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CANONICAL POINT MAPPINGS IN $H\overline{H}$

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We give a complete characterisation of univalent logharmonic mappings from the domain D of $\overline{C}$ such that $\overline{C} \setminus \{D\}$ has countable many components onto $\Omega = \overline{C} \setminus \bigcup_{j=1}^{\infty} \{p_j\}$ where p_j is a singleton in $\mathbf{C}$.

1. INTRODUCTION

Let D be an arbitrary domain in the extended complex plane $\overline{\mathbf{C}}$ which contains the point at infinity such that $\overline{\mathbf{C}} \setminus \{D\}$ has countably many components. It was shown in [**2**] that there exists a univalent harmonic and orientation-preserving mapping f which maps D onto a punctured plane Ω and it is normalised by $f(z) = z + o(1)$ as z approaches infinity. Any complex-valued harmonic map defined on D can be locally expressed as a sum of an analytic function h and an anti-analytic function g, that is, by $f = h + \overline{g}$. However, in general, h and g are not globally analytic on D. For instance, there is no pair of analytic functions h and g on $|z| > 1$ such that the univalent harmonic mapping $f(z) = z - (1/\overline{z}) + 2ln|z|$ can be written in the form $f = h + \overline{g}$. However, it was shown in [**2**] that one may add the additional hypothesis that $f = h + \overline{g}$; $h, g \in H(D)$ where $H(D)$ stands for the set of all analytic functions on D.

It is a natural question to ask if we may replace the sum $h + \overline{g}$ by the product $h.\overline{g}$. The answer is yes. Denote by $H\overline{H}$ the family of all mappings f of the form $f = h.\overline{g}$, where h and g belong to $H(D)$. Univalent mappings in $H\overline{H}$ have been studied in [**1**] for the case that D is a simply connected domain, for example, the unit disk.

Let $p \in D$ be a fixed given point, $p \neq \infty$, and let $\jmath_\theta(z,p)$ be a cononical conformal map from the domain D onto a helical domain of inclination θ with respect to the radial direction having the properties $\jmath_\theta(p,p) = 0$ and $\jmath_\theta(z,p) = z + O(1)$ as $z \longrightarrow \infty$. It was shown in [**3**] that $\jmath_\theta$ is uniquely determined and that we have for arbitrary θ

$$(1) \qquad \log \jmath_\theta(z,p) = e^{i\theta}\left[\cos(\theta)\log \jmath_0(z,p) - i\sin(\theta)\log \jmath_{\pi/2}(z,p)\right]$$

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where suitable branches of the logarithms are chosen. In other words, $\jmath_\theta(z,p)$ can be expressed in a unique way as a function of a univalent radial slit mapping, a univalent circular slit mapping and θ. Our purpose in this article is to show that

$$F(z) = \sqrt{\jmath_0 \cdot \jmath_{\pi/2}} \cdot \overline{\sqrt{\frac{\jmath_{\pi/2}}{\jmath_0}}} \tag{2}$$

is the desired univalent canonical point mapping in $H\overline{H}$.

2. Canonical point mappings

Theorem 2.1. *Let D be a domain of $\overline{\mathbf{C}}$ of countable connectivity containing the point infinity and let p be a given fixed point in D, $p \neq \infty$. Then there exists a unique mapping F of the form $F = H.\overline{G}$ where*

- *(i)* *H and G are in $H(D)$ such that $G = 1 + O(1/z)$ and $H = z + O(1)$ as $z \longrightarrow \infty$.*
- *(ii)* *$0 \notin HG(D \setminus \{p\})$ and $G(p) \neq 0$.*
- *(iii)* *$F(p) = 0$*
- *(iv)* *$\Omega = F(D) = \overline{C} \setminus \bigcup_{j=1}^{\infty} \{p_j\}$ where p_j is a singleton in $\mathbf{C}$. Furthermore, F is uniquely determined.*

Remark. For the case where $D = \{|z| > 1\}$, the mapping F can be written explicitly as

$$F(z) = \frac{z(1 - p/z)}{\overline{(1 - p/\overline{z})}}.$$

Proof: Let $\jmath_\theta(z,p)$ be the conformal canonical map from D onto a helical domain of inclination θ with repect to the radial direction, $0 \leqslant |\theta| \leqslant \pi/2$, normalised by $\jmath_\theta(z,p) = z + O(1)$ as $z \longrightarrow \infty$ and $\jmath_\theta(p,p) = 0$. Define

$$F(z) = \sqrt{\jmath_0(z,p)\jmath_{\pi/2}(z,p)} \cdot \overline{\left(\frac{\jmath_{\pi/2}(z,p)}{\jmath_0(z,p)}\right)}.$$

Then we have locally $F \in H\overline{H}(D)$, $F(p) = 0$, $F(z) = z + O(1)$ *as* $z \longrightarrow \infty$ and $F(D) = \overline{C} \setminus \bigcup_{j=1}^{\infty} \{p_j\}$. Indeed, we have $F = H\overline{G}$ where $H = \sqrt{\jmath_0 \jmath_{\pi/2}}$ and $G = \sqrt{\left(\jmath_{\pi/2}\right)/(\jmath_0)}$ are locally analytic in D. Furthermore, for each component of ∂D we have $|F| = |\jmath_{\pi/2}(z,p)| = \textit{constant}$ and $\arg F = \arg \jmath_0(z,p) = \textit{constant}$.

It remains to show that $F = H\overline{G}$ is a univalent mapping and it is uniquely determined.

Next, consider the locally analytic functions $\phi_\alpha = \left(H/(G)^{e^{-i\alpha}}\right)$, $\alpha \in \mathbf{R}$, defined on D. By choosing a suitable branch, we get

$$\begin{aligned}\log\,\phi_\alpha &= \log\,H - e^{-2i\alpha}\log\,G = \frac{1}{2}\log[\jmath_0(z,p)\jmath_{\pi/2}(z,p)] - \frac{1}{2}e^{-2i\alpha}\log\left[\frac{\jmath_{\pi/2}(z,p)}{\jmath_0(z,p)}\right] \\ &= e^{-i\alpha}\left[\cos(\alpha).\log\,\jmath_0 + i\sin(\alpha)\log\,\jmath_{\pi/2}\right].\end{aligned}$$

Therefore, there is locally a suitable branch, specifically, $\log\,\phi_\alpha \equiv \log\,\jmath_{-\alpha}$. This holds for any simply connected subdomain of D, from which we conclude that $\phi_\alpha \equiv \jmath_{-\alpha}$ and hence, ϕ_α is a univalent mapping in $H(D)$. Therefore, we get $\phi_\alpha{}' \neq 0$ for all $z \in D$ and

(3) $$\frac{\phi_\alpha{}'}{\phi_\alpha} = \frac{H'}{H} - e^{-2i\alpha}\frac{G'}{G} \neq 0 \text{ on D for all } \alpha \in \mathbf{R}.$$

Furthermore, on putting $\alpha = 0$ and $\alpha = \pi/2$, we conclude that H/G and $H.G$ are globally analytic functions on D and hence, H and G belong to $H(D)$.

Next, we consider a harmonic branch of $L(z) = \log\,F(z)$ in a simply connected subdomain of $D \setminus \{p\}$. Then we have

$$L_z = \frac{F_z}{F} = \frac{H'}{H} \text{ and } \overline{L_{\overline{z}}} = \frac{\overline{F_{\overline{z}}}}{F} = \frac{G'}{G}.$$

We show that the Jacobian of L,

$$J_L = |L_z|^2 - |L_{\overline{z}}|^2$$

does not vanish on D. Indeed, if $J_L(z_0) = 0$ then either $L_z(z_0) = L_{\overline{z}}(z_0) = 0$ or $H'/H = e^{i\gamma}(G'/G)$ for some $\gamma \in \mathbf{R}$. Both cases are excluded by (3). Therefore, we have $J_L \neq 0$ on D. But $J_L(\infty) > 0$ from which we conclude that $J_L > 0$ on D. In other words, L is locally a univalent orientation preserving map which implies that F is locally univalent and sense-preserving on D.

Let $\zeta = \xi + i\eta = \phi_0(z) = H(z)/G(z)$ and $B = \phi_0(D)$. Put $W(\zeta) = F \circ \phi_0^{-1}(\zeta)$. Then $W \in H\overline{H}(B)$ and is locally univalent. Furthermore, W maps each radial half-line onto itself. The local univalence of W and the fact that $W(B)$ is a punctured plane implies that W is globally univalent map from B onto Ω. Hence, F is univalent in D.

It remains to show that F is uniquely determined. Suppose F_1 and F_2 are two maps having the properties of the theorem. Put $Q = F_1/F_2$. Then Q is a bounded nonvanishing map in $H\overline{H}(D)$ and each component of ∂D is mapped to a point. The corresponding function $\psi_\alpha(z) = (H_1/H_2)\,(G_2/G_1)^{e^{-2i\alpha}}$ is a bounded nonvanishing analytic function defined on D and the property

$$\arg\,[\psi_\alpha]^{e^{i\alpha}} \equiv \arg\,[Q]^{e^{i\alpha}} \pmod{2\pi}$$

implies that $\arg\psi_\alpha$ is constant on each boundary component of D. If $\arg\psi_\alpha$ is not constant on D, then $\arg\psi_\alpha(D)$ is a bounded domain which misses all but countably

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many radial half-lines, which is impossible. Therefore, $\psi_\alpha \equiv constant$. Using the fact that $\psi_\alpha(\infty) = 1$, we conclude that $\psi_\alpha \equiv 1$ for all $\alpha \in \mathbf{R}$ and all $z \in D$. Therefore, we have

$$\left(\frac{H_1}{H_2}\right) = \left(\frac{G_2}{G_1}\right)^{e^{-2i\alpha}} \quad \text{for all } \alpha \in \mathbf{R}.$$

Since G_1 and G_2 are nonvanishing, we conclude that $H_1 = H_2$ and $G_1 = G_2$, that is, $F_1 = F_2$ and the theorem is proved. ∎

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