J. Austral. Math. Soc. (Series A) **63** (1997), 218–224

ON SOME REGULARITY CONDITIONS OF BOREL MEASURES ON $\mathbb{R}$

F. BEN NASR

(Received 1 May 1996; revised 3 December 1996)

Communicated by P. G. Dodds

Abstract

The aim of this paper is to resolve Taylor's question concerning certain regularity conditions on a Borel measure. The proposed solution is given in the framework of Brown, Michon and Peyrière, and Olsen.

1991 *Mathematics subject classification (Amer. Math. Soc.)*: primary 28A60; secondary 60G57.
Keywords and phrases: Hausdorff dimension, packing dimension.

Introduction

Let $\{\mathscr{F}_n\}_{n\geq 1}$ be a sequence of finite partitions of $[0, 1[$ by intervals, semi-open to the right. These partitions need not be nested. If $x \in [0, 1[$, $I_n(x)$ stands for the intervals of the family $\mathscr{F}_n$ which contains x. The length of an interval J is denoted by $|J|$. We suppose that, for any $x \in [0, 1[$, $\lim_{n\longrightarrow\infty} |I_n(x)| = 0$.

We consider two indices dim and Dim which are defined as Hausdorff and Tricot dimensions [7], but only considering coverings and packings by intervals in the family $\{\mathscr{F}_n\}_{n\geq 1}$.

A Borel probability measure μ is called *regular uni-dimensional* if

$$\exists\alpha : \qquad \lim_{n\longrightarrow+\infty} \frac{\log\mu(I_n(x))}{\log|I_n(x)|} = \alpha \qquad \mu\text{-a. e.}$$

For $q, t \in \mathbb{R}$, define

https://doi.org/10.1017/S1446788700000653 Published online by Cambridge University Press

$$H_\mu(q,t) = \lim_{\delta\longrightarrow 0} \inf \left\{ \sum_j{}' \mu(I_j)^{q+1} |I_j|^{-t} : [0,1[= \cup I_j,\ I_j \in \cup_n \mathscr{F}_n,\ |I_j| < \delta \right\},$$

$$P_\mu(q,t) = \lim_{\delta\longrightarrow 0} \sup \left\{ \sum_j{}' \mu(I_j)^{q+1} |I_j|^{-t} : (I_j)_j \text{ disjoint},\ I_j \in \cup_n \mathscr{F}_n,\ |I_j| < \delta \right\},$$

$$b_\mu(q) = \sup\{t \in \mathbb{R} : H_\mu(q,t) = 0\}, \qquad B_\mu(q) = \sup\{t \in \mathbb{R} : P_\mu(q,t) = 0\},$$

where Σ' is the sum over those j with $\mu(I_j) \neq 0$. The detailed properties of the functions b_μ can be found in [3, 4, 5], and detailed properties of the function B_μ can be found in [4].

For any function f, we consider the following Legendre transform of f:

$$f^*(x) = \inf_{y\in\mathbb{R}}(x(y+1) - f(y)).$$

If we put

$$\Delta_s = \left\{ x \in [0,1[:\ \lim_{n\longrightarrow+\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} = s \right\},$$

then the theorems in [4] imply that

$$\operatorname{Dim} \Delta_s \leq B_\mu^*(s) \quad \text{for} \quad a_1 = \sup_{q>-1} \frac{B_\mu(q)}{q+1} \leq s \leq a_2 = \inf_{q<-1} \frac{B_\mu(q)}{q+1},$$

and the theorems in [3, 4, 5] imply that

$$\dim \Delta_s \leq b_\mu^*(s) \quad \text{for} \quad c_1 = \sup_{q>-1} \frac{b_\mu(q)}{q+1} \leq s \leq c_2 = \inf_{q<-1} \frac{b_\mu(q)}{q+1}.$$

The aim of this paper is to resolve the following open problem of Taylor [6]: Find a regular uni-dimensional μ such that $a_1 < a_2,\ c_1 < c_2$ but

$$\dim \Delta_s \neq b_\mu^*(s), \qquad \operatorname{Dim} \Delta_s \neq B_\mu^*(s), \qquad \dim \Delta_s = \operatorname{Dim}\ \Delta_s \quad \text{for some } s.$$

Moreover the Borel measure μ which we propose satisfies the regularity condition suggested by Olsen [4]: $b_\mu(q) = B_\mu(q)$ for all q.

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Example

Let $\mathscr{A}$ be the set of finite words over the alphabet $\{0, 1\}$. The concatenation, just denoted by juxtaposition, endows $\mathscr{A}$ with the structure of a semigroup. The empty word, which is the unit, is denoted by ω. The set of words of length n is denoted by $\mathscr{A}_n$. For every $j \in \mathscr{A}$, we denote by $N_k(j)$ the number of times the letter k appears in the word j. Let $\mathscr{A} \cup \partial\mathscr{A}$ be the natural compactification of $\mathscr{A}$ ($\partial\mathscr{A}$ is the set of infinite words). For any $j \in \mathscr{A}$, we define C_j to be the cylinder formed by the elements of $\partial\mathscr{A}$ starting with j.

Take $\alpha, \beta \in \mathbb{R}$ such that $1/3 < \alpha < \beta < 1/2$. A cylinder C_j of order n ($j \in \mathscr{A}_n$) is called of *α-type* (respectively *β-type*) if we have:

$$|N_0(j)/n - \alpha| < 1/n \qquad \text{(respectively } |N_0(j)/n - \beta| < 1/n).$$

For any cylinder C_j, $j \in \mathscr{A}_n$, of α- or β-type, we define:

$$\widetilde{C}_j = \{C_l : l \in \mathscr{A}_{n+6}, C_l \subset C_j \text{ and } C_l \text{ is of the same type as } C_j\}.$$

It is easy to check that

$$\exists n_0 : \qquad \forall n \geq n_0, \quad \forall j \in \mathscr{A}_n, \quad \widetilde{C}_j \geq 2.$$

For each $k \in \mathbb{N}$ we select, in a random way, 2^{k+1} cylinders of order $n_0 + 6k$. The selection is done in steps. In the first step, we select two cylinders of order n_0, C_{j_0} and $C_{j_0'}$ with C_{j_0} of α-type and $C_{j_0'}$ of β-type. From the nth step to the $(n+1)$st step, we choose two elements of $\widetilde{C}_j$ for every C_j of the nth step.

Let l_0, l_1, p_0 and p_1 be a real numbers such that

(1) $$0 < l_1 < l_0, \qquad 0 < p_0 < p_1, \qquad l_0 + l_1 = 1, \qquad p_0 + p_1 = 1,$$
$$\frac{\beta \log(p_0/p_1) + \log p_1}{\beta \log(l_0/l_1) + \log l_1} < 1.$$

We construct a sequence $\{\mathscr{F}_n = \{I_j\}_{j \in \mathscr{A}_n}\}_{n \geq 0}$ of finite partitions of $[0, 1[$ in semi-open intervals in the following way. The first partition contains the unique interval $I_\omega = [0, 1[$. We obtain the $(n+1)$st partition from the nth one by cutting each interval I_j, $j \in \mathscr{A}_n$, into two intervals $\{I_{jk}\}_{k=0,1}$ such that:

$$|I_{jk}| = \begin{cases} l_k |I_j| & \text{if } C_j \text{ contains a selected cylinder,} \\ |I_j|/2 & \text{otherwise.} \end{cases}$$

Now define a measure μ in the following way. For $j \in \mathscr{A}$ and $k \in \{0, 1\}$ let

$$\mu(I_{jk}) = \begin{cases} p_k \mu(I_j) & \text{if } I_j \text{ contains a selected interval,} \\ \mu(I_j)/2 & \text{otherwise.} \end{cases}$$

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(I_j is selected if C_j is selected).

Then clearly μ is regular uni-dimensional of index 1:

$$\lim_{n\longrightarrow\infty}\frac{\log\mu(I_n(x))}{\log|I_n(x)|}=1 \qquad \mu\text{-a. e.}$$

Note that the measure μ is not quasi-Bernoulli, that is, there is no positive number M such that, for any j and k in $\mathscr{A}$, we have

$$M^{-1}\mu(I_j)\mu(I_k)\le\mu(I_{jk})\le M\mu(I_j)\mu(I_k).$$

Consider the following quantities.

$$C_n(q,t)=\frac{1}{n}\log\sum_{j\in\mathscr{A}_n}\mu(I_j)^{q+1}|I_j|^{-t},$$

$$C(q,t)=\lim_{n\longrightarrow\infty}\sup C_n(q,t) \quad \text{and} \quad \varphi(q)=\sup\{t;\,C(q,t)<0\}.$$

It is easy to check that φ is finite, strictly increasing on $\mathbb{R}$ and

(2) $$\varphi(0)=0, \qquad \varphi(q)\le q \quad \text{for all } q\in\mathbb{R}.$$

Since C is a convex finite function, the function φ is defined by the equality $C(x,\varphi(x)) = 0$. We prove that

(3) $$b_\mu=B_\mu=\varphi.$$

Property (3) results immediately from the following proposition.

PROPOSITION. *For* $q\in\mathbb{R}$,

(1) $\lim_{n\longrightarrow\infty}C_n(q,\varphi(q))=0$ *and* $\lim_{n\longrightarrow\infty}\inf nC_n(q,\varphi(q))>-\infty$.

(2) $b_\mu(q)\le\varphi(q)$.

PROOF. We introduce the following notation: for positive functions u and v, $u\approx v$ means that there exists a positive constant K such that $K^{-1}u\le v\le Ku$.

Fix $q\in\mathbb{R}$ and put

$$A_\alpha=\alpha\left((q+1)\log\frac{p_0}{p_1}-\varphi(q)\log\frac{l_0}{l_1}\right)+(q+1)\log p_1-\varphi(q)\log l_1,$$

$$A_\beta=\beta\left((q+1)\log\frac{p_0}{p_1}-\varphi(q)\log\frac{l_0}{l_1}\right)+(q+1)\log p_1-\varphi(q)\log l_1,$$

$$\lambda=2^{6(\varphi(q)-q)}, \qquad \lambda_\alpha=2e^{6A_\alpha}, \qquad \lambda_\beta=2e^{6A_\beta}.$$

Let Y_n denote the following mapping from $[0, 1[$ to $\mathbb{R}$:

$$Y_n(x) = \mu(I_n(x))^q |I_n(x)|^{-\varphi(q)}.$$

Obviously, we have

$$\int Y_n \, d\mu = e^{nC_n(q,\varphi(q))}.$$

For $k \in \mathbb{N}$, write $Z_k = Y_{n_0+6k}$, and if $j \in \mathscr{A}$, define

$$E_k^p(j) = \bigcup \{I_i : i \in \mathscr{A}_{n_0+6k},\ I_i \subset I_j \text{ and } n_0 + 6p \text{ is the largest order of a selected interval containing } I_i\}.$$

Note that $E_k^p(j)$ could be reduced to the empty set.

Proof of Assertion 1

In order to establish the first assertion, we only need to prove

$$\int Z_k \, d\mu \approx k \qquad \text{or} \qquad \int Z_k \, d\mu \approx 1. \tag{4}$$

We have

$$\int Z_k \, d\mu = \int_{I_{j_0}} Z_k \, d\mu + \int_{I'_{j'_0}} Z_k \, d\mu + \int_{\overline{I}_{j_0} \cap \overline{I}'_{j'_0}} Z_k \, d\mu. \tag{5}$$

It is easy to see that

$$\int_{\overline{I}_{j_0} \cap \overline{I}'_{j'_0}} Z_k \, d\mu \approx \lambda^k. \tag{6}$$

On the other hand,

$$\forall p \in \mathbb{N}, \qquad 0 \le p \le k, \qquad \int_{E_k^p(j_0)} Z_k \, d\mu \approx \lambda^{k-p} \lambda_\alpha^p.$$

Since $\{E_k^p(j_0)\}_{0 \le p \le k}$ is a family covering I_{j_0} whose elements are mutually disjoint, it follows that

$$\int_{I_{j_0}} Z_k \, d\mu \approx \sum_{p=0}^{k} \lambda^{k-p} \lambda_\alpha^p. \tag{7}$$

In a similar way, we can show that

$$\int_{I'_{j_0}} Z_k \, d\mu \approx \sum_{p=0}^{k} \lambda^{k-p} \lambda_\beta^p. \tag{8}$$

Since $C(q, \varphi(q)) = 0$, the relations (5), (6), (7) and (8) imply (4).

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Proof of Assertion 2

Let us define a family of functions from $[0, 1[$ to $\mathbb{R}_+$ in the following way:

$$g_t = \sum_{k=0}^{\infty} e^{-6kt} Z_k.$$

This allows us to define the family

$$P_t = \frac{g_t}{\int g_t \, d\mu} \mu \qquad (t > 0)$$

of probability measures on $[0, 1[$.

Let $j \in \mathscr{A}_{n_0+6n}$; then we have

$$P_t(I_j) = \frac{1}{\int g_t \, d\mu} \left[\sum_{k=0}^{n-1} e^{-6kt} \int_{I_j} Z_k \, d\mu + \sum_{k=n}^{\infty} e^{-6kt} \int_{I_j} Z_k \, d\mu \right]. \tag{9}$$

For a fixed $k \geq n$, in order to evaluate the integral $\int_{I_j} Z_k \, d\mu$, we need to distinguish three cases.

1st case (I_j is not selected).

$$\int_{I_j} Z_k \, d\mu = \lambda^{k-n} \mu(I_j)^{q+1} |I_j|^{-\varphi(q)}. \tag{10}$$

2nd case (I_j selected, $I_j \subset I_{j_0}$). We have

$$\forall p \in \mathbb{N}, \; n \leq p \leq k, \; \int_{E_k^p(j)} Z_k \, d\mu \approx \lambda^{k-p} \lambda_\alpha^{p-n} \mu(I_j)^{q+1} \, |I_j|^{-\varphi(q)}.$$

Since $I_j = \cup_{p=n}^k E_k^p(j)$ we get

$$\int_{I_j} Z_k \, d\mu \approx \left[\sum_{p=n}^{k} \lambda^{k-p} \lambda_\alpha^{p-n} \right] \mu(I_j)^{q+1} |I_j|^{-\varphi(q)}. \tag{11}$$

3rd case (I_j selected, $I_j \subset I_{j_0'}$).

$$\int_{I_j} Z_k \, d\mu \approx \left[\sum_{p=n}^{k} \lambda^{k-p} \lambda_\beta^{p-n} \right] \mu(I_j)^{q+1} |I_j|^{-\varphi(q)}. \tag{12}$$

When t goes to 0, P_t has at least a weak limit ν. By using the first assertion and the relations (4), (9), (10), (11) and (12), this weak limit ν satisfies

$$\nu(I_j) \leq K \; \mu(I_j)^{q+1} |I_j|^{-\varphi(q)}$$

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where K is a constant which does not depend on I_j. Since the intervals of order $n_0 + 6k$, $k \in \mathbb{N}$, allow us to construct b_μ, we conclude that $b_\mu(q) \leq \varphi(q)$.

This concludes the proof.

Put
$$s = \frac{\beta \log(p_0/p_1) + \log p_1}{\beta \log(l_0/l_1) + \log l_1}$$
and observe that $\Delta_s \neq \phi$, $\Delta_s \subset I_{j_0'}$. Then (1) and (2) imply that $a_1 = c_1 < a_2 = c_2$. On the other hand, due to the theorem in [1], it follows from (1) and (2) that $\mathrm{Dim}\,\Delta_s < B_\mu^*(s)$.

Now, let us consider the Borel probability measure w on $[0, 1[$ such that $w(I_j) = 2^{-k}$ for each selected interval I_j of order $n_0 + 6k$, $I_j \subset I_{j_0'}$. Then w is concentrated on Δ_s. By using Billingsley's theorem [2] for dim and the associated result [6] for Dim, we obtain $\mathrm{Dim}\,\Delta_s = \dim \Delta_s$.

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Faculté des Sciences de Monastir
Département de Mathématiques
5000 Monastir
Tunisie

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