_r(V)$$

with $\phi(a_j) = a_j$, $1 \leq j \leq r$, such that simultaneously $\phi(s_i) \neq \phi(t_1)$,

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$1 \le i \le k$, in $F_r(V)$.

Let E be the trivial variety of groups and A the variety of Abelian groups. Clearly the infinite cyclic group $F_1(A)$ strongly discriminates A , and $A = EN_1 = N_1 \wedge A^2$.

Gupta and Levin proved the existence of such a retraction

$$F_s(UN_c) \xrightarrow{\phi} F_r(UN_c)$$

where $r = \min \{n \in \underset{\sim}{N} \mid F_n(UN_c)$ $(B+3N)$ discriminates $UN_c\}$

$= \max \{2, c-1\}$ (unless $U = E$ and $c = 1$) .

See [5] for details.

The authors have shown the existence of such a retraction

$$F_s(N_c \wedge A^2) \xrightarrow{\phi} F_2(N_c \wedge A^2) .$$

$2 = \min \{n \in \underset{\sim}{N} \mid F_n(N_c \wedge A^2)$ $(B+3N)$ discriminates $N_c \wedge A^2\}$

(unless $c = 1$) .

See [3] for details.

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