ON THE RING OF QUOTIENTS OF A BOOLEAN RING

B. Brainerd and J. Lambek[1)]

(received August 12, 1958)

Two important mathematical constructions are: the construction of the rationals from the integers and the construction of the reals from the rationals. The first process can be carried out for any ring, producing its maximal ring of quotients [4, 5]. The second process can be carried out for any partially ordered set producing its Dedekind-MacNeille completion [2, p. 58]. We will show that for Boolean rings, which are both rings and partially ordered sets, the two constructions coincide.

In what follows, R denotes a Boolean ring, that is, a ring in which every element is an idempotent. (Such a ring is necessarily commutative.) Furthermore M_R denotes an R-module, and R_R denotes the ring R regarded as an R-module. By a partial endomorphism of M_R we mean a homomorphism φ of a submodule $D_R = \mathrm{dom}\varphi$ of M_R into M_R, that is, a mapping satisfying the conditions:

$$\varphi(d + d') = \varphi d + \varphi d', \quad \varphi(dr) = (\varphi d)r$$

for $d, d' \in \mathrm{dom}\varphi$ and $r \in R$. We call φ irreducible if it cannot be extended to a larger domain.

PROPOSITION 1. If φ is a partial endomorphism of R_R, then the image $\mathrm{im}\varphi$ of φ is contained in $\mathrm{dom}\varphi$ and $\varphi^2 = \varphi$.

Proof. Let $d \in \mathrm{dom}\varphi$. Then $\varphi d = \varphi(d^2) = (\varphi d)\, d \in d\,R$; hence $\mathrm{im}\varphi \subseteq \mathrm{dom}\varphi$. Moreover $\varphi^2 d = \varphi(\varphi d) = \varphi(\varphi(d^2)) = \varphi((\varphi d)d) = \varphi(d(\varphi d)) = (\varphi d)\,(\varphi d) = \varphi d$.

An ideal D of R is called dense [2; p. 160] if for all $r \in R$, $rD = 0$ implies $r = 0$. By [4; 6.4], the fractional endomorphisms

1) This paper was written while both authors were Fellows of the Summer Research Institute of the Canadian Mathematical Congress.

Can. Math. Bull., vol. 2, No. 1, Jan. 1959.

[4;3] are precisely the partial endomorphisms of R_R with dense domains.

COROLLARY 2. The maximal ring of quotients of a Boolean ring is also a Boolean ring.

Proof. We first observe that for any $r \in R$, $rR = 0$ implies $r = 0$. This follows because $r = r^2$. Now by [4; 6.1] the maximal ring Q of quotients of R may be constructed as the ring of all irreducible fractional endomorphisms of R. Hence by Proposition 1, all elements of Q are idempotents and so Q is also a Boolean ring.

A Boolean ring R is partially ordered by the relation $r \leq r'$ if and only if $rr' = r$ $(r, r' \in R)$.

If S is a Boolean ring which contains R as a subring, S may be called a <u>completion</u> of R provided

(1) S is complete, that is, every subset of S has a supremum relative to " $\leq$ ";
(2) for every $s \in S$, $s = \sup \{r \in R \mid r \leq s\}$.

For example, the Dedekind-MacNeille completion [2; p. 58] is such a completion.

PROPOSITION 3. If φ is a partial endomorphism of R_R, then any completion S of R contains an element s such that $\varphi d = sd$ for all $d \in \mathrm{dom}\varphi$.

Proof. Let $s = \sup \{d' \in \mathrm{dom}\varphi \mid \varphi d' = d'\}$, and take any $d \in \mathrm{dom}\varphi$. We have $\varphi(\varphi d) = \varphi^2 d = \varphi d$ by Proposition 1, hence $\varphi d \leq s$, and therefore $\varphi d = \varphi(d^2) = (\varphi d)d \leq sd$.

On the other hand, take any $d' \in \mathrm{dom}\varphi$ such that $\varphi d' = d'$. Then $d'd = (\varphi d')d = \varphi(d'd) = (\varphi d)d' \leq \varphi d$. Thus $d'd \leq \varphi d$ for every d' such that $\varphi d' = d'$, and so $sd \leq \varphi d$. The result now follows.

COROLLARY 4. If R is a complete Boolean ring, then R_R is injective.

Proof. In view of [3; p. 8, 3.2], R_R is injective if and only if for each partial endomorphism φ of R_R there exists an element r of R such that $\varphi d = rd$ for all $d \in \mathrm{dom}\varphi$. The result follows from Proposition 3.

Actually, the theorem quoted in the proof just given presumes that R contains a unity element. This condition is satisfied here, since we can show that the supremum of all elements of R is a unity element of R.

THEOREM 5. The Boolean ring S is a completion of the Boolean ring R if and only if it is a maximal ring of quotients of R.

Proof. Let S be a completion of R. We may construct a quotient ring Q of R from the irreducible fractional endomorphisms φ of R_R. By Proposition 3, each such φ can be realized by multiplication with an element s of S, hence we have a homorphism $\varphi \rightarrow s$ of Q into S. This mapping is a faithful embedding; for its kernel consists of all φ with image 0, and being irreducible any such φ must be the zero mapping of R.

Without loss in generality, we may therefore regard Q as a subring of S containing R. An element $s \in Q$ induces an irreducible fractional endomorphism of R_R. Conversely, suppose $s \in S$ induces a fractional endomorphism φ of R_R. This has an irreducible extension φ' which is still fractional, hence there is an $s' \in Q$ such that $\varphi' d = s'd$ for all $d \in \operatorname{dom} \varphi'$. Therefore $(s + s')d = sd + s'd = 0$ for all $d \in \operatorname{dom}\varphi$. Now $s + s' = \sup\{r \in R \mid r \leqslant s + s'\}$, and so $r \operatorname{dom}\varphi = 0$ for all $r \leqslant s + s'$. Since $\operatorname{dom}\varphi$ is dense in R, $r = 0$ for all $r \leqslant s + s'$, and therefore $s + s' = 0$, that is $s = s' \in Q$.

We have thus shown that $s \in Q$ if and only if s induces a fractional endomorphism φ_s in R_R, that is, if and only if $\operatorname{dom} \varphi_s = \{r \in R \mid sr \in R\}$ is dense in R. It is easily seen that this last condition can be written as follows[2]:

$$(\ast) \ldots \qquad \forall_{r \in R} \quad r \neq 0 \Rightarrow \exists_{d \in R} \quad rd \neq 0 \text{ and } sd \in R.$$

Given any $r \in R$, we distinguish two cases.

Case 1. $r \leqslant s$. Choose $d = r$, then $rd = r^2 = r \neq 0$ and $sd = sr = r \in R$.

2) This condition is the same as that used by Utumi [5; 1.1] to define the ring of quotients.

Case 2. $r\bar{s} \neq 0$.[3] Now $\bar{s} = \sup \{ r' \in R \mid r' \leqslant \bar{s} \}$, hence $0 \neq r\bar{s} = \sup \{ rr' \mid r' \in R \text{ and } r' \leqslant \bar{s} \}$, therefore there exists $d = r' \in R$ such that $d \leqslant \bar{s}$ and $rd \neq 0$. Since $d \leqslant \bar{s}$ is equivalent to $ds = 0 \in R$, the condition (A) is satisfied.

Condition (A) shows that dom φ_s is dense for any $s \in S$, hence $S = Q$, and so any completion is also a maximal ring of quotients of R. Now, the maximal ring of quotients of R is known to be unique up to isomorphism over R [4; 5.3] , hence any maximal ring of quotients of R is isomorphic over R to a given completion of R, say the Dedekind-MacNeille completion.

This last remark also shows the validity of the following, which is probably well known [eg. 1; p. 123] .

COROLLARY 6. The completion of a Boolean ring R is unique up to an isomorphism over R.

In view of Theorem 5, the construction of the ring of quotients given in [4] can be used in place of the Dedekind-MacNeille cut construction. In particular, the following may be of interest.

PROPOSITION 7. If a Boolean ring R contains a smallest dense ideal F, then its completion is the ring of endomorphisms of F_R.

Proof. This follows from Theorem 5 and [4; 8.3] .

EXAMPLE. Let R be an atomic Boolean ring, F the ideal consisting of all finite sums of atoms. This is easily shown to be the smallest dense ideal in R. In this case the completion is clearly isomorphic to the Boolean ring of all subsets of the set of atoms of R. This could also have been deduced from Proposition 7.

REFERENCES

1. B. Banaschewski, Hüllensysteme and Erweiterung von Quasiordnungen, Zeitschr. f. math. Logik and Grundlagen d. Math. 2 (1956), 117-130.

---

3) Here $\bar{s}$ denotes the complement $1 + s$ of s, where 1 is the unity element of S.

https://doi.org/10.4153/CMB-1959-006-x Published online by Cambridge University Press

2. G. Birkhoff, Lattice theory, (New York, 1948).

3. H. Cartan and S. Eilenberg, Homological algebra, (Princeton, 1956).

4. G.D. Findlay and J. Lambek, A generalized ring of quotients, Can. Math. Bull. 1 (1958), 77-85, 155-167.

5. Y. Utumi, On quotient rings, Osaka Math. J. 8 (1956), 1-18.

The University of Western Ontario

McGill University

https://doi.org/10.4153/CMB-1959-006-x Published online by Cambridge University Press