Can. J. Math., Vol. XXVIII, No. 1, 1976, pp. 112–115

POSSIBILITY OF UNIFORM RATIONAL APPROXIMATION IN THE SPHERICAL METRIC

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Let f be a mapping defined on a compact subset K of the finite complex plane $\mathbf{C}$ and taking its values on the extended plane $\mathbf{C} \cup \{\infty\}$. For χ a metric on the extended plane, we consider the possibility of approximating f χ-uniformly on K by rational functions. Since all metrics on $\mathbf{C} \cup \{\infty\}$ are equivalent, we shall consider that χ is the chordal metric on the Riemann sphere of diameter one resting on a finite plane at the origin. Thus,

$$\chi(a, b) \leqq |a - b|,$$

for all values a, b, and hence the chordal metric is less descriminating than the Euclidean metric. On the other hand, away from ∞, they are equivalent. Thus it is natural to expect that results on χ-uniform approximation resemble those on uniform approximation, and we shall indeed see that this is the case.

Our motivation is twofold. Firstly, there is a widespread feeling among function theorists that the point at infinity is no different from any other point. In our framework the usual approximation problem then becomes a problem with an auxiliary condition. Namely, one adds the restriction that f omits some value, say ∞ on K. We place no such restriction on our functions. Thus our problem is truly more general. Indeed, there exist highly surjective functions which can be approximated in the present framework. For example, there exists a continuous mapping f from the closed unit disc to $\mathbf{C} \cup \{\infty\}$ and meromorphic on the open disc such that f assumes every value infinitely often. Such a function can be constructed by mapping the unit disc to an appropriate Riemann surface which resembles a Peano curve. Moreover such an f can be χ-uniformly approximated by rational functions.

Our second motivation: approximation theory is usually concerned with the approximation of functions taking their values in a linear space. But one can consider the more general problem of approximating functions taking their values on a locally linear space, that is, a manifold. Grauert and Kerner [**1**, p. 146], for example, have obtained beautiful Runge type theorems for functions taking their values in complex projective spaces. We may think of $\mathbf{C} \cup \{\infty\}$ as projective one-space. In this case, Runge's theorem is trivial, but we shall obtain much stronger theorems of the Vitushkin type.

We dedicate this paper to the memory of J. L. Walsh who died before its completion.

Received January 2, 1975 and in revised form, April 28, 1975.
This research was supported in part by the N.R.C. of Canada and the Québec Government.

For K a compact subset of the closed plane, $R(K)$ denotes, as usual, the uniform limits on K of rational functions without poles on K. $A(K)$ denotes the set of continuous functions f from K to $\mathbf{C}$ such that the restriction $f|K^0$ of f to the interior of K is holomorphic.

Analogously, we denote by $R_\chi(K)$ the χ-uniform limits on K of rational functions (with or without poles on K), and by $A_\chi(K)$ the χ-continuous functions f from K to $\mathbf{C} \cup \{\infty\}$ such that $f|K^0$ is meromorphic (or constant).

All of our results are based on the following fundamental lemma of Alice Roth [**2**].

LEMMA 1. (Euclidean fusion). *Let $K_1 K_2$, and K be compact subsets of the extended plane with K_1 and K_2 disjoint. If r_1 and r_2 are any two rational functions satisfying, for some $\epsilon > 0$,*

(1) $\quad |r_1(z) - r_2(z)| < \epsilon, \quad$ *for* $z \in K$,

then there is a positive number a, depending only on K_1 and K_2 and a rational function r such that for $j = 1, 2$,

(2) $\quad |r(z) - r_j(z)| < a\epsilon, \quad$ *for* $z \in K_j \cup K$.

We remark that in Lemma 1, r_1 and r_2 are allowed to have poles on the sets in question.

Remark. In Lemma 1 it is clear that by Runge's theorem we may replace r_1 and r_2 by functions f_1 and f_2 meromorphic on $K_1 \cup K$ and $K_2 \cup K$ respectively. Similarly, if

$$A(K_j \cup K) = R(K_j \cup K), \quad j = 1, 2,$$

we may take $f_j \in A(K_j \cup K)$, $j = 1, 2$.

Another consequence of the fusion lemma is the following well known theorem of E. Bishop [**3**, p. 97].

THEOREM 1. (Euclidean localization, Bishop). *Let f be given on a compact set K and suppose that for each $z \in K$ there exists a closed disc D_z with center z such that*

$$f|(K \cap D_z) \in R(K \cap D_z).$$

Then $f \in R(K)$.

To prove the localization theorem one can cover K by a fine grid of squares and apply the fusion lemma a finite number of times. We omit the details.

THEOREM 2. (Spherical localization). *Let f be given on a compact set K and suppose that for each $z \in K$ there exists a closed disc D_z with center z such that*

$$f|(K \cap D_z) \in R_\chi(K \cap D_z).$$

Then $f \in R_\chi(K)$.

https://doi.org/10.4153/CJM-1976-013-0 Published online by Cambridge University Press

Proof. Set

(3) $K_1 = \{z \in K : |f(z)| \leqq 1/2\}$,

(4) $K_0 = \{z \in K : 1/2 \leqq |f(z)| \leqq 1\}$, and

(5) $K_2 = \{z \in K : |f(z)| \geqq 1\}$.

Let $a > 1$ be a positive constant associated with K_1 and K_2 by Lemma 1.

Since f is continuous and finite-valued in a neighbourhood of $K_1 \cup K_0$, for each $z \in K_1 \cup K_0$, there is a closed disc D_z for which

$$f|(K \cap D_z) \in R(K \cap D_z).$$

By Theorem 1 for each $\epsilon > 0$, there exists a rational function r_1 such that

(6) $|r_1 - f| < \epsilon/4a < \epsilon/2$ on $K_1 \cup K_0$.

Analogously, there is a rational function r_2 such that

(7) $|r_2 - 1/f| < \epsilon/8a < \epsilon/2$ on $K_2 \cup K_0$.

From (14) it follows that

(8) $\chi(1/r_2, f) = \chi(r_2, 1/f) \leqq |r_2 - 1/f| < \epsilon/2$ on $K_2 \cup K_0$.

By (14) and (11),

(9) $|r_2| \geqq 1/|f| - \epsilon/2 \geqq 1 - \epsilon/2 > 1/2$ on K_0,

provided $\epsilon < 1$.

By (14), (11), and (16), we have

(10) $|1/r_2 - f| < \dfrac{\epsilon}{8a}\dfrac{|f|}{|r_2|} < \dfrac{\epsilon}{4a}$ on K_0,

and by (13) and (17),

$$|r_1 - 1/r_2| < \epsilon/2a \quad \text{on } K_0.$$

By Lemma 1, there is a rational function r with

(11) $|r - r_1| < \epsilon/2$ on $K_1 \cup K_0$, and

(12) $|r - 1/r_2| < \epsilon/2$ on $K_2 \cup K_0$.

Thus, on $K_1 \cup K_0$, (18) and (13) yield

$$\chi(r, f) \leqq |r - f| \leqq |r - r_1| + |r_1 - f| < \epsilon,$$

and on $K_2 \cup K_0$, (19) and (15) yield

$$\chi(r, f) \leqq \chi(r, 1/r_2) + \chi(1/r_2, f) < \epsilon.$$

This completes the proof.

The following theorem states (among other things) that spherical uniform

https://doi.org/10.4153/CJM-1976-013-0 Published online by Cambridge University Press

approximation by rational functions is possible for the same compact sets as uniform approximation is possible.

THEOREM 3. *For a compact set K, the following are equivalent:*

(I) $A(K \cap D) = R(K \cap D)$, *for each closed disc D.*

(II) $A(K) = R(K)$.

(III) $A_\chi(K \cap D) = R_\chi(K \cap D)$, *for each closed disc D.*

(IV) $A_\chi(K) = R_\chi(K)$.

Proof. (I) $\Leftrightarrow$ (II): This equivalence is well-known. The implication (I) $\Rightarrow$ (II) follows immediately from the localization Theorem 1, and (II) $\Rightarrow$ (I) follows from Vitushkin's theorem and properties of continuous analytic capacity [**3**].

(II) $\Rightarrow$ (IV): Let $f \in A_\chi(K)$. Then for each $z \in K$, there is a disc D_z such that either

$$f|(K \cap D_z) \in A(K \cap D_z), \quad (1/f)|(K \cap D_z) \in A(K \cap D_z).$$

Since II $\Rightarrow$ I, either $f|(K \cap D_z) \in R(K \cap D_z)$ or $(1/f)|(K \cap D_z) \in R(K \cap D_z)$. In either case $f|(K \cap D_z) \in R_\chi(K \cap D_z)$. Now IV follows from the spherical localization theorem 2.

(IV) $\Rightarrow$ (II): This is obvious.

(III) $\Leftrightarrow$ (I): This follows from (IV) $\Leftrightarrow$ (II), replacing K by $K \cap D$.

This completes the proof of Theorem 3.

Open problems.

(1) In a later paper we shall consider the problem of spherical uniform approximation by meromorphic functions on closed sets.

(2) We have considered the case where $K \subset \mathbf{C} \cup \{\infty\}$. What about the case where K is a compact subset of a Riemann surface S and one asks for spherical uniform approximation by functions meromorphic in a neighbourhood of K?

(3) Suppose $f|K^0$ has no poles. Can one approximate f spherically uniformly by rational functions with no poles on K^0?

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