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ORIGINAL RESEARCH PAPER

# Generalized Poisson random variable: its distributional properties and actuarial applications

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**Abstract**

Generalized Poisson (GP) distribution was introduced in Consul & Jain ((1973). *Technometrics*, 15(4), 791–799.). Since then it has found various applications in actuarial science and other areas. In this paper, we focus on the distributional properties of GP and its related distributions. In particular, we study the distributional properties of distributions in the $\mathcal{H}$ family, which includes GP and generalized negative binomial distributions as special cases. We demonstrate that the moment and size-biased transformations of distributions within the $\mathcal{H}$ family remain in the same family, which significantly extends the results presented in Ambagaspitiya & Balakrishnan ((1994). *ASTINBulletin: the Journal of the IAA*, 24(2), 255–263.) and Ambagaspitiya ((1995). *Insurance Mathematics and Economics*, 2(16), 107–127.). Such findings enable us to provide recursive formulas for evaluating risk measures, such as Value-at-Risk and conditional tail expectation of the compound GP distributions. In addition, we show that the risk measures can be calculated by making use of transform methods, such as fast Fourier transform. In fact, the transformation method showed a remarkable time advantage over the recursive method. We numerically compare the risk measures of the compound sums when the primary distributions are Poisson and GP. The results illustrate the model risk for the loss frequency distribution.

**Keywords:** Generalized Poisson distribution; generalized negative binomial distribution; compound random variables; moment transforms; recursive formulas; risk measures

## 1. Introduction

A discrete random variable (rv) $N$ is said to follow a generalized Poisson (GP) distribution with parameters $(\lambda, \theta)$, i.e., $N \sim \mathrm{GP}(\lambda, \theta)$, if its probability mass function (pmf) is given by

$$p_n(\lambda, \theta) = \mathbb{P}(N = n) = \frac{\lambda(\lambda + n\theta)^{n-1}}{n!} \exp\{-\lambda - n\theta\}, \qquad n = 0, 1, 2, \ldots,$$

where $\lambda > 0$ and $\max(-1, -\lambda/4) < \theta < 1$. The mean and variance of the GP rv are $\mathbb{E}[N] = \frac{\lambda}{1-\theta}$ and $\mathbb{V}\mathrm{ar}[N] = \frac{\lambda}{(1-\theta)^3}$, respectively. Thus, it exhibits "overdispersion" or "underdispersion" when $\theta > 0$ or $\theta < 0$, respectively.

The GP distribution was introduced in Consul & Jain (1973) as a limiting form of a generalized negative binomial (GNB) distribution whose pmf is

$$p_n(a, b, \alpha) = \frac{a}{a + bn} \binom{a + bn}{n} \alpha^n (1 - \alpha)^{a+bn-n}, \qquad n = 0, 1, 2, \ldots,$$

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for $0 < \alpha < 1$ and $|\alpha b| < 1$. Note that when $b = 0$, it reduces to a binomial distribution; when $b = 1$, it reduces to the NB distribution. When the parameters $(a, b, \alpha)$ are chosen such that $a\alpha = \lambda$ and $b\alpha = \theta$, as $\alpha$ goes to 0, the limiting distribution of GNB becomes GP. Another relationship between GP and GNB distributions is that they both belong to the so-called $\mathcal{H}$ family of distributions introduced in Ambagaspitiya (1995), whose pmf are characterized by a recursive relationship.

From the probability point view, Consul & Shoukri (1988) showed that the GP distribution can be viewed as the distribution of the number of served customers in a busy period of a queue with Poisson arrival and constant service time. The GP distribution can also be viewed as the distribution of the total progeny in a branching process, where the initial number of a species follows a Poisson($\lambda$) distribution and the number of offspring an individual produce follow a Poisson($\theta$) distribution.

The GP rv has found applications in actuarial science. For instance, Gerber (1990) showed that the number of jumps it takes for a classical compound Poisson risk process with constant claim size $\mu$ to reach a level $x > 0$ follows GP($cx, c\mu$), where $c$ is the ratio of the Poisson arrival rate and the constant premium rate. Consul (1989) compared the GP distribution with several well-known distributions and concluded that GP is a plausible model for claim frequency data. Goovaerts & Kaas (1991) made use of the connection between the GP distribution and the Galton-Watson branching process to derive many distributional properties of GP and a recursive method to compute the distribution of a compound GP distribution. Later, Ambagaspitiya & Balakrishnan (1994) presented a different recursive method for the compound GP distribution.

The concept of the moment transform of distributions has a long history and is widely used in statistics (see, e.g., Patil & Ord, 1976; Arratia & Goldstein, 2010, and references therein). Its relevance to the study of actuarial risk measures has been exploited in the risk theory literature. Furman & Landsman (2005) showed that moment transforms can be used in computing the conditional tail expectation (CTE). More recently, Denuit (2020) presented formulas for the first-moment transform of compound distribution and illustrated their applications in determining the CTE. Ren (2022) further studied the moment transform of multivariate compound sums.

In this paper, we first show that the moment transform of distributions in the $\mathcal{H}$ family is still in the family. Then we apply the results in determining the CTE and higher tail moment for the compound GP rv. These results extend those in Ambagaspitiya & Balakrishnan (1994) and Ambagaspitiya (1995). In addition, we show that the risk measures can also be calculated by making use of transform methods, such as fast Fourier transform (FFT). In fact, the transformation method showed a remarkable time advantage over the recursive method. The applications of FFT in the compound models were discussed in, e.g., Wang (1998) and Embrechts & Frei (2009), and FFT's applications in capital allocation were studied in detail in Blier-Wong et al. (2022).

The rest of the paper is organized as follows. Section 2 reviews various distributional properties of the GP rv and its compound sums. Section 3 studies the moment transform (and size-biased transform) of the GP distribution and more general distributions in the $\mathcal{H}$ family, defined herein. Section 4 applies the results in computing the tail moments of compound GP distribution. Section 5 provides methods for evaluating tail probability and risk measures of the compound GP distribution, with applications in performing the CTE and Euler capital allocations. Finally, numerical examples are provided in Section 6 to illustrate our results.

## 2. The generalized Poisson random variable

### 2.1. The pgf and moments of the GP distribution

Consider a rv $N$ follow the distribution of GP($\lambda, \theta$). Ambagaspitiya & Balakrishnan (1994) showed that when $\theta > 0$, its probability generating function (pgf) could be written as

$$G_N(z) = \mathbb{E}[z^N] = \exp\left\{-\frac{\lambda}{\theta}\left[W(-\theta e^{-\theta} z) + \theta\right]\right\}, \tag{1}$$

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where $W$ is the Lambert's $W$ function defined as

$$W(x)\exp(W(x)) = x.$$

For more details about the properties of $W$ function, see Corless et al. (1993).

Let $\mu_N^{(\alpha)} := \mathbb{E}[N^{(\alpha)}]$ be the $\alpha$th factorial moment of $N$ (see Definition 2). Because $\mathbb{E}[N^{(\alpha)}] = \frac{d^\alpha}{dz^\alpha} G_N(z)\big|_{z=1}$, it is easy to see by differentiating that

$$\mu_N^{(1)} = \mathbb{E}[N] = \frac{\lambda}{1-\theta},$$

$$\mu_N^{(2)} = \mathbb{E}[N \cdot (N-1)] = \frac{\lambda}{(1-\theta)^3} + \left(\frac{\lambda}{1-\theta}\right)^2 - \frac{\lambda}{1-\theta},$$

and thus the variance of $N$ is

$$\mathbb{V}\mathrm{ar}(N) = \frac{\lambda}{(1-\theta)^3}.$$

### 2.2. Recursive formula for GP and a family of $\mathcal{H}$ distributions

**Definition 1** (Ambagaspitiya, 1995). *A rv $N$ is said to belong to the $\mathcal{H}$ family characterized by $(h_1(a,b), h_2(a,b))$, i.e., $X \in \mathcal{H}(h_1(a,b), h_2(a,b))$, if its pmf satisfies the recursive formula*

$$p_n(a,b) = \left(h_1(a,b) + \frac{h_2(a,b)}{n}\right) p_{n-1}(a+b,b), \qquad n = 1,2,\ldots. \tag{2}$$

It is easy to check that both GP and GNB belong to the $\mathcal{H}$ family. For GP$(a,b)$, $h_1(a,b) = \frac{ab}{a+b}$ and $h_2(a,b) = \frac{a^2}{a+b}$, while for GNB$(a,b,\alpha)$, $h_1(a,b) = \frac{a(b-1)\alpha}{(1-\alpha)(a+b)}$ and $h_2(a,b) = \frac{a(a+1)\alpha}{(1-\alpha)(a+b)}$. More specifically, the pmf of GP$(a,b)$ satisfies the recursive relation

$$p_n(a,b) = \frac{a}{a+b}\left(b + \frac{a}{n}\right) p_{n-1}(a+b,b), \qquad n = 1,2,\ldots, \tag{3}$$

and the pmf of GNB$(a,b,\alpha)$ satisfies the recursive relation

$$p_n(a,b,\alpha) = \frac{a\alpha}{(1-\alpha)(a+b)}\left(b - 1 + \frac{a+1}{n}\right) p_{n-1}(a+b,b,\alpha), \qquad n = 1,2,\ldots.$$

**Remark 1.** The $\mathcal{H}$ family reduces to the traditional Panjer family by setting the parameter $b = 0$ in Equation (2).

### 2.3. The GP distribution as a compound Poisson distribution itself

The GP distribution is closely related to the Galton-Watson branching process (Goovaerts & Kaas, 1991), which models the spreading of certain objects (e.g., family names, viruses) as follows. For example, suppose that there are $M$ individuals originally. Each of these gives rise to $L_i$ other individuals, $i = 1,2,\ldots,M$. These, in turn, give rise to $L_{ij}$ new victims, $j = 1,2,\ldots,L_i$, and so on. Now if $M$ is a Poisson$(\lambda)$ distributed rv, and $L_i, L_{ij},\ldots$ are independent Poisson$(\theta)$ $(\theta < 1)$ rv's, the total number $N$ of people infected before extinction has a GP$(\lambda,\theta)$ distribution.

If the original number of infected is fixed at one, then the total number of infected, $B$, has a Borel $(\theta)$ distribution, with pmf

$$\mathbb{P}(B = n) = \frac{(n\theta)^{n-1} e^{-n\theta}}{n!}, \qquad n = 1,2,\ldots,$$

mean $\mathbb{E}(B) = \frac{1}{1-\theta}$ and variance $\mathbb{V}\mathrm{ar}(B) = \frac{\theta}{(1-\theta)^3}$.

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Therefore, a GP$(\lambda, \theta)$ is a compound Poisson sum of independent Borel$(\theta)$ rv's, i.e.,

$$N = \sum_{i=1}^{M} B_i, \tag{4}$$

where the primary rv $M$ is Poisson distributed with parameter $\lambda$, and the secondary rv is independent and identically distributed (iid) as Borel$(\theta)$.

Note that one relationship between the Borel and GP distributions is

$$B \stackrel{d}{=} 1 + N^*,$$

where $N^* \sim \text{GP}(\theta, \theta)$. Therefore, the pgf of $B \sim \text{Borel}(\theta)$ is

$$G_B(z) = zG_{N^*}(z) = ze^{\theta(G_B(z)-1)}, \tag{5}$$

where the compound form of $N^*$ using Equation (4) is applied here.

### 2.4. The compound GP random variable

Denote a compound GP rv $X$ as

$$X = \sum_{i=1}^{N} C_i,$$

where $N$ follows GP$(\lambda, \theta)$ and $C_i$ are iid rv's independent of $N$.

Goovaerts & Kaas (1991) derived a recursive algorithm for computing the distribution of the compound sum when $C_i$'s take positive integer-valued using the representation in Equation (4). Their derivation starts by rewriting the compound GP rv as

$$X = \sum_{i=1}^{M} Y_i, \qquad Y_i = \sum_{j=1}^{B_i} C_{ij},$$

where $M$ is a Poisson$(\lambda)$ rv, $B_i$ is a Borel$(\theta)$ rv, and $C_{ij}$ is an iid sequence of claim amounts having the same distributions as $C_i$. Each $Y_i$ has a compound Borel distribution. Hence, the pgf of $X$ becomes (see Equation (12) of Goovaerts & Kaas, 1991)

$$\begin{aligned} G_X(u) &= \exp(\lambda[G_Y(u) - 1]) \\ &= \exp(\lambda[G_B(G_C(u)) - 1]) \\ &= \exp(\lambda(t-1)), \end{aligned}$$

with $t = t(u)$ such that $G_C(u) = te^{-\theta(t-1)}$ via Equation (5). This gives a two-step algorithm for calculating the distribution of $X$. The first step is to compute the coefficients of $G_B(G_C(u))$ (which amounts to the pdf of $Y_i$); and the second step is to evaluate the coefficients of $G_X(u)$ by Panjer's recursion formula.

Ambagaspitiya & Balakrishnan (1994) made use of the recursion formula for GP in Equation (3) and showed that, when claim sizes are discrete on positive integers (or have absolutely continuous pdf), the pmf (or pdf) of the compound GP rv also satisfies a recursive formula. In particular, their Theorem 5.1 stated that for the discrete claim size with pmf $f(x) = \mathbb{P}(C = x)$ for $x = 1, 2, \ldots$, the pmf of $X$ can be calculated as

$$\mathbb{P}(X = x; \lambda, \theta) = \frac{\lambda}{\lambda + \theta} \sum_{j=1}^{x} \left(\theta + \lambda \frac{j}{x}\right) \mathbb{P}(X = x - j; \lambda + \theta, \theta) f(j), \tag{6}$$

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with the initial values $\mathbb{P}(X=0; \lambda + j\theta, \theta) = \exp(-\lambda - j\theta)$ for $j = 0, 1, 2, \ldots$. Later on, Ambagaspitiya (1995) extended the recursive formula for the pmf of compound distribution when $N$ is in $\mathcal{H}$ family and $C$ has support on positive integers,

$$\mathbb{P}(X=x; a, b) = \sum_{j=1}^{x} \left( h_1(a, b) + h_2(a, b)\frac{j}{x} \right) \mathbb{P}(X=x-j; a+b, b) f(j). \tag{7}$$

## 3. Moment transforms of generalized Poisson distribution

In this section, we show that the factorial moment transform of distributions in family $\mathcal{H}$ is still in the family. This is, their distribution satisfies the recursive relation (2). We begin with the definition of the factorial moment transform of discrete distributions.

**Definition 2.** *Let N be a discrete rv having probability function $p_k$, $k = 0, 1, 2 \cdots$. Denote the $\alpha$**th factorial moment of** N by*

$$\mu_N^{(\alpha)} := \mathbb{E}\big[N^{(\alpha)}\big],$$

*where for an integers I and $\alpha$, $I^{(\alpha)} = I(I-1)\cdots(I-\alpha+1)$ if $\alpha \leq I$ and zero otherwise. Then the $\alpha$**th factorial moment transform** of N is a discrete rv $\tilde{N}_\alpha$ with probability function*

$$g_k^\alpha := \mathbb{P}\big(\tilde{N}_\alpha = k\big) = \frac{\mathbb{E}\big[N^{(\alpha)} 1(N=k)\big]}{\mathbb{E}[N^{(\alpha)}]} = \frac{k^{(\alpha)} p_k}{\mu_N^{(\alpha)}}, \quad k = 0, 1, 2, \cdots,$$

*where $1(A)$ is the indicate function of event A.*

*In the case of $\alpha = 1$, we simply write $\tilde{N}_1 = \tilde{N}$, $\mu_N^{(1)} = \mu_N$ and, etc. The rv $\tilde{N}$ is referred to as the **size-biased transform** of N.*

Note that in this section, we sometimes use the notation $\mu_N^{(\alpha)}(a, b)$, when we need to specify the parameters $a$ and $b$ in the distribution of $N$.

### 3.1. Recursive formula for the pmf of the moment transforms of GP

We have the following theorem regarding the size-biased transform of a general distribution in the $\mathcal{H}$ family.

**Theorem 1.** *For $N \sim \mathcal{H}(h_1(a, b), h_2(a, b))$, let $\tilde{N}$ be its size-biased transform and $M = \tilde{N} - 1$. Then $M \sim \mathcal{H}(h_1^*(a, b), h_2^*(a, b))$ with*

$$h_1^*(a, b) = 1 - \frac{h_1(a, b) + h_2(a, b)}{\mu_N(a, b)},$$

*and*

$$h_2^*(a, b) = \frac{h_1(a, b) + h_2(a, b)}{h_1(a, b)} h_1^*(a, b).$$

*Proof.* Equation (2) can be written as, for $k = 1, 2, \ldots$,

$$\begin{aligned} k p_k(a, b) &= (h_1(a, b)k + h_2(a, b)) p_{k-1}(a+b, b) \\ &= \big[h_1(a, b)(k-1) + (h_1(a, b) + h_2(a, b))\big]\, p_{k-1}(a+b, b). \end{aligned} \tag{8}$$

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Dividing both sides of Equation (8) by $\mu_N(a,b)$, we have, for $k=1,2,\ldots$

$$g_k(a,b)=h_1(a,b)\frac{\mu_N(a+b,b)}{\mu_N(a,b)}g_{k-1}(a+b,b)\\+\left[h_1(a,b)+h_2(a,b)\right]\frac{\mu_N(a+b,b)}{\mu_N(a,b)}\frac{p_{k-1}(a+b,b)}{\mu_N(a+b,b)}.$$

For $k=2,3,\ldots$, multiplying both side by $(k-1)$ yields

$$(k-1)g_k(a,b)=h_1(a,b)\frac{\mu_N(a+b,b)}{\mu_N(a,b)}(k-1)g_{k-1}(a+b,b)\\+\left[h_1(a,b)+h_2(a,b)\right]\frac{\mu_N(a+b,b)}{\mu_N(a,b)}g_{k-1}(a+b,b),$$

which can be written as

$$kg_{k+1}(a,b)=h_1(a,b)\frac{\mu_N(a+b,b)}{\mu_N(a,b)}kg_k(a+b,b)+\left[h_1(a,b)+h_2(a,b)\right]\frac{\mu_N(a+b,b)}{\mu_N(a,b)}g_k(a+b,b),$$

for $k=1,2,3,\ldots$.

Now, since $M:=\tilde{N}-1$, we have its pmf denoted as $f_{k-1}(a,b):=\mathbb{P}(M=k-1)=\mathbb{P}(\tilde{N}=k)=g_k(a,b)$ for $k=1,2,3,\ldots$, thus

$$kf_k(a,b)=h_1(a,b)\frac{\mu_N(a+b,b)}{\mu_N(a,b)}kf_{k-1}(a+b,b)+\left[h_1(a,b)+h_2(a,b)\right]\frac{\mu_N(a+b,b)}{\mu_N(a,b)}f_{k-1}(a+b,b),$$

which is equivalent to

$$f_k(a,b)=\left(h_1(a,b)\frac{\mu_N(a+b,b)}{\mu_N(a,b)}+\left[h_1(a,b)+h_2(a,b)\right]\frac{\mu_N(a+b,b)}{\mu_N(a,b)}\frac{1}{k}\right)f_{k-1}(a+b,b)\\=:\left(h_1^*(a,b)+\frac{h_2^*(a,b)}{k}\right)f_{k-1}(a+b,b),\qquad k=1,2,3,\ldots.$$

Hence, the distribution of $M$ still satisfies the general recursive formula (2) as desired.

To further simplify the representation of $h_1^*(a,b)$, we write Equation (2) as

$$kp_k(a,b)=h_1(a,b)(k-1)p_{k-1}(a+b,b)+(h_1(a,b)+h_2(a,b))p_{k-1}(a+b,b),\qquad k=1,2,\ldots.$$

Summing up both sides from $k=1$ to $\infty$ yields

$$\mu_N(a,b)=h_1(a,b)\mu_N(a+b,b)+\left[h_1(a,b)+h_2(a,b)\right],$$

or equivalently

$$\mu_N(a+b,b)=\frac{\mu_N(a,b)-\left[h_1(a,b)+h_2(a,b)\right]}{h_1(a,b)}.$$

Therefore,

$$h_1^*(a,b)=h_1(a,b)\frac{\mu_N(a+b,b)}{\mu_N(a,b)}=1-\frac{h_1(a,b)+h_2(a,b)}{\mu_N(a,b)},$$

which completes the proof. □

Since both GP and GNB distributions are in the family $\mathcal{H}$, we have the following corollary as the direct application of the above theorem.

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**Corollary 1.**

*(i)* If $N\sim GP(a,b)$, then $M=\tilde{N}-1\in\mathcal{H}(h_1^*(a,b),h_2^*(a,b))$ with $h_1^*(a,b)=b$ and $h_2^*(a,b)=a+b$.
*(ii)* If $N\sim GNB(a,b,\alpha)$, then $M=\tilde{N}-1\in\mathcal{H}(h_1^*(a,b),h_2^*(a,b))$ with $h_1^*(a,b)=\frac{(b-1)\alpha}{1-\alpha}$ and $h_2^*(a,b)=\frac{(a+b)\alpha}{1-\alpha}$.

Actually, Theorem 1 could be generalized to the $\alpha$th factorial moment transform in a similar manner. The result is stated in the following Theorem. Its proof is provided in Appendix A.

**Theorem 2.** *Let $N\sim\mathcal{H}(h_1(a,b),h_2(a,b))$. For an integer $\alpha\geq 1$, let $\tilde{N}_\alpha$ be its $\alpha$th factorial moment transform and $M_\alpha=\tilde{N}_\alpha-\alpha$. Then $M_\alpha\sim\mathcal{H}(h_1^*(a,b),h_2^*(a,b))$ with*

$$h_1^*(a,b)=h_1(a,b)\frac{\mu_N^{(\alpha)}(a+b,b)}{\mu_N^{(\alpha)}(a,b)}\qquad \textit{and}\qquad h_2^*(a,b)=\frac{\alpha\cdot h_1(a,b)+h_2(a,b)}{h_1(a,b)}h_1^*(a,b).$$

### 3.2. Generating function for the moment transforms of GP

As illustrated in Remark 2.5 of Blier-Wong et al. (2022), another way to study the moment transform of distributions is through the generating function. This is due to the fact that the pgf of the factorial moment transform of a discrete rv $N$ is given by

$$G_{\tilde{N}_\alpha}(z)=\frac{z^\alpha}{\mathbb{E}[N^{(\alpha)}]}\cdot\frac{d^\alpha}{dz^\alpha}G_N(z).$$

In particular, the pgf for a first-moment transformed GP distribution, $\tilde{N}$, is given by

$$\begin{aligned}G_{\tilde{N}}(z)&=\frac{zG_N'(z)}{\mathbb{E}[N]}=\frac{z}{\mathbb{E}[N]}\left\{\frac{\lambda}{\theta}\theta e^{-\theta}W'(-\theta e^{-\theta}z)G_N(z)\right\}\\&=(1-\theta)e^{-\theta}W'(-\theta e^{-\theta}z)zG_N(z)\\&=-\frac{1-\theta}{\theta}\frac{W(-\theta e^{-\theta}z)}{1+W(-\theta e^{-\theta}z)}G_N(z),\end{aligned}$$

where we have made use of the identity for Lambert's $W$ function that

$$W'(x)=\frac{W(x)}{x(1+W(x))}.$$

Then the pgf for $M=\tilde{N}-1$ is

$$G_M(z)=-\frac{1-\theta}{\theta}\frac{W(-\theta e^{-\theta}z)}{1+W(-\theta e^{-\theta}z)}\frac{G_N(z)}{z}. \tag{9}$$

## 4. Size-biased transform of compound distributions

In this section, we discuss the size-biased transform of the corresponding compound GP rv's. To proceed, we provide a general definition of the moment transform which accounts for all types of nonnegative rv's.

**Definition 3.** *Consider a nonnegative rv $C$ with cumulative distribution function (cdf) $F_C$ and moments $\mathbb{E}[C^\alpha]<\infty$ for some positive integer $\alpha$. A rv $\tilde{C}_\alpha$ is said to be a copy of the $\alpha$th moment transform of $C$ if its cdf is given by*

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$$F_{\tilde{C}_\alpha}(c) = \frac{\mathbb{E}[C^\alpha 1(C \le c)]}{\mathbb{E}[C^\alpha]} = \frac{\int_0^c t^\alpha dF_C(t)}{\mathbb{E}[C^\alpha]}, \quad c > 0.$$

*The first moment transform of $C$ is commonly referred to as **the size-biased transform**, denoted by $\tilde{C}$ in the sequel.*

We now consider a compound rv $X$ defined as

$$X = \sum_{i=1}^{N} C_i,$$

where $N$ is a counting rv and $C_i$ are iid claim number (or claim szie) rv's having the same distribution as the common rv $C$ whose cdf is denoted as $F_C$.

**Theorem 3.** *For a compound rv $X = \sum_{i=1}^{N} C_i$,*

*(i) Let $\tilde{X}$ denote its size-biased transform, then*

$$\tilde{X} \stackrel{d}{=} \tilde{C} + Y, \tag{10}$$

*where $Y = \sum_{i=1}^{M} C_i$ with $M = \tilde{N} - 1$ and $\tilde{C}$ being the size-biased version of $C$.*

*(ii) Let $\tilde{X}_2$ denote its second-moment transform, then its distribution is given by*

$$\mathbb{P}(\tilde{X}_2 \le x) = \frac{1}{\mathbb{E}[X^2]}\left(\mathbb{E}[N^{(2)}](\mathbb{E}[C])^2\mathbb{P}(Y_2 + \widetilde{C}^{*2} \le x) + \mathbb{E}[N]\mathbb{E}[C^2]\mathbb{P}(Y_1 + \widetilde{C}_{1,2} \le x)\right), \tag{11}$$

*where $Y_1 = \sum_{i=1}^{M} C_i$ and $Y_2 = \sum_{i=1}^{M_2} C_i$; $\widetilde{C}^{*2}$ is the twofold convolution of $\widetilde{C}$ (the first-moment transform of $C$), and $\widetilde{C}_{1,2}$ is a copy of the second-moment transform of $C$.*

This result follows directly from Ren (2021) and Denuit and Robert (2022)). Here we provide an alternative proof via moment generating function (mgf) arguments provided existence.

*Proof.* Without loss of generality, we assume that $C$ is a continuous rv. (The discrete or mixture versions could be similarly argued.) Note that, for the size-biased and second moment transforms of $N$ and $C$, (denoted as $\tilde{N}$, $\tilde{N}_2$ and $\tilde{C}$, $\widetilde{C}_{1,2}$, with $M = \tilde{N} - 1$ and $M_2 = \tilde{N}_2 - 2$), we have

$$G_{\tilde{N}}(t) = \frac{tG'_N(t)}{\mathbb{E}[N]}, \qquad G_{\tilde{N}_2}(t) = \frac{t^2 G''_N(t)}{\mathbb{E}[N^{(2)}]},$$

$$G_M(t) = \frac{G_{\tilde{N}}(t)}{t} = \frac{G'_N(t)}{\mathbb{E}[N]}, \qquad G_{M_2}(t) = \frac{G_{\tilde{N}_2}(t)}{t^2} = \frac{G''_N(t)}{\mathbb{E}[N^{(2)}]}, \tag{12}$$

$$M_{\tilde{C}}(t) = \frac{M'_C(t)}{\mathbb{E}[C]}, \qquad M_{\tilde{C}_{1,2}}(t) = \frac{M''_C(t)}{\mathbb{E}[C^2]}. \tag{13}$$

The mgf of $\tilde{X}$, the size-biased version of $X$, is given as

$$\begin{aligned} M_{\tilde{X}}(t) &= \frac{M'_X(t)}{\mathbb{E}[X]} \\ &= \frac{G'_N(M_C(t)) \cdot M'_C(t)}{\mathbb{E}[N] \cdot \mathbb{E}[C]} \qquad \text{since } M_X(t) = G_N(M_C(t)),\ \mathbb{E}[X] = \mathbb{E}[N]\mathbb{E}[C], \\ &= G_M(M_C(t)) \cdot M_{\tilde{C}}(t) \qquad \text{by Equations (12) and (13)}. \end{aligned} \tag{14}$$

This leads to Equation (10).

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Similarly, the mgf of $\tilde{X}_2$, the second moment transform of $X$, is given as

$$
\begin{aligned}
M_{\tilde{X}_2}(t) &= \frac{M''_X(t)}{\mathbb{E}[X^2]} = \frac{G''_N(M_C(t))}{\mathbb{E}[X^2]} \\
&= \frac{G''_N(M_C(t))(M'_C(t))^2 + G'_N(M_C(t))M''_C(t)}{\mathbb{E}[X^2]} \\
&= \frac{\mathbb{E}[N^{(2)}]\mathbb{E}^2[C]}{\mathbb{E}[X^2]}\left(G_{M_2}(M_C(t))M^2_{\tilde{C}}(t)\right) + \frac{\mathbb{E}[N]\mathbb{E}[C^2]}{\mathbb{E}[X^2]}\left(G_M(M_C(t))M_{\tilde{C}_{1,2}}(t)\right),
\end{aligned}
$$

which results in Equation (11). This completes the proof. □

**Remark 2.** For a convolution $Z = Z_1 + Z_2$ we have that

$$
\begin{aligned}
\mathbb{P}(\tilde{Z} \le t) &= \frac{\mathbb{E}[Z 1_{\{Z \le t\}}]}{\mathbb{E}[Z]} = \frac{\mathbb{E}[(Z_1 + Z_2)1_{\{Z_1+Z_2 \le t\}}]}{\mathbb{E}[Z_1 + Z_2]} \\
&= \frac{\mathbb{E}[Z_1]}{\mathbb{E}[Z_1] + \mathbb{E}[Z_2]}\mathbb{P}\big(\tilde{Z}_1 + Z_2 \le t\big) + \frac{\mathbb{E}[Z_2]}{\mathbb{E}[Z_1] + \mathbb{E}[Z_2]}\mathbb{P}\big(Z_1 + \tilde{Z}_2 \le t\big),
\end{aligned}
$$

hence, from the probability point of view, we could interpret $\tilde{Z}$ being a discrete mixture of $\tilde{Z}_1 + Z_2$ and $Z_1 + \tilde{Z}_2$ with weights $\frac{\mathbb{E}[Z_1]}{\mathbb{E}[Z_1]+\mathbb{E}[Z_2]}$ and $\frac{\mathbb{E}[Z_2]}{\mathbb{E}[Z_1]+\mathbb{E}[Z_2]}$, respectively. Equivalently, we could write

$$
\tilde{Z} = I(\tilde{Z}_1 + Z_2) + (1 - I)(Z_1 + \tilde{Z}_2),
$$

with $I$ being an independent Bernoulli rv with mean $\frac{\mathbb{E}[Z_1]}{\mathbb{E}[Z_1]+\mathbb{E}[Z_2]}$. This could be easily generalized to the $n$-fold convolution, which results in Property 1 in Denuit (2020).

Furthermore, this provides an alternative proof for the above Theorem 3 from (i) to (ii) that

$$
\begin{aligned}
\tilde{X}_2 = \tilde{\tilde{X}} &= \widetilde{\tilde{C} + Y} \\
&= I(\tilde{\tilde{C}} + Y) + (1 - I)(\tilde{C} + \tilde{Y}) \\
&= I(\tilde{C}_{1,2} + Y) + (1 - I)(\tilde{C} + \tilde{C} + Y_2) \\
&= I(\tilde{C}_{1,2} + Y_1) + (1 - I)(\tilde{C}^{*2} + Y_2), \qquad (15)
\end{aligned}
$$

where $I$ is an independent Bernoulli rv with mean $\frac{\mathbb{E}[\tilde{C}]}{\mathbb{E}[\tilde{C}]+\mathbb{E}[Y]} = \frac{\mathbb{E}[C^2]\mathbb{E}[N]}{\mathbb{E}[X^2]}$.

Therefore, noticing that

$$
\mathbb{E}[X^2] = \mathbb{E}[N]\mathbb{E}[C^2] + \mathbb{E}[N^{(2)}](\mathbb{E}[C])^2,
$$

we see from Equation (15) that the distribution of $\tilde{X}_2$ is a mixture of $Y_2 + \tilde{C}^{*2}$ and $Y_1 + \tilde{C}_{1,2}$.

Since the modified size-biased Poisson($\lambda$) distribution (i.e., in the form $M = \tilde{N} - 1$) is again a Poisson($\lambda$) distribution, we have the following corollary directly from Equation (10).

**Corollary 2.** *The size-biased transform of a compound Poisson distribution $X$ is given by*

$$
\tilde{X} \overset{d}{=} \tilde{C} + X.
$$

Since GP distribution is a compound Poisson distribution itself, we could apply the above corollary to the size-biased transform of GP as below.

**Corollary 3.** *The size-biased transform of $N \sim GP(\lambda, \theta)$ is*

$$
\tilde{N} \overset{d}{=} \tilde{B} + N,
$$

*the convolution of $\tilde{B}$ (the size-biased transform of $B \sim Borel(\theta)$) and an independent copy of $N$.*

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Furthermore, for the compound sums with GP as the primary distribution, we have the following result, whose proof is postponed to Appendix A.

**Corollary 4.** *Let $X=\sum_{i=1}^{N} C_i$ with $N\sim GP(\lambda,\theta)$. Its size-biased transform is given by*

$$\begin{aligned}\tilde{X} &\overset{d}{=} \tilde{C}+Y\\ &\overset{d}{=} \tilde{C}+Z+X,\end{aligned} \tag{16}$$

*where $Y=\sum_{i=1}^{\tilde{N}-1} C_i$ and $Z=\sum_{i=1}^{\tilde{B}-1} C_i$ with $\tilde{B}$ being the size-biased transform of $B\sim Borel(\theta)$.*

## 5. The actuarial applications of GP and its moment transforms

As discussed before, the GP distribution has been applied in different actuarial fields. In this section, we discuss two applications for the compound GP distribution. First, we study the computation methods for risk measures of the compound GP distribution, and then we discuss the risk allocation for a portfolio of compound GP risks.

### 5.1. Application I: risk measures on compound GP distribution

Consider the aggregate loss model

$$X=\sum_{i=1}^{N} C_i,$$

where $N$ is the counting rv and $C_i$ follows some discrete distribution with $\mathbb{P}(C_i=k)=f_k$. In this section, we provide methods for computing the tail probability $\mathbb{P}(X>x)$ and the CTE defined as $\mathbb{E}[X|X>x]$.

To calculate the CTE, we use the result from Denuit (2020), which states that, for any rv $X$ and a measurable function $g$,

$$\mathbb{E}[Xg(X)]=\mathbb{E}[X]\mathbb{E}[g(\tilde{X})], \tag{17}$$

where $\tilde{X}$ is the size-biased transform of $X$. When $X$ follows a compound distribution, the distribution $\tilde{X}$ is given by Theorem 3 in Equation (10). Hence, to evaluate the CTE, one could apply Equation (17) with $g(x)=\mathbb{I}(X>x)$ being an indicate function and obtain

$$\mathbb{E}[X|X>x]=\mathbb{E}[X]\frac{\mathbb{P}(\tilde{X}>x)}{\mathbb{P}(X>x)}.$$

Thus, the calculation boils down to the two tail probabilities $\mathbb{P}(\tilde{X}>x)$ and $\mathbb{P}(X>x)$. In what follows, we provide three methods to carry out the calculation, namely, the recursive method, the transformation method via FFT, and the simulation method. Note that the recursive method has the restriction that the claim size distribution has positive support. The other two methods do not have such a restriction.

#### 5.1.1. Recursive method

The recursive formula for computing the distribution of a compound GP $X$ with positive discrete severity is given in Equation (6).

To compute the distribution of $\tilde{X}$, we need to use our main result in Theorem 3. Recall from Equation (10) that

$$\tilde{X}\overset{d}{=}\tilde{C}+Y,$$

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where $Y=\sum_{i=1}^{M} C_i$ with $M=\tilde{N}-1$ and $\tilde{C}$ being the size-biased version of $C$. When $N$ is GP distributed, we showed in Corollary 1 that the distribution of $M$ still in $\mathcal{H}$ family, therefore, the distribution of $Y=\sum_{i=1}^{M} C_i$ can be evaluated recursively using Equation (7). Then, the distribution of $\tilde{X}$ can be computed through a direct convolution of $\tilde{C}$ and $Y$.

#### 5.1.2. Transformation method via FFT

The mgf of a compound GP rv $X$ is given by

$$M_X(t)=\mathbb{E}[e^{tX}]=G_N(M_C(t)).$$

Applying the pgf of the GP distribution in Equation (1), we have

$$M_X(t)=\exp\left\{-\frac{\lambda}{\theta}\left[W(-\theta e^{-\theta}M_C(t))+\theta\right]\right\}. \tag{18}$$

Then by Equation (14), the mgf of $\tilde{X}$ becomes

$$\begin{aligned} M_{\tilde{X}}(t) &= G_M(M_C(t))\cdot M_{\tilde{C}}(t) \\ &= -\frac{1-\theta}{\theta}\,\frac{W(-\theta e^{-\theta}M_C(t))}{1+W(-\theta e^{-\theta}M_C(t))}\,\frac{G_N(M_C(t))}{M_C(t)}\,\frac{M_C{}'(t)}{\mathbb{E}[C]}, \end{aligned} \tag{19}$$

where we applied the pgf of $M$ given in Equation (9).

With the mgf of $X$ and $\tilde{X}$ given by Equations (18) and (19), respectively, both the tail probability and the CTE can be computed through FFT. An algorithm is provided in the following; see Wang (1998) and Embrechts & Frei (2009).

1. **Parameter Definition:** Define the parameters for the GP claim frequency and the claim severity distributions.
2. **Fourier Transform of Severity Distribution:** Compute the FFT of the claim severity distribution. If the severity distribution is continuous, it needs to be discretized.
3. **Fourier Transform of Compound Distribution:** The FFT of compound distribution $X$ (or $\tilde{X}$) can be computed according to Equation (18) (or Equation (19)). The Lambert $W$ function can be evaluated using, for example, the `emdbook` package in R programming language.
4. **Inverse Fourier Transform:** In this step, we recover the aggregate loss distribution by applying inverse FFT.

These steps together provide a practical and computationally efficient way of calculating the compound GP distribution.

#### 5.1.3 Simulation method via the branching process

Since the cdf of the GP distribution cannot be inverted analytically, when using the direct inversion method to simulate GP rv's, we use the recursive relation in Equation (2) to obtain the cdf and then numerically invert it to generate pseudo numbers.

Another approach to generate GP rv's is to utilize the branching process-based algorithm, as described in Chapter 16 of Consul & Famoye (2006). Recall that a branching process generates the GP rv as follows. Suppose that there are $Y$ individuals originally. Each of these gives rise to $L_i$ other individuals, $i=1,2,\ldots,Y$. These, in turn, give rise to $L_{ij}$ new victims, $j=1,2,\ldots,L_i$, and so on. Now if $Y$ is a Poisson($\lambda$) distributed rv, and $L_i, L_{ij},\ldots$ are independent Poisson($\theta$) ($\theta<1$) rv's, the total number $N$ of people infected before extinction has a GP distribution with parameters $(\lambda,\theta)$. Thus, the algorithm for the branching process simulation method for GP is given by:

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1. Generate $Y$ from Poisson distribution with mean $\lambda$;
2. Let $N = Y$;
3. While $Y > 0$, generate $Z$ from Poisson distribution with mean $\theta Y$, and update $N = N + Z$ and $Y = Z$. Repeat until $Y = 0$;
4. Return $N$.

Note that this simulation method does not require calculating any GP probabilities.

### 5.2. Application II: capital allocations for a portfolio of risks

In this section, we consider a capital allocation problem of a portfolio of $n$ independent losses $(X_1, X_2, \ldots, X_n)$, each of which is a compound distribution. Specifically,

$$X_i = \sum_{k=1}^{N_i} C_{ik}, \qquad i = 1, \ldots, n,$$

where $N_i \sim \mathrm{GP}(\lambda_i, \theta_i)$ and $C_{ik}$, $k = 1, 2, \ldots$, are iid with the common distribution $C_i$. All rv's are independent. For $i = 1, \ldots, n$, let $\tilde{X}_i$ and $\tilde{C}_i$ be the size-biased version of $X_i$ and $C_i$, respectively, and let

$$Y_i = \sum_{j=1}^{M_i} C_{ij},$$

where $M_i = \tilde{N}_i - 1$. Then as shown earlier, we have

$$\tilde{X}_i \stackrel{d}{=} Y_i + \tilde{C}_i.$$

Now consider the aggregate risk of the portfolio, denoted as

$$S = \sum_{i=1}^{n} X_i.$$

We look into details of the CTE and Euler allocation rules in the following.

**CTE allocation rule**

Under the CTE risk measure, the total capital requirement for the portfolio is $\mathbb{E}[S|S > s]$, which can be calculated using

$$\mathbb{E}[S|S > s] = \mathbb{E}[S]\frac{\mathbb{P}(\tilde{S} > s)}{\mathbb{P}(S > s)},$$

where $\tilde{S}$ denotes the size-biased transform of $S$.

Furthermore, from Property 1 in Section 3.1 of Denuit (2020), we have

$$\tilde{S} \stackrel{d}{=} S - X_K + \tilde{X}_K = \sum_{j \neq K}^{n} X_j + \tilde{X}_K,$$

where $K$ is a discrete rv, independent of $\{X_i\}$ and $\{\tilde{X}_i\}$, with pmf

$$\mathbb{P}(K = i) = \frac{\mathbb{E}[X_i]}{\mathbb{E}[S]}, \qquad i = 1, 2, \ldots, n.$$

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Putting these together, the capital requirement for the portfolio becomes

$$\begin{aligned}\mathbb{E}[S|S>s] &= \mathbb{E}[S]\frac{\mathbb{P}(\tilde{S}>s)}{\mathbb{P}(S>s)} \\ &= \sum_{i=1}^{n}\frac{\mathbb{E}(X_i)}{\mathbb{P}(S>s)}\mathbb{P}\left(\sum_{j\neq i}^{n}X_j+\tilde{X}_i>s\right). \end{aligned} \tag{20}$$

In addition, according to the CTE capital allocation rule, the capital allocated to risk $X_i$ is

$$\mathbb{E}[X_i|S>s] = \frac{\mathbb{E}(X_i)}{\mathbb{P}(S>s)}\mathbb{P}(S-X_i+\tilde{X}_i>s) = \frac{\mathbb{E}(X_i)}{\mathbb{P}(S>s)}\mathbb{P}\left(\sum_{j=1,j\neq i}^{n}X_j+\tilde{X}_i>s\right). \tag{21}$$

**Euler allocation rule**

When the distribution of $S$ is discrete, we may study the conditional mean allocation $\mathbb{E}[X_i|S=s]$. This quantity is based on the Euler allocation rule and plays a unique role in peer-to-peer insurance prices. It has been studied in the literature. For example, Denuit & Robert 2022) derived a list of desirable properties of this allocation principle. Gribkova et al. (2023) proposed an empirical estimator of $\mathbb{E}[X_i|S=s]$ and established its asymptotic normality under minimal conditions. Blier-Wong et al. (2022) presented an ordinary generating function of $\mathbb{E}[X_i \times I_{\{S=s\}})]$, which leads a method to compute it using FFT.

Here, because the distribution of $X_i$ and its moment transform $\tilde{X}_i$ can be evaluated using recursive or FFT methods as shown in Section 4, we took a different approach from Blier-Wong et al. (2022) and propose to compute conditional mean allocation $\mathbb{E}[X_i|S=s]$ by

$$\mathbb{E}[X_i|S=s] = \frac{\mathbb{E}(X_i)}{\mathbb{P}(S=s)}\mathbb{P}\left(\sum_{j\neq i}^{n}X_j+\tilde{X}_i=s\right). \tag{22}$$

Numerically, in order to apply the recursive methods to evaluate Equations (21) and (22), all the claim size $C_{ij}$ has to have positive support, which is an inconvenient constraint. Therefore, we recommend using FFT method because it can be flexibly used to compute the distribution function of $X_i$, its moment transform, as well as the required convolution.

## 6. Numerical examples

This section provides numerical examples illustrating the practicality of our results. The first example considers the computation of the tail probabilities and CTE of a compound GP distribution; while the second one considers capital allocations for a portfolio of compound GP losses.

### 6.1. Example 1

Consider a compound GP distribution

$$X = \sum_{i=1}^{N} C_i,$$

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**Table 1.** Computation of tail probabilities and CTE of compound GP distribution using different methods

| $x$ | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{P}(X>x)$[a] | 0.945 | 0.836 | 0.694 | 0.548 | 0.417 | 0.308 | 0.223 | 0.159 | 0.111 | 0.078 |
| $\mathbb{P}(X>x)$[b] | 0.945 | 0.838 | 0.694 | 0.549 | 0.418 | 0.307 | 0.221 | 0.158 | 0.110 | 0.078 |
| $\mathbb{P}(X>x)$[c] | 0.946 | 0.837 | 0.694 | 0.550 | 0.418 | 0.308 | 0.223 | 0.159 | 0.111 | 0.078 |
| $\mathbb{P}(X>x)$[d] | 0.945 | 0.836 | 0.694 | 0.548 | 0.417 | 0.308 | 0.223 | 0.159 | 0.111 | 0.078 |
| $\mathbb{E}(X\|X>x)$[a] | 52.91 | 57.74 | 64.32 | 72.03 | 80.44 | 89.31 | 98.49 | 107.88 | 117.42 | 127.06 |
| $\mathbb{E}(X\|X>x)$[b] | 52.91 | 57.65 | 64.31 | 71.90 | 80.19 | 89.18 | 98.48 | 107.82 | 117.48 | 127.01 |
| $\mathbb{E}(X\|X>x)$[c] | 52.88 | 57.69 | 64.33 | 71.88 | 80.30 | 89.16 | 98.30 | 107.51 | 117.23 | 126.54 |
| $\mathbb{E}(X\|X>x)$[d] | 52.91 | 57.73 | 64.30 | 72.00 | 80.39 | 89.25 | 98.41 | 107.79 | 117.32 | 126.95 |

[a] FFT method.
[b] $N$ Simulated by Branching process based simulation.
[c] $N$ Simulated by direct inversion simulation method.
[d] Recursive method.

**Table 2.** Comparison of tail probability results

| Method | $\mathbb{P}(X>100)$ | Standard deviation | Computation time (s) |
|---|---|---|---|
| FFT | 0.077637 | – | 0.01 |
| Simulation via branching | 0.077819 | 0.000827 | 16.23 |
| Simulation via inversion | 0.077487 | 0.000734 | 19.40 |
| Recursive | 0.077637 | – | 9.14 |

**Table 3.** Comparison of CTE results

| Method | $\mathbb{E}(X\|X>100)$ | Standard deviation | Computation time (s) |
|---|---|---|---|
| FFT | 127.068243 | – | 0.01 |
| Simulation via branching | 127.013518 | 0.269607 | 23.51 |
| Simulation via inversion | 126.955244 | 0.299739 | 20.53 |
| Recursive | 126.949588 | – | 10.30 |

where $N$ follows $\text{GP}(\lambda = 5, \theta = 0.5)$ and $C_i$'s are iid following zero truncated Poisson distribution with parameter $\mu = 5$, that is

$$p_C^T(k) = \frac{1}{1-e^{-\mu}} \frac{\mu^k e^{-\mu}}{k!}, \qquad k = 1, 2, 3, \ldots.$$

This assumption of a zero-truncated claim size distribution allows the application of the recursive method for computing the compound GP distribution.

In this example, we calculate the tail probability $\mathbb{P}(X > x)$ and CTE $\mathbb{E}[X|X > x]$ for different risk levels $x$.

The computation is carried out using the recursive and FFT methods. The results are compared with those obtained from the simulation methods with a sample size of $n = 10^6$. The simulations were performed using the inversion method and the branching algorithm. The results are shown in Table 1, from which we observe that all methods result in nearly identical results.

In addition, in Tables 2 and 3, we report the computation time of the different methods and the standard error of the simulation methods (estimated by repeating the simulation 100 times). All computations are performed on a personal laptop with Intel Core i7 CPU and 16 GB RAM. It is seen that the FFT method is by far the fastest, whereas the simulation methods are the slowest. Based on these results, we conclude that the FFT method is reliable and efficient for evaluating the risk measures of the compound GP distributions.

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**Table 4.** Tail probabilities and CTE of compound Poisson distribution

| $x$ | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{P}(X > x)$ **(Poi)** | 0.997 | 0.971 | 0.878 | 0.698 | 0.470 | 0.266 | 0.127 | 0.051 | 0.018 | 0.005 |
| $\mathbb{E}(X\|X > x)$ **(Poi)** | 50.46 | 51.36 | 54.00 | 58.70 | 65.08 | 72.60 | 80.87 | 89.61 | 98.68 | 107.96 |

**Table 5.** The portfolio of three compound GP risks

| Risk | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $\text{GP}(\lambda_i, \theta_i)$ | (5, 0.5) | (7.5, 0.5) | (7.5, 0.625) |
| $\mathbb{P}(C_i = 1)$ | 0.7 | 0.3 | 0.3 |
| $\mathbb{P}(C_i = 2)$ | 0.2 | 0.5 | 0.5 |
| $\mathbb{P}(C_i = 3)$ | 0.1 | 0.2 | 0.2 |

Furthermore, we compare our results with the compound Poisson case with a Poisson parameter 10 such that the mean of the frequency distribution equals. As seen in Table 4, the tail of compound GP distribution is heavier, and its CTE risk measures are larger than the compound Poisson case.

### 6.2. Example 2

Consider the capital allocation example in Section 4.2.4 in Denuit (2020). Let $(X_1, X_2, X_3)$ be three independent risks follow the compound GP's whose parameters are given in Table 5. The severity parameters are the same as those adopted in Denuit (2020), whereas the claim frequencies are assumed to be GP distributed instead of Poisson distributed while keeping the same mean. The significance of using GP is that it has over-dispersion: the means of the frequencies are $(10, 15, 20)$, whereas their variances are $(40, 60, 142)$. Consequently, compared with the case of Poisson frequency, $X_3$ is much more risky than $X_2$, which in turn is more risky than $X_1$.

The portfolio aggregate risk is thus given by $S = \sum_{i=1}^{3} X_i$.

#### CTE allocation rule

The CTE-based total capital requirement is given as $\mathbb{E}(S|S > s)$, and the proportions allocated to individual risk are

$$\alpha_i := \frac{\mathbb{E}(X_i|S > s)}{\mathbb{E}(S|S > s)}, \quad i = 1, 2, 3.$$

We implemented the capital allocation computation described in Section 5.2 using the FFT method. The results are shown in Table 6, from which we conclude that:

- When the threshold $s$ increases, the proportions of risk capital allocated to $X_1$ and $X_2$ slightly decrease, whereas those allocated to $X_3$ increase.
- The risks $X_2$ and $X_3$ have the same loss severity but the frequency component in $X_3$ is more volatile. As a result, their proportions of risk capital allocated change with the risk levels $s$; more precisely, a greater proportion is assigned to $X_3$ as $s$ increases. The ratio $\alpha_2/\alpha_3$ decreases from 75% to around 60% as $s$ increases from 0 to 100. Note that $\alpha_2/\alpha_3 = \mathbb{E}(N_2)/\mathbb{E}(N_3) = 0.75$ when $s = 0$.

Table 7 shows the CTE capital allocation and tail behavior for the compound Poisson sums under the parameter setup in Table 5 (except changing the primary distributions to Poisson with the same means), which is also calculated in Table 2 of Ren (2022). Comparing Tables 6 and 7, we have the following observations:

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**Table 6.** CTE capital allocation for the portfolio of three compound GP risks

| $s$ | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha_1$ | 0.174 | 0.174 | 0.174 | 0.173 | 0.171 | 0.168 | 0.163 | 0.158 | 0.152 | 0.146 |
| $\alpha_2$ | 0.354 | 0.354 | 0.354 | 0.353 | 0.352 | 0.348 | 0.344 | 0.338 | 0.330 | 0.322 |
| $\alpha_3$ | 0.472 | 0.472 | 0.472 | 0.474 | 0.478 | 0.484 | 0.493 | 0.504 | 0.517 | 0.532 |
| $\mathbb{E}(S\|S>s)$ | 80.50 | 80.57 | 81.13 | 82.92 | 86.45 | 91.66 | 98.20 | 105.72 | 113.93 | 122.63 |
| $\mathbb{P}(S>s)$ | 1.000 | 0.999 | 0.989 | 0.950 | 0.867 | 0.742 | 0.593 | 0.446 | 0.318 | 0.217 |

**Table 7.** CTE capital allocation and tail risk for the portfolio of three compound Poisson risks

| $s$ | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha_1$ **(Poi)** | 0.174 | 0.174 | 0.174 | 0.174 | 0.174 | 0.173 | 0.172 | 0.170 | 0.168 | 0.166 |
| $\alpha_2$ **(Poi)** | 0.354 | 0.354 | 0.354 | 0.354 | 0.354 | 0.354 | 0.355 | 0.356 | 0.357 | 0.357 |
| $\alpha_3$ **(Poi)** | 0.472 | 0.472 | 0.472 | 0.472 | 0.472 | 0.472 | 0.473 | 0.474 | 0.475 | 0.477 |
| $\mathbb{E}(S\|S>s)$ **(Poi)** | 80.50 | 80.50 | 80.50 | 80.51 | 80.72 | 81.96 | 85.36 | 91.08 | 98.45 | 106.80 |
| $\mathbb{P}(S>s)$ **(Poi)** | 1.00 | 1.00 | 1.00 | 1.00 | 0.993 | 0.944 | 0.776 | 0.488 | 0.216 | 0.066 |

**Table 8.** Euler capital allocation for the portfolio of three compound GP risks

| $s$ | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\beta_1$ | 0.272 | 0.245 | 0.230 | 0.218 | 0.208 | 0.199 | 0.190 | 0.182 | 0.173 | 0.165 |
| $\beta_2$ | 0.382 | 0.390 | 0.391 | 0.388 | 0.385 | 0.380 | 0.374 | 0.366 | 0.358 | 0.349 |
| $\beta_3$ | 0.346 | 0.365 | 0.379 | 0.393 | 0.407 | 0.421 | 0.436 | 0.452 | 0.469 | 0.486 |
| $\mathbb{P}(S=s)$ | 9.8e-06 | 2.9e-04 | 1.9e-03 | 5.7e-03 | 1.0e-02 | 1.4e-02 | 1.5e-02 | 1.4e-02 | 1.2e-02 | 8.8e-03 |

- Overall, the tail of the compound GP distribution is heavier than compound Poisson, and so is the CTE risk measure.
- When the threshold $s$ increases, the proportion of risk capital allocated to $X_2$ decreases under the compound GP assumption, while it increases slightly under the compound Poisson case.
- At the same risk level $s$, more capital is allocated to the risk $X_3$ under the compound GP case.

These differences show the model risk for the frequency assumption, which is one of the motivations for future analysis on statistical modeling of the loss frequency distribution based on real data. This will be a future research topic.

### Euler allocation rule

We re-do the above calculation with Euler allocation rule, where the proportions allocated to individual risks are

$$\beta_i := \frac{\mathbb{E}(X_i|S=s)}{\mathbb{E}(S|S=s)} = \frac{\mathbb{E}(X_i|S=s)}{s}, \quad i=1,2,3,$$

where the numerator is given in Equation (22). The results for the GP case are shown in Table 8 and those for the Poisson case are shown in Table 9. We can observe that the capital allocation patterns for the three risks are similar under the Euler and CTE capital allocation rules.

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**Table 9.** Euler capital allocation and tail risk for the portfolio of three compound Poisson risks

| $s$ | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\beta_1$ **(Poi)** | 0.271 | 0.232 | 0.212 | 0.199 | 0.191 | 0.184 | 0.179 | 0.174 | 0.171 | 0.168 |
| $\beta_2$ **(Poi)** | 0.312 | 0.329 | 0.338 | 0.343 | 0.347 | 0.350 | 0.352 | 0.354 | 0.355 | 0.357 |
| $\beta_3$ **(Poi)** | 0.417 | 0.439 | 0.450 | 0.457 | 0.463 | 0.466 | 0.469 | 0.472 | 0.474 | 0.476 |
| $\mathbb{P}(S=s)$ **(Poi)** | 6.6e-13 | 4.4e-09 | 1.6e-06 | 9.4e-05 | 1.5e-03 | 9.0e-03 | 2.4e-02 | 3.1e-02 | 2.2e-02 | 9.6e-03 |

**Data availability statement.** Data availability is not applicable to this article as no new data were created or analyzed in this study. The main results are theoretical. The algorithms for numerical examples are described/presented in the manuscript. The code for numerical examples that support the findings of this study are available from the corresponding author, SL, upon reasonable request.

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**Competing interests.** The authors declare none.

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## Appendix A. Proofs

*Proof of Theorem* 2. Equation (2) can be written as

$$\begin{aligned} kp_k(a,b) &= (h_1(a,b)k + h_2(a,b))p_{k-1}(a+b,b) \\ &= \left[h_1(a,b)(k-\alpha) + (\alpha h_1(a,b) + h_2(a,b))\right] p_{k-1}(a+b,b), \qquad k=1,2,\ldots. \end{aligned} \tag{A.1}$$

For $k \geq \alpha + 1$, multiplying both sides by $(k-1)^{(\alpha-1)}$ yields

$$k^{(\alpha)} p_k(a,b) = \left(h_1(a,b)(k-1)^{(\alpha)} + (\alpha h_1(a,b) + h_2(a,b))(k-1)^{(\alpha-1)}\right) p_{k-1}(a+b,b),$$

Dividing both sides by $\mu_N^{(\alpha)}(a,b)$, we have, for $k = \alpha+1, \alpha+2, \ldots,$

$$\begin{aligned} g_k^\alpha(a,b) &= h_1(a,b)\frac{\mu_N^{(\alpha)}(a+b,b)}{\mu_N^{(\alpha)}(a,b)} g_{k-1}^\alpha(a+b,b) \\ &\quad + (\alpha h_1(a,b) + h_2(a,b))\frac{\mu_N^{(\alpha)}(a+b,b)}{\mu_N^{(\alpha)}(a,b)} \frac{(k-1)^{(\alpha)} p_{k-1}(a+b,b)}{\mu_N^{(\alpha)}(a+b,b)}. \end{aligned}$$

Multiplying both side by $(k-\alpha)$ yields

$$\begin{aligned} (k-\alpha)g_k^\alpha(a,b) &= h_1(a,b)\frac{\mu_N^{(\alpha)}(a+b,b)}{\mu_N^{(\alpha)}(a,b)}(k-\alpha) g_{k-1}^\alpha(a+b,b) \\ &\quad + (\alpha h_1(a,b) + h_2(a,b))\frac{\mu_N^{(\alpha)}(a+b,b)}{\mu_N^{(\alpha)}(a,b)} g_{k-1}^\alpha(a+b,b). \end{aligned}$$

Now, since $M_\alpha = \tilde{N}_\alpha - \alpha$, we have its pmf denoted as $f_{k-\alpha}^\alpha(a,b) := \mathbb{P}(M_\alpha = k-\alpha) = \mathbb{P}(\tilde{N}_\alpha = k) = g_k^\alpha(a,b)$ for $k \geq \alpha$, we have

$$\begin{aligned} kf_k^\alpha(a,b) &= h_1(a,b)\frac{\mu_N^{(\alpha)}(a+b,b)}{\mu_N^{(\alpha)}(a,b)} kf_{k-1}^\alpha(a+b,b) \\ &\quad + (\alpha h_1(a,b) + h_2(a,b))\frac{\mu_N^{(\alpha)}(a+b,b)}{\mu_N^{(\alpha)}(a,b)} f_{k-1}^\alpha(a+b,b), \qquad k=1,2,3,\ldots, \end{aligned}$$

which is equivalent to

$$f_k^\alpha(a,b) = \left(h_1^*(a,b) + \frac{h_2^*(a,b)}{k}\right) f_{k-1}^\alpha(a+b,b), \qquad k=1,2,3,\ldots,$$

where

$$h_1^*(a,b) = h_1(a,b)\frac{\mu_N^{(\alpha)}(a+b,b)}{\mu_N^{(\alpha)}(a,b)},$$

and

$$h_2^*(a,b) = (\alpha h_1(a,b) + h_2(a,b))\frac{\mu_N^{(\alpha)}(a+b,b)}{\mu_N^{(\alpha)}(a,b)}.$$

Hence, the distribution of $M_\alpha$ still satisfies the general recursive formula (2) as desired. □

*Proof of Corollary* 4. Since $X$ is a compound distribution, by Theorem 3 (i), we have

$$\tilde{X} \stackrel{d}{=} \tilde{C} + Y,$$

where $Y=\sum_{i=1}^{\tilde{N}-1} C_i$. Furthermore, thanks to the property of GP given in Corollary 3, we have $\tilde{N} \stackrel{d}{=} \tilde{B}+N$, we have

$$Y=\sum_{i=1}^{\tilde{N}-1} C_i \stackrel{d}{=} \sum_{i=1}^{N+\tilde{B}-1} C_i \stackrel{d}{=} \sum_{i=1}^{\tilde{B}-1} C_i+\sum_{i=1}^{N} C_i \stackrel{d}{=} Z+X.$$

where $Z=\sum_{i=1}^{\tilde{B}-1} C_i$ and $\tilde{B}$ is the size-biased transform of $B \sim$ Borel $(\theta)$. This completes the proof. □

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