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A PARAMETRIC GAUSS-GREEN THEOREM IN SEVERAL VARIABLES

BY

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ABSTRACT. We present a short, computational proof of the parametric Gauss-Green theorem for a broad class of closed chains. The proof involves only measure theory and the basic theory of differential forms: in particular, no constructions from topology are used. For completeness, the standard properties of winding numbers are also established by methods from analysis.

1. **Introduction.** The terms appearing in Theorem 1 and the rest of this introduction are defined in section 2.

THEOREM 1. *Suppose $N \in \{2,3,\cdots\}, j \in \{1,2,\cdots N\}, \Omega$ is an open subset of $\mathbf{R}^N, f : \Omega \longrightarrow \mathbf{R}$ is Lipschitzian, and (γ, T) is a Lipschitzian $N-1$ chain in Ω which is both closed and homologous to zero modulo Ω. Then*

$$\int_{(\gamma,T)} f \cdot (DX)_{\hat{j}} = (-1)^{j-1} \int_{\Omega} \operatorname{ind}(\gamma, z) \cdot D_j f(z) d\mathcal{L}^N(z).$$

In this paper we give a short, computational proof of Theorem 1, which turns on an interchange in order of integration. Other proofs are considerably longer and make heavy use of constructions from combinatorial topology.

The homology version of the Cauchy Integral Theorem is an immediate corollary of Theorem 1 in the case $N = 2$; in this case our proof can be considerably simplified by use of the operator $\bar{\partial}$, as in ([OS 1]).

The primary achievement of modern research on integral theorems is the Gauss-Green theorem of H. Federer ([FED 1], p. 478), which is in non-parametric form: that is, the theorem does not involve winding numbers and, for an admissible set $A \subset \mathbf{R}^N$, equates integrals over A and Bndry A with respect to $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$. Federer's theorem is the *optimal* statement of that form with regard to the scope of admissible sets, and it easily implies Theorem 1 in many cases: for instance, the case that $\Omega = \mathbf{R}^N$ and (γ, T) properly parametrizes a finite family of closed surfaces. Of course, Federer's theorem is also useful in many situations beyond the scope of Theorem 1.

Important research, specifically on the *parametric* Gauss-Green theorem, is due to J. H. Michael ([M 2]), who studies the question of the *joint* regularity of chains

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and integrands required for the fundamental equation. His theorem is stronger than Theorem 1 when (γ, T) is equivalent to a chain whose parametric domain is a compact submanifold of $\mathbf{R}^N$.

2. **Preliminary material.**

2.1. We shall follow the terminology and notation of [FED 1] for measure theory, Grassmann algebra, and differential forms.

Throughout, N denotes a fixed element of $\{2,3,4\cdots\}$, $\mathcal{L}^N$ denotes Lebesgue N dimensional measure on $\mathbf{R}^N$, $\mathcal{H}^{N-1}$ denotes Hausdorff $N-1$ dimensional measure on $\mathbf{R}^N$, $S^{N-1} \equiv \{x \in \mathbf{R}^N : |x| = 1\}$, and $\alpha_{N-1} \equiv \mathcal{H}^{N-1}(S^{N-1})$. The terms *Lipschitzian map, Lipschitz constant* (*Lip(f)*), and *function of class k* have standard meanings, and we denote the *support* of a continuous function f as *spt f* ([FED 1], pages 63, 64, 220, 106).

If $(V, |\cdot|)$ is normed vector space, and $0 \neq x \in V$, we define sign $x \equiv x|x|^{-1}$ and sign $0 \equiv 0$. The Euclidean norm on $\mathbf{R}^N$ and the induced norms on $\wedge_k\mathbf{R}^N$ and $\wedge^k\mathbf{R}^N$ are denoted $|\cdot|$. If L is a linear transformation on V we may write $\langle x, L\rangle$ in place of $L(x)$, and we regard $\beta \in \wedge^k\mathbf{R}^N$ as both an alternating multilinear function on k-tuples of vectors and as a linear function on $\wedge_k\mathbf{R}^N$ without change of notation.

The terms *differential form, closed form, exact form*, and the notations $f^{\#}\beta$ (the *pull-back* of a form β by a differentiable map f), and $d\beta$ (the exterior derivative of β) are explained in section 4.1.6 of [FED 1]. Every closed form on $\mathbf{R}^N$ is also exact on $\mathbf{R}^N$ ([FED 1], 4.1.10).

The standard basis vectors and coordinate functions on $\mathbf{R}^N$ are denoted $e_1,\ e_2, \cdots e_N$ and $X_1,\ X_2, \cdots X_N$, and we denote the basic $N-1$ forms on $\mathbf{R}^N$ as follows:

$$(DX)_{\hat{n}} \equiv DX_1 \wedge \cdots \wedge D\hat{X}_n \wedge \cdots \wedge DX_N$$

for $n = 1, 2, \cdots N$, wherein the sign $\hat{}$ denotes omission. The linear map $* : \wedge_1\mathbf{R}^N \longrightarrow \wedge_{N-1}\mathbf{R}^N$ is defined in section 1.7.8 of [FED 1]. If $v \in \mathbf{R}^N$ then $*v$ is a simple $N-1$ vector associated with the subspace of $\mathbf{R}^N$ orthogonal to v: also, $v \wedge *v = |v|^2 e_1 \wedge \cdots \wedge e_N$, and $|*v| = |v|$.

2.2. When $\Omega \subset \mathbf{R}^N$, T is a closed and bounded subset of $\mathbf{R}^{N-1}$, $\gamma : \mathbf{R}^{N-1} \longrightarrow \mathbf{R}^N$ is Lipschitzian, and $\gamma[T] \subset \Omega$, we say (γ, T) is a *Lipschitzian* $N-1$ *chain in* Ω. If (γ, T) is a Lipschitzian $N-1$ chain in $\mathbf{R}^N$, then $\gamma[T]$ is a compact subset of $\mathbf{R}^N$ and $\mathcal{L}^N(\gamma[T]) = 0$.

2.3. We cite two results from [*FED*1]: Suppose $F : \mathbf{R}^P \longrightarrow \mathbf{R}^Q$ is Lipschitzian and $j \in \{1, \cdots N\}$. Then $DF(x)$ exists and $|D_jF(x)| \leqq Lip(F)$ for $\mathcal{L}^P$ almost all $x \in \mathbf{R}^P$ (p. 216). Also, D_jF is $\mathcal{L}^P$ measurable on $\mathbf{R}^P$ (p. 73, (6)).

2.4. If (γ, T) is a Lipschitzian $N-1$ chain in $\mathbf{R}^N$, and β is an $N-1$ form of class 0 defined in a neighborhood of $\gamma[T]$, we denote

$$\int_{(\gamma,T)} \beta \equiv \int_T \langle D_1\gamma(t) \wedge \cdots \wedge D_{N-1}\gamma(t), \beta(\gamma(t))\rangle d\mathcal{L}^{N-1}(t):$$

summability follows since, by 2.3, the expressions

$$d_n(\gamma, t) \equiv \langle D_1\gamma(t) \wedge \cdots \wedge D_{N-1}\gamma(t),\ (DX)_{\hat{n}}(\gamma(t))\rangle$$

are all $\mathcal{L}^{N-1}$ measurable and essentially bounded.

If (γ, T) is a Lipschitzian $N-1$ chain in Ω_1, β is an $N-1$ form on Ω_2 of class 0, and $f : \Omega_1 \longrightarrow \Omega_2$ is of class ∞, we have

$$\int_{(f\circ\gamma,T)} \beta = \int_{(\gamma,T)} f^{\#}\beta.$$

2.5. If (γ, T) is a Lipschitzian $N-1$ chain in Ω and $\int_{(\gamma,T)} d\beta = 0$ whenever β is an $N-2$ form of class ∞ in a neighborhood of $\gamma[T]$, we say (γ, T) is a *closed* $N-1$ *chain in* Ω. By using regularizations of characteristic functions, we see that (γ, T) is a closed $N-1$ chain in Ω if and only if $\int_{(\gamma,T)} \alpha = 0$ whenever α is a closed $N-1$ form of class ∞ in $\mathbf{R}^N$. If (γ, T) is a closed $N-1$ chain in Ω and $f : \Omega \longrightarrow \mathbf{R}^N$ is of class ∞, then $(f\circ\gamma, T)$ is a closed $N-1$ chain, as follows from the pull-back formula in 2.4

2.6. For each $z \in \mathbf{R}^N$ and each $w \in \mathbf{R}^N - \{z\}$ define

$$\omega_z(w) \equiv \alpha_{N-1}^{-1} \sum_{k=1}^{N} (-1)^{k-1} \frac{w_k - z_k}{|w-z|^N} (DX)_{\hat{k}}(w)$$

(where $(DX)_{\hat{k}}$ is defined in 2.1). Then, if (γ, T) is a Lipschitzian $N-1$ chain in $\mathbf{R}^N$, we define

$$\operatorname{ind}(\gamma, z) \equiv \int_{(\gamma,T)} \omega_z \qquad \text{if } z \in \mathbf{R}^N - \gamma[T]$$

and $\operatorname{ind}(\gamma, z) \equiv 0$ if $z \in \gamma[T]$, and we call $\operatorname{ind}(\gamma, z)$ the *winding number of* (γ, T) *with respect to* z.

Clearly, $\operatorname{ind}(\gamma, \cdot)$ is a continuous function in $\mathbf{R}^N - \gamma[T]$. Nevertheless, $\operatorname{ind}(\gamma, \cdot)$ *may be unbounded.*

2.7. If (γ, T) is a closed $N-1$ chain in Ω and $\operatorname{ind}(\gamma, T) = 0$ for all $z \in \mathbf{R}^N - \Omega$, we say (γ, T) *is homologous to zero modulo* Ω.

3. **Lemmas.**

LEMMA 3.1. *Suppose h is function of class ∞ on $\mathbf{R}^N$*, spt *f is compact, and $t \in \mathbf{R}^N$. Then*

$$\int_{\mathbf{R}^N} \sum_{k=1}^{N} D_k h(t-u) u_k |u|^{-N} d\mathcal{L}^N(u) = \alpha_{N-1} h(t).$$

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PROOF. Define $g(u) \equiv h(t-u)$ for all $u \in \mathbf{R}^N$. Then

$$\begin{aligned}
&\alpha_{N-1} h(t)\\
&= \int_{S^{N-1}} g(0) d\mathcal{H}^{N-1}(s)\\
&= -\int_{S^{N-1}} \int_{(0,\infty)} \sum_{k=1}^{N} D_k g(rs) s_k d\mathcal{L}^1(r) d\mathcal{H}^{N-1}(s)\\
&= -\int_{(0,\infty)} \int_{S^{N-1}} \sum_{k=1}^{N} D_k g(rs) r^{-N+1} s_k d\mathcal{H}^{N-1}(s) r^{N-1} d\mathcal{L}^1(r)\\
&= -\int_{\mathbf{R}^N} \sum_{k=1}^{N} D_k g(u) u_k |u|^{-N} d\mathcal{L}^N(u)\\
&= \int_{\mathbf{R}^N} \sum_{k=1}^{N} D_k h(t-u) u_k |u|^{-N} d\mathcal{L}^N(u)
\end{aligned}$$

LEMMA 3.2. *Suppose f is a function of class ∞ on $\mathbf{R}^N$,* spt f *is compact, $j \in \{1, 2, \cdots N\}$, and (γ, T) is a closed $N-1$ chain in $\mathbf{R}^N$. Then*

$$\begin{aligned}
&\int_{(\gamma,T)} \sum_{k=1}^{N} \left[(-1)^{k-1} \int_{\mathbf{R}^N} D_j f(z) \frac{w_k - z_k}{|w-z|^N} d\mathcal{L}^N(z) \right] \cdot (DX)_{\hat{k}}(w)\\
&= \int_{(\gamma,T)} (-1)^{j-1} \alpha_{N-1} f(w) \cdot (DX)_{\hat{j}}(w).
\end{aligned}$$

PROOF. For each $w \in \mathbf{R}^N$ and each $k \in \{1, 2, \cdots N\}$, define

$$\begin{aligned}
\rho_k(w) &\equiv \int_{\mathbf{R}}^{N} D_j f(z) \frac{w_k - z_k}{|w-z|^N} d\mathcal{L}^N(z); \text{ then consider the } N-1 \text{ forms}\\
\lambda &\equiv \alpha_{N-1} (-1)^{j-1} f \cdot (DX)_{\hat{j}},\\
\beta &\equiv \sum_{k=1}^{N} (-1)^{k-1} \rho_k \cdot (DX)_{\hat{k}}.
\end{aligned}$$

Each ρ_k is of class ∞ since it is the convolution of a locally integrable function and a compactly supported function of class ∞. By direct computation, we get (for all $w \in \mathbf{R}^N$)

$$\begin{aligned}
d\lambda(w) - d\beta(w) &= \left[\alpha_{N-1} D_j f(w) - \sum_{k=1}^{N} D_k \rho_k(w) \right] DX_1(w) \wedge \cdots \wedge DX_N(w),\\
\rho_k(w) &\equiv \int_{\mathbf{R}^N} D_j f(w-u) u_k |u|^{-N} d\mathcal{L}^N(u), \text{ and}\\
D_k \rho_k(w) &= \int_{\mathbf{R}^N} D_{kj} f(w-u) u_k |u|^{-N} d\mathcal{L}^N(u).
\end{aligned}$$

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Application of 3.1 to $D_j f$ shows that the bracketed expression directly above is zero, and we conclude $d(\lambda-\beta)(w)=0$ for all $w \in \mathbf{R}^N$. Thus $\lambda-\beta$ is an *exact* $N-1$ form on $\mathbf{R}^N$, and $\int_{(\gamma,T)} \lambda-\beta=0$ since (γ,T) is a closed $N-1$ chain. This completes the proof. □

LEMMA 3.3. *If f is a function of class ∞ on $\mathbf{R}^N$, $\operatorname{spt} f$ is compact, (γ,T) is a closed $N-1$ chain in $\mathbf{R}^N$, and $j \in \{1,2,\cdots N\}$, then*

$$\text{(1)} \qquad \int_{\mathbf{R}^N} |D_j f(z)|\,|\operatorname{ind}(\gamma,z)| d\mathcal{L}^N(z)<\infty.$$

$$\text{(2)} \qquad \int_{(\gamma,T)} f\cdot(DX)_{\hat{j}} = (-1)^{j-1}\int_{\mathbf{R}^N} \operatorname{ind}(\gamma,z)\cdot D_j f(z) d\mathcal{L}^N(z).$$

PROOF. Recalling the notation $d_k(\gamma,\cdot)$ from 2.4, set

$$F(z,t)\equiv D_j f(z)\cdot\sum_{k=1}^N(-1)^{k-1}\frac{\gamma_k(t)-z_k}{|\gamma(t)-z|^N}\,d_k(\gamma,t)$$

for $(z,t)\in\mathbf{R}^N\times\mathbf{R}^{N-1}$ where the sum is defined. Since $\mathcal{L}^N\times\mathcal{L}^{N-1}(\gamma[\mathbf{R}^{N-1}]\times\mathbf{R}^{N-1})=0$, the functions

$$(z,t)\longrightarrow\frac{\gamma_k(t)-z_k}{|\gamma(t)-z|^N}$$

are $\mathcal{L}^N\times\mathcal{L}^{N-1}$ measurable: hence F is $\mathcal{L}^N\times\mathcal{L}^{N-1}$ measurable. Moreover, if E is a *bounded* open subset of $\mathbf{R}^N$, Fubini's theorem and well known estimates give (since T is also bounded)

$$\int_{E\times T}\frac{|\gamma_k(t)-z_k|}{|\gamma(t)-z|^N}\,d\mathcal{L}^N\times\mathcal{L}^{N-1}(z,t)\leqq\int_T\int_E\frac{d\mathcal{L}^n(z)}{|\gamma(t)-z|^{N-1}}\,d\mathcal{L}^{N-1}(t)<\infty.$$

Since $\operatorname{spt} f$ is compact and each function $d_k(\gamma,\cdot)$ is uniformly bounded, we conclude $\int_{\mathbf{R}^N\times T}|F(z,t)|d\mathcal{L}^N\times\mathcal{L}^{N-1}(z,t)<\infty$, which allows us to evaluate integrals of F by use of Fubini's Theorem.

Our first observation is

$$\begin{aligned}&\int_{\mathbf{R}^N}|D_j f(z)|\,|\alpha_{N-1}\operatorname{ind}(\gamma,z)|d\mathcal{L}^N(z)\\ &=\int_{\mathbf{R}^N}|D_j f(z)|\left|\int_T\sum_{k=1}^N(-1)^{k-1}\frac{\gamma_k(t)-z_k}{|\gamma(t)-z|^N}\,d_k(\gamma,t)d\mathcal{L}^{N-1}(t)\right|d\mathcal{L}^N(z)\\ &\leqq\int_{\mathbf{R}^N\times T}|F(z,t)|d\mathcal{L}^N\times\mathcal{L}^{N-1}(z,t)<\infty,\end{aligned}$$

which completes the proof of (1).

Also, by Fubini, and 3.2 applied at the last step, we conclude

$$\int_{\mathbf{R}^N} D_j f(z)\alpha_{N-1} \text{ ind } (\gamma, z) d\mathcal{L}^N(z)$$

$$= \int_{\mathbf{R}^N} D_j f(z) \int_T \sum_{k=1}^{N} (-1)^{k-1} \frac{\gamma_k(t) - z_k}{|\gamma(t) - z|^N} d_k(\gamma, t) d\mathcal{L}^{N-1}(t) d\mathcal{L}^N(z)$$

$$= \int_T \sum_{k=1}^{N} (-1)^{k-1} \int_{\mathbf{R}^N} D_j f(z) \frac{\gamma_k(t) - z_k}{|\gamma_k(t) - z|^N} d\mathcal{L}^N(z) d_k(\gamma, t) d\mathcal{L}^{N-1}(t)$$

$$= \int_{(\gamma,T)} \sum_{k=1}^{N} (-1)^{k-1} \int_{\mathbf{R}^N} D_j f(z) \frac{w_k - z_k}{|w - z|^N} d\mathcal{L}^N(z) \cdot (DX)_{\hat{k}}(w)$$

$$= \int_{(\gamma,T)} (-1)^{j-1} \alpha_{N-1} f(w) \cdot (DX)_{\hat{j}}(w),$$

which completes the proof of (2). □

LEMMA 3.4. *Suppose that $\langle \gamma, T \rangle$ is a closed $N-1$ chain in $\mathbf{R}^N$. Then:*
(1) $\text{ind}(\gamma, \cdot)$ *is constant in each component of* $\mathbf{R}^N - \gamma[T]$, *and* $\text{ind}(\gamma, z) = 0$ *for all* z *in the unbounded component of* $\mathbf{R}^N - \gamma[T]$.
(2) $\int_{\mathbf{R}^N} |\text{ind}(\gamma, z)| d\mathcal{L}^N(z) < \infty$.

PROOF. If B is a connected component of $\mathbf{R}^N - \gamma[T]$ and f is a function of class ∞ on $\mathbf{R}^N$ with $\text{spt} f \subset B$, we have (by 3.3(2)), $\int_B \text{ind}(\gamma, z) D_j f(z) d\mathcal{L}^N(z) = 0$, for each $j = 1, 2, \cdots N$. So, by well known arguments, there is a real number c such that $\text{ind}(\gamma, z) = c$ for $\mathcal{L}^N$ almost all $z \in B$. Since $\text{ind}(\gamma, \cdot)$ is continuous in B, it follows that $\text{ind}(\gamma, z) = c$ for all $z \in B$. Since $\lim_{|z| \to \infty} \text{ind}(\gamma, z) = 0$, the proof of (1) is complete.

By (1), $\text{ind}(\gamma, z) = 0$ for z outside a compact set E. Choose a compactly supported function f of class ∞ such that $D_1 f(z) = 1$ at all points of E. By 3.3.(1),

$$\int_{\mathbf{R}^N} |\text{ind}(\gamma, z)| d\mathcal{L}^N(z) \leqq \int_{\mathbf{R}^N} |D_1 f(z)| \, |n(\gamma, z)| d\mathcal{L}^N(z) < \infty$$

which completes the proof of (2). □

4. **Proof of Theorem 1.**

We refer now to the terms of Theorem 1.

First, suppose $\Omega = \mathbf{R}^N$, f is Lipschitzian on $\mathbf{R}^N$, $\text{spt} f$ is compact, and let U denote a bounded open set containing *sptf*. By basic smoothing arguments ([FED 1], p. 347) and 2.3, there is a sequence $(f_n)_{n=1}^{\infty}$ such that:
(1) Each f_n is of class ∞ on $\mathbf{R}^N$ and $spt f_n \subset U$.
(2) $f_n \longrightarrow f$ uniformly on $\mathbf{R}^N$ as $n \longrightarrow \infty$.
(3) $\sup\{|D_j f_n(z)| : z \in \mathbf{R}^N, j = 1, 2, \cdot \cdot N, n = 1, 2, 3, \cdot\} < \infty$.
(4) $Df_n(z) \longrightarrow Df(z)$ for $\mathcal{L}^N$ almost all $z \in \mathbf{R}^N$ as $n \longrightarrow \infty$.

(At this point difficulties arise if we attempt to admit the more general integrands of Michael ([M 2]), since we make essential use of the hypothesis that f is Lipschitzian in (3) and (4).) By (1) through (4) above, *and special reference to 3.4(2)*, we may apply the Lebesgue dominated convergence theorem: the integrals of f_n and $D_j f_n$ converge to the integrals of f and $D_j f$. By 3.3(2), Theorem 1 is true for $\Omega = \mathbf{R}^N$ and each function f_n. The proof of Theorem 1, for compactly supported Lipschitzian functions on $\mathbf{R}^N$, is complete.

Now assume that Ω is an open subset of $\mathbf{R}^N$ and f is Lipschitzian on Ω. Since $\text{ind}(\gamma, z) = 0$ for each $z \in \mathbf{R}^N - \Omega$ (this is the first point at which this hypothesis appears), 3.4 implies $\text{Clos}\{z \in \mathbf{R}^N : \text{ind}(\gamma, z) \neq 0\}$ is a *compact* subset of Ω: hence, there is a Lipschitzian function g, of class ∞ on $\mathbf{R}^N$, such that $\text{spt}\, g$ is compact, and $g(z) = f(z)$ whenever $\text{ind}(\gamma, z) \neq 0$ or $z \in \gamma[T]$. As Theorem 1 is true if f is replaced by g, and as both integrals in Theorem 1 are unchanged if f is replaced by g, the proof of Theorem 1 is complete.

5. **Winding numbers and closed chains.**

5.1. Here we use Federer's *area theorem* to prove that winding numbers of closed chains are integers, and that they may be computed by examining the orientations of certain frames. More about winding numbers may be found in [FED 2] (p. 377).

When (γ, T) is a closed $N-1$ chain in $\mathbf{R}^N$ and $D\gamma(t)$ exists, set

$$J_{N-1}\gamma \equiv |D_1\gamma(t) \wedge \cdots \wedge D_{N-1}\gamma(t)|$$
$$\text{Or}(\gamma, t) \equiv \text{ sign det}\left[\gamma(t), D_1\gamma(t), \cdots D_{N-1}\gamma(t)\right] :$$

then define

$$\text{Cross}(\gamma, s) \equiv \sum_{\{t \in T:\, \gamma(t) = s\}} \text{Or}(\gamma, t)$$

at $s \in \mathbf{R}^N$ where the indicated terms exist and are summable. Also for use here, define $P_z(x) \equiv \text{sign } (x - z)$ for $z \in \mathbf{R}^N$ and all $x \in \mathbf{R}^N - \{z\}$ (see 2.1)

THEOREM 2. *Suppose* (γ, T) *is a closed* $N-1$ *chain in* $\mathbf{R}^N$ *and* $z \notin \gamma[T]$. *Then there is an integer* c *such that* $\text{ind}(\gamma, z) = c = \text{Cross}(P_z \circ \gamma, s)$ *for* $\mathcal{H}^{N-1}$ *almost all* $s \in S^{N-1}$.

PROOF. By computation, we see $\langle h, DP_z(x)\rangle = 0$ if h is a real multiple of $x - z$, and $\langle h, DP_z(x)\rangle = |x - z|^{-1}h$ if $h \cdot (x - z) = 0$. It follows that $P_z^{\#}\omega_0 = \omega_z$, and hence that $\text{ind}(\gamma, z) = \text{ind}(P_z \circ \gamma, 0)$. Therefore, since $(P_z \circ \gamma, T)$ is closed when (γ, T) is closed, it suffices to establish Theorem 2 under the assumption (adopted from now on) that $\gamma[T] \subset S^{N-1}$ and $z = 0$.

Since $\gamma[T] \subset S^{N-1}$, we obtain, from 2.1 and 2.3, $D_1\gamma(t) \wedge \cdots \wedge D_{N-1}\gamma(t) = *\gamma(t) \cdot \text{Or}(\gamma, t) \cdot J_{N-1}\gamma(t) \in \wedge_{N-1}\mathbf{R}^N$ for $\mathcal{L}^{N-1}$ almost all $t \in T$: also $(*\gamma) \cdot \text{Or}(\gamma, \cdot)$ is $\mathcal{L}^{N-1}$ integrable on T. Thus, assuming β is an $N-1$ form of class ∞ in a neighborhood of S^{N-1}, we obtain by use of Federer's area theorem ([FED 1], Theorem 3.2.3),

$$\begin{aligned}\int_{(\gamma,T)}\beta &= \int_T \langle D_1\gamma(t)\wedge\cdots\wedge D_{N-1}\gamma(t), \beta(\gamma(t))\rangle d\mathcal{L}^{N-1}(t)\\ &= \int_T \langle *\gamma(t), \beta(\gamma(t))\rangle\cdot \mathrm{Or}(\gamma,t)\cdot J_{N-1}\gamma(t) d\mathcal{L}^{N-1}(t)\\ &= \int_{S^{N-1}}\Bigg[\sum_{\{t\in T:\,\gamma(t)=s\}}\langle *\gamma(t),\beta(\gamma(t))\rangle\cdot\mathrm{Or}(\gamma,t)\Bigg]d\mathcal{H}^{N-1}(s)\\ &= \int_{S^{N-1}}\Bigg[\sum_{\{t\in T:\,\gamma(t)=s\}}\langle *s,\beta(s)\rangle\cdot\mathrm{Or}(\gamma,t)\Bigg]d\mathcal{H}^{N-1}(s),\end{aligned}$$

and we also conclude that $\mathrm{Cross}(\gamma, s) \in \mathbf{Z}$ (integers) for $\mathcal{H}^{N-1}$ almost all $s \in S^{N-1}$ such that $\langle *s, \beta(s)\rangle \neq 0$. Thus $\mathrm{Cross}(\gamma, s) \in \mathbf{Z}$ for $\mathcal{H}^{N-1}$ almost all $s \in S^{N-1}$, and

$$(1)\qquad \int_{(\gamma,T)}\beta = \int_{S^{N-1}}\langle *s,\beta(s)\rangle\cdot\mathrm{Cross}(\gamma,s)d\mathcal{H}^{N-1}(s)$$

whenever β is an $N-1$ form of class ∞ in a neighborhood of S^{N-1}.

Now set $U \equiv \{u \in \mathbf{R}^{N-1} : |u| < 1\}, \mathbf{R}^N(+) \equiv \{x \in \mathbf{R}^N : x_N > 0\}$, define $\rho(u) = (u_1, \cdots, u_{N-1}, (1-|u|^2)^{1/2})$ for all $u \in U$, and let f be a function of class ∞ on U with compact support. Then choose a corresponding function F, of class ∞ on $\mathbf{R}^N$, so that $\mathrm{spt}F$ is a compact subset of $\mathbf{R}^N(+), D_N F(\rho(u)) = 0$, and $D_j F(\rho(u)) = D_j f(u)$ for all $u \in U$. Finally, define

$$\beta(x) = (-1)^{N-1}D_jF(x)(DX)_{\hat N}(x) - (-1)^{j-1}D_NF(x)(DX)_{\hat j}(x)$$

for all $x \in \mathbf{R}^N$. Noting that β is an exact $N-1$ form of class ∞ on $\mathbf{R}^N$ and that $(-1)^{N-1}D_1\rho(u)\wedge\cdots\wedge D_{N-1}\rho(u) = *\rho(u)\cdot J_{N-1}\rho(u)$ for all $u \in U$, we obtain, from (1) and the area theorem,

$$\begin{aligned}\int_{(\gamma,T)}\beta &= \int_{S^{N-1}}\langle *s,\beta(s)\rangle\cdot\mathrm{Cross}(\gamma,s)d\mathcal{H}^{N-1}(s)\\ &= \int_U\langle *\rho(u),\beta(\rho(u))\rangle\cdot\mathrm{Cross}(\gamma,\rho(u))\cdot J_{N-1}\rho(u)d\mathcal{L}^{N-1}(u)\\ &= \int_U D_jF(\rho(u))\cdot\mathrm{Cross}(\gamma,\rho(u))d\mathcal{L}^{N-1}(u)\\ &= (-1)^{N-1}\int_U\langle D_1\rho(u)\wedge\cdots\wedge D_{N-1}\rho(u),\beta(\rho(u))\rangle\cdot\mathrm{Cross}(\gamma\cdot\rho(u))d\mathcal{L}^{N-1}(u)\\ &= \int_U D_jf(u)\cdot\mathrm{Cross}(\gamma,\rho(u))d\mathcal{L}^{N-1}(u) = 0, \text{ since } (\gamma,T) \text{ is closed.}\end{aligned}$$

Because f is an arbitrary function of class ∞ on U with compact support, we conclude (as in 3.4) there exists $c \in \mathbf{R}$ such that $\mathrm{Cross}(\gamma, \rho(u)) = c$ for $\mathcal{L}^{N-1}$ almost all

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$u \in U$. It is now clear that $\text{Cross}(\gamma, s) = c$ for $\mathcal{H}^{N-1}$ almost all $s \in s^{N-1}$ and, since $\text{Cross}(\gamma, s) \in \mathbf{Z}$ for $\mathcal{H}^{N-1}$ almost all $s \in s^{N-1}$, we conclude that $c \in \mathbf{Z}$.

Now apply (1) with $\beta = \omega_0$, noting that $\langle *s, \omega_0(s)\rangle = \alpha_{N-1}^{-1}$ for all $s \in S^{N-1}$: we obtain $\text{ind}(\gamma, 0) = \int_{S^{N-1}} \alpha_{N-1}^{-1} \cdot c d\mathcal{H}^{N-1}(s) = c$. □

5.2. Here we show how Federer's Gauss-Green theorem implies a natural characterization of closed chains. However, to avoid auxilliary complications which lead beyond our scope, we must limit the context of our discussion. If $T \subset \mathbf{R}^{N-1}$, then $n(T, b)$ denotes Federer's general *exterior normal* of T at b ([FED 1], p. 477). For $N > 2$, we say T is an *elementary subset* of $\mathbf{R}^{N-1}$ if T is compact and there exists a corresponding pair (ρ, S) such that $\rho : \mathbf{R}^{N-2} \longrightarrow \mathbf{R}^{N-1}$ is Lipschitzian, S is a bounded $\mathcal{L}^{N-2}$ measurable subset of $\mathbf{R}^{N-2}$, $\rho|S$ is univalent, $\rho[S] \subset$ Bndry T, $\mathcal{H}^{N-2}(\text{Bndry } T - \rho[S]) = 0$, $\langle e_1 \wedge \cdots \wedge e_{N-2}, \wedge_{N-2} D\rho(s)\rangle = *n(T, \rho(s)) \cdot J_{N-2}\rho(s)$ for all $s \in S$: when $N = 2$, T must be the union of a finite, disjointed family of compact intervals. We say (γ, T) is an *elementary* $N-1$ *chain* in $\mathbf{R}^N$ if $\gamma : \mathbf{R}^{N-1} \longrightarrow \mathbf{R}^N$ is of class 2 and T is an elementary subset of $\mathbf{R}^{N-1}$.

THEOREM 3. *Suppose (γ, T) is an elementary $N-1$ chain in $\mathbf{R}^N$. Then (γ, T) is closed if and only if*

$$\sum_{\{b \in \text{Bndry} T : \gamma(b) = x\}} \text{sign } \langle *n(T, b), \wedge_{N-2} D\gamma(b)\rangle = 0$$

for $\mathcal{H}^{N-2}$ almost all $x \in \mathbf{R}^N$.

PROOF. Suppose β is an $N-2$ form of class ∞ in $\mathbf{R}^N$. Since γ is of class 2 on $\mathbf{R}^{N-1}$, we have $\gamma^{\#} d\beta = d\gamma^{\#}\beta$, and (by 2.4),

$$\begin{aligned} \int_{(\gamma,T)} d\beta &= \int_T \langle e_1 \wedge \cdots \wedge e_{N-1}, \gamma^{\#} d\beta(t)\rangle d\mathcal{L}^{n-1}(t) \\ &= \int_T \langle e_1 \wedge \cdots \wedge e_{N-1}, d\gamma^{\#}\beta(t)\rangle d\mathcal{L}^{N-1}(t). \end{aligned}$$

Since $\mathcal{H}^{N-2}(\text{Bndry } T) < \infty$, we may transform the last integral by means of Federer's Gauss-Green theorem ([FED 1], p. 478 (4)), obtaining

$$\int_{(\gamma,T)} d\beta = \int_{\text{Bndry} T} \langle *n(T, b), \gamma^{\#}\beta(b)\rangle d\mathcal{H}^{N-2}(b).$$

By use of Federer's area theorem and the properties of the parametrization (ρ, S), we transform the integral on the right: from the chain rule and the properties of (ρ, S), we obtain sign $\langle e_1 \wedge \cdots \wedge e_{N-2}, \wedge_{N-2} D\gamma \circ \rho(s)\rangle = \text{sign}\langle *n(T, b), \wedge_{N-2} D\gamma(b)\rangle$ when $b = \rho(s)$ and $s \in S$; the result of the transformation is

$$\int_{(\gamma,T)} d\beta = \int_{\mathbf{R}^N} \left\langle \sum_{\{b \in \text{Bndry} T : \gamma(b) = x\}} \text{sign } \langle *n(T, b), \wedge_{N-2} D\gamma(b)\rangle, \beta(x) \right\rangle d\mathcal{H}^{N-2}(x)$$

Thus, (γ, T) is closed if and only if the integral on the right vanishes for all choices of β, and our theorem follows. □

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