THE NUMBER OF HEXAGONS AND THE SIMPLICITY OF GEODESICS ON CERTAIN POLYHEDRA

B. GRÜNBAUM AND T. S. MOTZKIN

1. Introduction. The problem of determining the possible morphological types of convex polyhedra in three-dimensional Euclidean space E^3 is well known to be quite hopeless. We lack not only any general way of determining whether there exists a convex polyhedron having as faces f_3 triangles, f_4 quadrangles, . . . , and f_n n-gons, but even much more special questions of this kind seem to be rather elusive.

Restricting the attention to the class of convex and trivalent polyhedra (i.e. convex polyhedra in which every vertex is incident on three faces), the following well-known necessary condition for the existence of such a polyhedron having f_k k-gonal faces, $k = 3, 4, 5, \ldots, n$, is easily derivable from Euler's formula:

$$\sum (6 - k) f_k = 12. \tag{1}$$

This necessary condition is not sufficient; for example, there exists no polyhedron[1] with $f_3 = 4$, $f_6 = 1$ or 2, and $f_k = 0$ for $n \neq 3, 6$.

Equation (1) does not impose any restriction on the value of f_6. Nevertheless, the last remark shows that f_6 may not be taken arbitrarily. As an example we mention also the fact (Brückner **1**, p. 119) that among the 19 different solutions of (1) with $f_k = 0$ for $k \geqslant 7$, only 11 are realizable as polyhedra with $f_6 = 0$; other solutions are realizable only if f_6 is at least 1, 2, or 3, depending on the solution.

Some time ago, Professor H. S. M. Coxeter posed the question whether for every $n \neq 1$ there exist polyhedra P_n with $f_5 = 12, f_6 = n, f_k = 0$ for $k \neq 5, 6$. Denoting by Q_n polyhedra with $f_4 = 6, f_6 = n, f_k = 0$ for $k \neq 4, 6$, the answer to Coxeter's problem is contained in the following theorem.

THEOREM 1. *Polyhedra P_n and Q_n exist for every non-negative integer n satisfying $n \neq 1$.*

On further inquiry we found that the situation is different with regard to polyhedra R_n with $f_3 = 4$, $f_6 = n$, $f_k = 0$ for $k \neq 3, 6$. We have the following theorem.

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[1]Here and in what follows "polyhedron" will always mean "convex trivalent polyhedron in E^3."

THEOREM 2. *Polyhedra R_n exist if and only if n is a non-negative even integer different from* 2.

This fact seems to be new, and somewhat surprising, especially in view of the contrast with the result concerning P_n or Q_n. On the other hand, Theorem 2 gives a partial answer to a rather old question (Eberhard **2**, p. 84): Do there exist polyhedra with an odd number of faces such that the number of edges of each face is a multiple of 3?

We shall prove Theorem 1 (in § 2) by indicating the construction of polyhedra P_n and Q_n, $n \neq 1$. Polyhedra P_1 and Q_1 obviously do not exist. Theorem 2 will be proved in § 3 using elementary considerations which may be of possible use also in other problems.

Throughout the paper we shall make use of the following consequence of Steinitz's **(4)** "Fundamentaltheorem der konvexen Typen" (Grünbaum–Motzkin **3**): A graph G is 3-polyhedral (i.e. may be realized by the edges and vertices of a convex polyhedron P in E^3) if and only if G is a 3-connected planar graph. Moreover, the faces of P are uniquely determined and correspond to the faces of any realization of G on the 2-sphere.

2. Proof of Theorem 1. Theorem 1 will be established by describing two families of 3-connected trivalent planar graphs P_n^* and Q_n^*, $n \neq 1$, imbedded in the 2-sphere S^2, the faces of P_n^* being 12 pentagons, and those of Q_n^*, 6 quadrangles and n hexagons.

For $k = 0$ the desired graphs are those of a dodecahedron and a cube.

Q_3^* is represented in Figure 1.

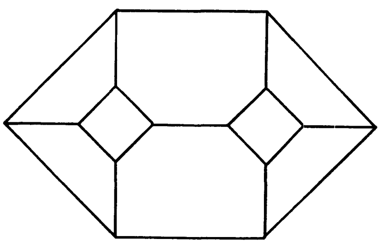

FIGURE 1

The other graphs P_n^* and Q_n^* are constructed as follows. Assume the graphs A, B, C, D (Fig. 2) each drawn on a hemisphere, with the heavy lines as equator. Combining two hemispheres with A on each we obtain P_2^*; combining similarly

A and B,	we obtain P_3^*,
A and C, or B and B,	P_4^*,
A and D, or B and C,	P_5^*,
B and D, or C and C,	P_6^*,
C and D,	P_7^*.

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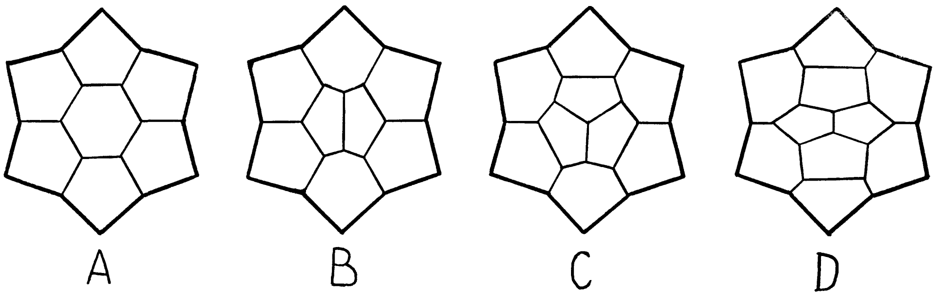

FIGURE 2

In order to obtain P^*_{j+6i} $(2 \leqslant j \leqslant 7;\ i = 1, 2, \ldots)$, we proceed as above except that the two hemispheres are separated by i "belts" (Fig. 3), each consisting of six hexagons. As an illustration, P_{15}^* is represented in Figure 4.

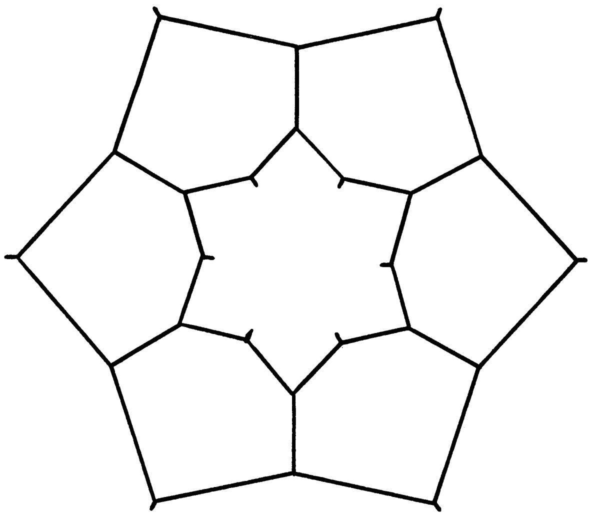

FIGURE 3

In an analogous manner the graphs Q_n^* are constructed starting from K, L, M, N (Fig. 5). Thus

K and K	yield	Q_2^*,
K and L		Q_4^*,
K and M		Q_5^*,
L and L		Q_6^*,
L and M, or K and N,		Q_7^*,
L and N		Q_9^*.

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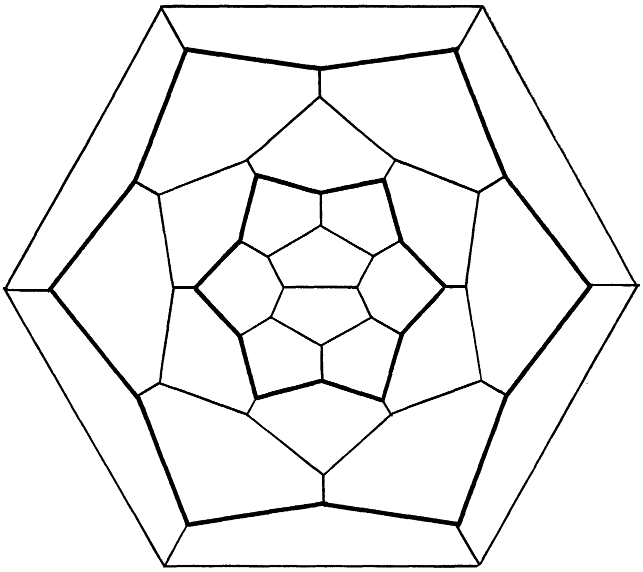

FIGURE 4

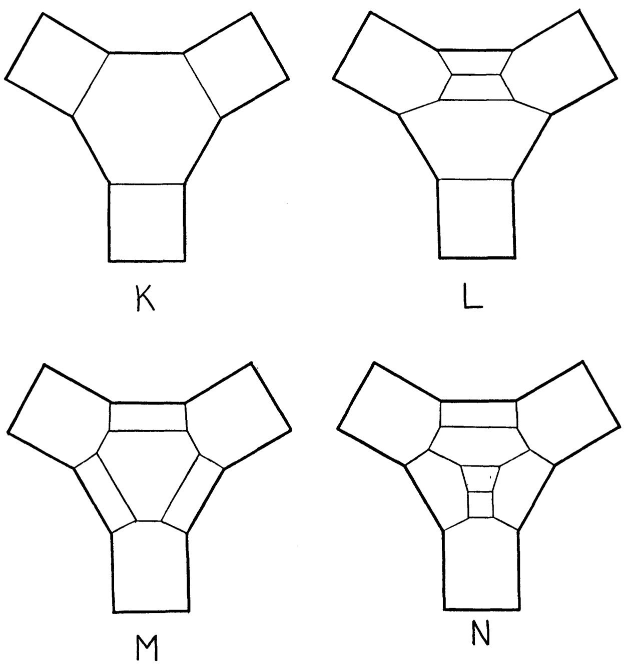

FIGURE 5

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The remaining graphs, Q_8^*, Q_k^*, $k \geqslant 10$, may be obtained by inserting the appropriate number of "belts" represented in Figure 6. As an illustration, Q_{13}^* is represented in Figure 7.

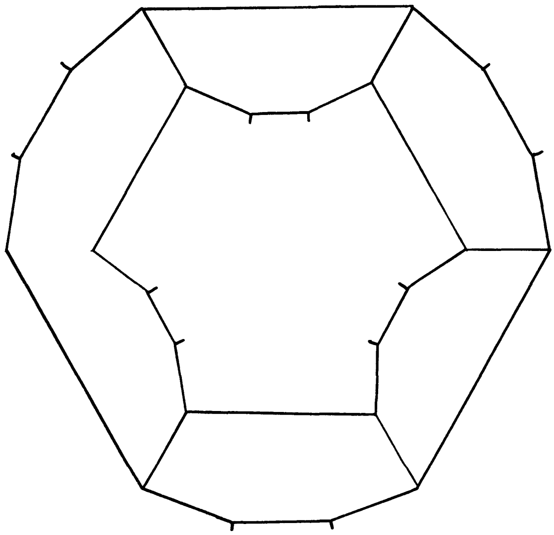

FIGURE 6

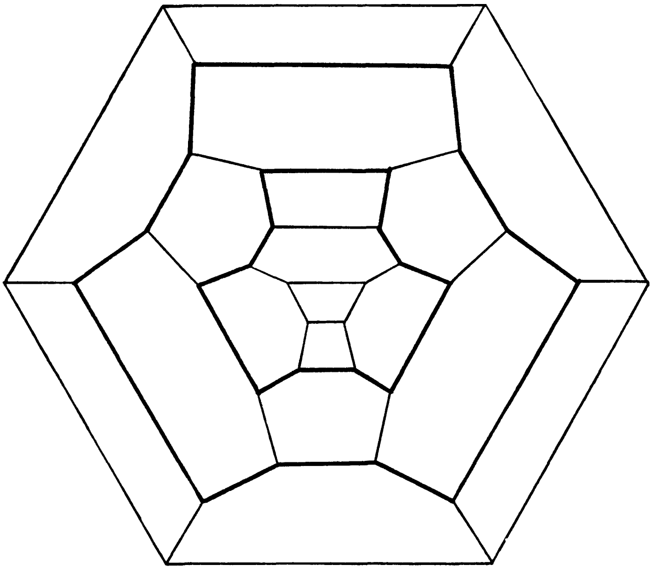

FIGURE 7

3. Proof of Theorem 2. Let P be a planar graph imbedded in a 2-sphere, and let S be an oriented simple circuit in P. We define the *leftness* $\lambda(S)$ as the excess of the number of edges of P branching off from S to the right over the number of edges branching off to the left. Thus in a trivalent graph, if F is the boundary of a k-gonal face taken in the positive direction, then $\lambda(F) = k$.

If P is a trivalent planar graph,[2] a (not necessarily simple) circuit G is called a *geodesic* if "right-turn" and "left-turn" vertices alternate on G. E.g., the heavy lines in Figure 4 are geodesics, while those in Figure 7 are not.

An intersection of two branches of a geodesic has one of the two forms represented in Figure 8 (both forms do occur). We shall call I an "acute" and II an "obtuse" self-intersection.

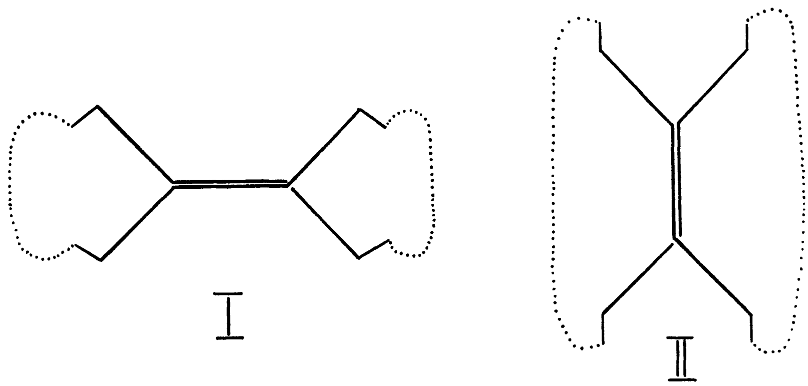

FIGURE 8

The following statement is easily proved by induction on the number of faces enclosed.

LEMMA 1. *If P is such that $\lambda(F)$ is divisible by 3 for every F, then $\lambda(S)$ is divisible by 3 for every simple circuit S in P.*

Crucial in the proof of Theorem 2 is the following lemma.

LEMMA 2. *If P is such that $\lambda(F)$ is divisible by 3 for every F, then all the geodesics of P are simple.*

Proof. If G is a geodesic of any graph such that G is not simple, let G^* be any simple loop of G. If G^* is oriented in the positive sense, then, as can be easily checked, $\lambda(G^*) = 2$ if G^* is determined by an acute self-intersection of G, and $\lambda(G^*) = 1$ if G^* is determined by an obtuse self-intersection of G. But in view of Lemma 1, $\lambda(G^*) \equiv 0 \pmod 3$, and therefore each geodesic G is simple.

[2]In the remainder of this section all graphs are understood to be connected, trivalent, and planar (imbedded in a 2-sphere). F will always denote the boundary of a face of P, taken in the positive orientation.

Now we are able to describe completely the structure of P_n^* graphs. Let P be such a graph and let G_0 be a geodesic with positive orientation, containing one (and therefore two) edges of one of the triangles T^1 of P, T^1 being to the left of the (simple) geodesic G_0. It is then obvious (see Fig. 9) that the region L to the left of G_0 can be described as follows. The triangle T^1 is followed by

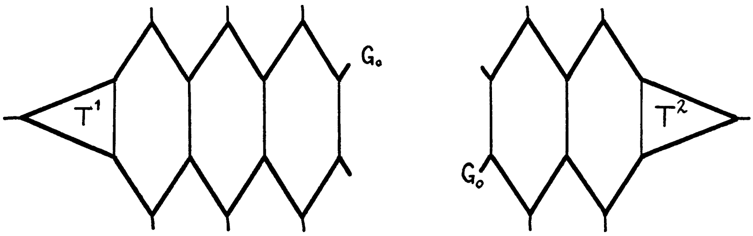

FIGURE 9

a row consisting of a certain number l, $l \geqslant 0$, of hexagons, beyond which follows another triangle T^2. (Thus L consists of 2 triangles and of l hexagons between them.) If $l = 0$, P can be 3-connected only if P is the graph of a tetrahedron. In the remainder of the proof we shall therefore assume $l \geqslant 1$.

Now, if G_0 contains also the edge of another triangle T^3 (which is necessarily to the right of G_0) the above reasoning (applied to G_0 with the negative orientation) shows that the region R to the right of G_0 consists of two triangles, T^3 and T^4, and l' hexagons between them. Since $\lambda(G_0) = 0$, it follows that $l = l'$ and P consists of 4 triangles and $2l$ hexagons.

If, on the other hand, G_0 does not meet any other triangle besides T^1 and T^2, the boundary of R consists exclusively of edges of hexagons (see Fig. 10). Then the edges of these hexagons which are non-incident to G_0 form another geodesic G_1, enclosing G_0 and of the same length as G_0. Note that the "belt" between G_0 and G_1 consists of $2l + 2$ hexagons. With regard to G_1 the above

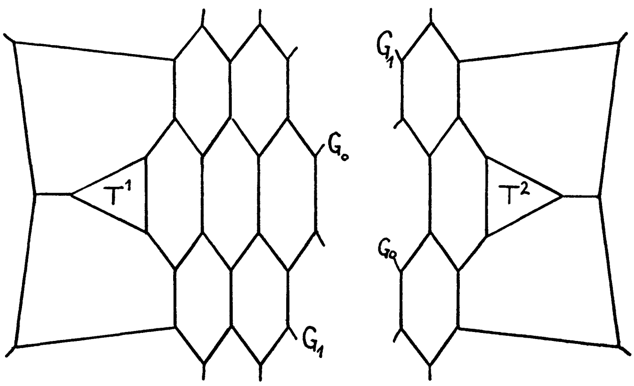

FIGURE 10

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alternative holds: either there is a triangle meeting G_1, in which case the "outside" region of G_1 consists of two triangles and l hexagons; or G_1 is adjacent to hexagons only and there exists a second belt of $2l + 2$ hexagons incident to G_1 and bounded on the "outside" by another geodesic G_2.

After a certain number w of steps ($w \geqslant 0$) obviously the first alternative must take place, and then P may be described as follows: P contains two strips (one to the left of G_0, and the other to the right of G_w) each of which consists of two triangles and l hexagons; the two strips are separated by w belts, each consisting of $2l + 2$ hexagons. Therefore P contains altogether $2(l + w + lw)$ hexagons, and thus the number of hexagons in P is even, as claimed in Theorem 2. Moreover, since every even number $2k$ can be obtained in the above form (e.g. for $l = k$, $w = 0$), and since the corresponding graph is 3-connected except for $l = 0$, $w \geqslant 0$, and $l = 1$, $w = 0$, Theorem 2 is proved completely.

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