A GENERALIZATION OF COX'S CHAIN OF THEOREMS

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1. Introduction. In [5, p. 105] attention has been called to a set of propositions, due to H. Cox [3, p. 67], which are related to another set, due to Clifford [2, p. 145; 4, p. 447], concerning points and circles in the plane or on the sphere. One may state Cox's chain of theorems as follows:

In a projective 3-space, S_3, let (1), (2), (3), (4) be four points lying in a plane α such that no three of them are collinear. Every two determine a line; let one plane such as [12], pass through each line. There are six such planes. The planes [12], [23], [13] determine a point (123); there are four such points. The first theorem of the chain states that they all lie in one plane [1234]. It is not difficult to see that this is, in fact, a rewording of Möbius's theorem on mutually inscribed pairs of tetrahedra [4, p. 444].

Now if we take a fifth point (5) in α, then any four of them give rise by the first theorem to a plane, so that we have five planes [1234], [1235], [1245], [1345], [2345]. The second theorem of the chain states that these planes pass through the same point (12345).

Continuing in this manner, by introducing a new point in each step, we obtain Cox's general theorem [4, pp. 446-447] to the effect that d coplanar points with arbitrary planes through their lines of of intersections determine an incomplete (Cox's) configuration which is, in fact, complete. The configuration consists of 2^{d-1} points and 2^{d-1} planes with d points on each plane and d planes on each point.

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The purpose of this note is to extend this chain of theorems to higher spaces. We shall be more concerned with the propositions of the chain than their corresponding configurations. We develop a chain of theorems in odd-dimensional spaces which specializes to Cox's chain in 3-space. In the next section, we give a detailed treatment of the developed chain in 5-space; and in the last section, we establish the chain in its general form.

2. Five-dimensional analogues. Let (1), (2), (3), (4), (5), (6) be six points on a hyperplane α of a projective 5-space, S_5, such that no subset, of this set of points, is linearly dependent. Every four of them determine a 3-space; let an arbitrary hyperplane, different from α, such as [1234], pass through each. There are fifteen such hyperplanes. The hyperplanes [1234], [1235], [1245], [1345], [2345] determine a point (12345); there are six such points. Then these points lie in one hyperplane [123456].

The figure involved in this generalization of the first statement of Cox's chain is (unlike its 3-space analogue) not symmetric. However, it admits a natural projection, from a line joining any two of the given points, into an arbitrary 3-space not contained in α, which yields the corresponding Cox's configuration in S_3. Indeed, if we project, for example, from the line joining (5), (6), by planes passing through it, we obtain the four points (1)', (2)', (3)', (4)', which are the images of (1), (2), (3), (4) respectively, lying in one plane. The six hyperplanes, each containing a pair of the latter four, are projected into six planes, each passing through a line determined by a pair of the former four. Discarding the two points (12345) and (12346), the remaining four points (12356), (12456), (13456), (23456) are mapped into the points (123)', (124)', (134)', (234)' respectively (we erase the digits 5, 6). These incidences obviously define an incomplete Cox's configuration corresponding to the first statement. The configuration is complete, that is, the four points (123)', (124)', (134)', (234)' are coplanar, if and only if the resulting six points lie in one hyperplane [123456].

To obtain, in S_5, an analogue to the second step of the chain, let (7) be a seventh point in α. Every six of them, by the first step, give rise to a hyperplane like [123456]; there are seven such hyperplanes. We shall prove that they all meet in one point (1234567). Let $\beta \not\subset \alpha$ be an arbitrary 3-space. Project the figure from the line (6) · (7) into β. The five points (1), ... , (5) are mapped onto five points (1)', ... , (5)' respectively, lying in a plane common to α and β. The ten hyperplanes, each passing through a pair of the first set, are projected into ten planes, each passing through a line joining a pair of the second. Discarding the two hyperplanes [123456] and [123457], the remaining five hyperplanes, namely [123467], [123567], [124567], [134567], [234567] are projected into the five planes [1234]', [1235]', [1245]', [1345]', [2345]' respectively (here also we erase the digits 6, 7). The primed five points with the ten planes, each passing through a line joining a pair of them, together with the five primed planes, determine an incomplete Cox's configuration in $\beta = S_3$, corresponding to the second statement of the chain. Hence the configuration closes with the point (12345)'. The two hyperplanes [123456] and [123457] intersect in a 3-space γ. The plane, determined by the point (12345)' and the line joining (6), (7), intersects γ in a point. This point is incident to the projected five hyperplanes because it is the image of the point of intersection of their projection planes. Thus the seven hyperplanes intersect in one point (1234567).

It is interesting to observe that one could have obtained the result directly by projecting into the three-space of intersection of the two hyperplanes [123456] and [123457]. In fact, let us denote the hyperplane [123456] by α_7, ... , [234567] by α_1. The five hyperplanes α_i (i = 1, 2, ... , 5) determine a point P_{67}. We shall prove that P_{67} is incident to the 3-space α_{67} common to α_6, α_7; and then the result follows. Project the whole figure from the line joining (6), (7) into α_{67}. The five hyperplanes α_i intersect α_{67} in five planes α_{i67} (i = 1, 2, ... , 5), and the arbitrary hyperplanes intersect α_{67} in planes, each passing through a line joining a pair of the images of (1), (2), ... , (5). As before, this defines an incomplete Cox's configuration corresponding to the second statement. Hence the configuration closes by the point of intersection of α_{i67} provided that the α_i, all, contain this point; and conversely.

3. Odd-dimensional analogues.

THEOREM 1. Let (1), (2), ... , (n + 1) be a set of points, in general position, lying in a hyperplane α of a projective

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odd-dimensional space $S_n (n > 1)$. Every $n - 1$ of them determine an $(n - 2)$-space, S_{n-2}; let an arbitrary hyperplane, such as $[12 \ldots n - 1]$ different from α, pass through each. There are $\binom{n+1}{2}$ such hyperplanes. The hyperplanes $[123 \ldots n - 1]$, $[123 \ldots n - 2\ n]$, $[123 \ldots n - 3\ n - 1\ n]$, $\ldots$, $[23 \ldots n]$ determine a point $(123 \ldots n)$; there are $n + 1$ such points. Then these points lie in one hyperplane $[123 \ldots n + 1]$.

THEOREM 2. Let another point $(n + 2)$ be added to the set of points in α. By the procedure followed in theorem 1, we have $n + 2$ hyperplanes. Then these hyperplanes are incident to one point $(123 \ldots n + 2)$.

THEOREM 3. If a further point $(n + 3)$ is added, then $n + 3$ points would be obtained. These points will be contained in one hyperplane $[123 \ldots n + 3]$.

And so on . . . .

Proof of theorem 1. Project the figure from the line $n.(n + 1)$ into an arbitrary $S_{n-2} \not\subset \alpha$. The points $(1)'$, $(2)'$, $\ldots$, $(n - 1)'$, which are the images of (1), (2), $\ldots$, $(n - 1)$ respectively, lie in one hyperplane of S_{n-2}. The $\binom{n-1}{2}$ hyperplanes passing through the projecting line are projected into hyperplanes of S_{n-2} passing through $(n - 3)$-spaces of the primed set of $n - 1$ points. Thus we obtain an incomplete figure, corresponding to the same statement, in S_{n-2}; and therefore the completeness of either is necessary and sufficient for the completeness of the other. As n is odd, successive projections of this type will eventually lead to an incomplete Cox's configuration, corresponding to the same statement, in S_3. Hence the closure of the last is equivalent to that of the first in S_n.

We remark that an analytic proof of this theorem is given in [1, p. 226]. There, it arose as a special case, in odd dimensions, of a theorem in all dimensions. Incidentally, since the 2-dimensional theorem of Pappus is equivalent to the 3-dimensional theorem of Möbius [4, p. 445], we have proved the following

COROLLARY. Theorem 1 characterizes the commutativity of multiplication in odd-dimensional projective spaces

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defined over (not necessarily commutative) fields.

Proof of theorem 2. Denote the $n + 2$ resulting hyperplanes by α_i $(i = 1, 2, \ldots, n + 2)$, where i indicates the missing integer in the symbol of each. The first n of them determine a point $P_{n+1, n+2}$, while the last two determine an $(n - 2)$-space $\alpha_{n+1, n+2}$. Projecting the whole figure from the line $(n + 1).(n + 2)$ into $\alpha_{n+1, n+2}$, we have the α_i $(i = 1, 2, \ldots, n)$ mapped into $\alpha_{i, n+1, n+2}$, hyperplanes of $\alpha_{n+1, n+2}$, which are $(n - 3)$-spaces in $S_{n-2} = \alpha_{n+1, n+2}$. The arbitrary hyperplanes passing through the projecting line are projected into hyperplanes, of S_{n-2}, each passing through an $(n - 4)$-subspace determined by every $n - 3$ points of the n images $(1)'$, $(2)'$, $\ldots$, $(n)'$. These incidences define an incomplete (Cox's) second figure in $\alpha_{n+1, n+2}$, which is of odd-dimension. Continuing the process of projection in this manner, into the next lower odd-dimensional space in each, we would arrive at Cox's second configuration in S_3; the last configuration is closed if, and only if, the first figure, in S_n, is closed. This completes the proof of the theorem.

The proofs of theorem 3 and the successive theorems of the chain are clear now. In each, we project, from a line joining two of the given points, into the next lower odd-dimensional space, to get the same figure in that space; and hence the corresponding configuration in 3-space. Thus, we reduce the theorem to the 3-dimensional case.

Finally, we note that Cox's original chain of propositions is self-dual. It remains to be seen whether it is possible to obtain a self-dual extension.

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