Canad. Math. Bull. Vol. **21** (3), 1978

ON EARLE'S mod n RELATIVE TEICHMÜLLER SPACES

BY
ROBERT ZARROW

§1. In this paper we answer an open question of C. J. Earle ([2] §3.3 remarks (a) and (b)) in several cases. We first give some definitions and state some results which are given in greater detail in [2].

We let X be a smooth surface of genus $g \geq 2$ and let $M(X)$ be the space of smooth complex structures with the C^∞ topology. If $\mu \in M(X)$ let X_μ denote the Riemann surface determined by μ. The group $(\mathrm{Diff}^+(X))$ $\mathrm{Diff}(X)$ is the group of (orientation preserving) diffeomorphisms of X. Also $\mathrm{Diff}_n^+(X) = \{f \in \mathrm{Diff}^+(X)$: f induces the identity on $H_1(X, \mathbb{Z}/n\mathbb{Z})\}$.

The group $\mathrm{Diff}(X)$ acts on $M(X)$ by pullback: If $\mu \in M(X)$, $f \in \mathrm{Diff}(X)$ then $\mu \cdot f \in M(X)$ has the property that f: $X_{\mu \cdot f} \to X_\mu$ is conformal. With this action $\mathrm{Diff}_n^+(X)$ acts freely on $M(X)$ ([3], [4] or [6]). If H is a finite subgroup of $\mathrm{Diff}(X)$ then let $M(X)^H = \{\mu \in M(X)$: $\mu \cdot f = \mu\}$. We let $N_n(H)$ be the normalizer of H in $\mathrm{Diff}_n^+(X)$ and let $N^+(H)$ be the normalizer of H in $\mathrm{Diff}^+(X)$. Then we define $T_n(X) = M(X)/\mathrm{Diff}_n^+(X)$, $T_n(X, H) = M(X)^H/N_n(H)$, $R(X) = M(X)/\mathrm{Diff}^+(X)$, and $R(X, H) = M(X)^H/N^+(H)$. These spaces we call the mod n Teichmüller space, the mod n relative Teichmüller space, the Riemann space and the relative Riemann space. The space $T_n(X)$ and $T_n(X, H)$ are finite branched coverings of $R(X)$ and $R(X, H)$ respectively. We define $\theta_n : \mathrm{Diff}(X) \to \mathrm{Diff}(X)/\mathrm{Diff}_n^+(X) = \Gamma_n(X)$. The group $\theta_n(H)$ acts on $T_n(X)$ and the set of fixed points is denoted by $T_n(X)^{\theta_n(H)}$. We let $\Gamma_n(H)$ be the normalizer of $\theta_n(H)$ in $\theta_n(\mathrm{Diff}^+(X)) = \Gamma_n^+(X)$. Then Earle [2] process the following.

THEOREM A. *If $n > 2$, then*

(a) *$\Gamma_n(H)$ is a group of automorphisms of $T_n(X)^{\theta_n(H)}$*

(b) *The quotient space $T_n(X)^{\theta_n(H)}/\Gamma_n(H)$ is the disjoint union of Riemann spaces $R(X, H')$. The union is over the* $\mathrm{Diff}^+(X)$ *conjugacy classes of finite groups H' such that $\theta_n(H') = \theta_n(H)$.*

In the present paper we determine the number of components in (b) in several cases when H has order two. We thus denote by $\Psi(n, H)$ the number of components of $T_n(H)^{\theta_n(H)}/\Gamma_n(H)$. Our results are the following.

THEOREM 1. *If H is of order two and generated by an orientation reversing map then*

(a) *$\Psi(n, H) = 2$, if $H = \langle\sigma_1\rangle$ or $H = \langle\sigma_2\rangle$ and n is even, where $X/\langle\sigma_1\rangle$ is a*

Received by the editors September 21, 1977 and in revised form, January 11, 1978.

sphere with $g+1$ *cross caps and no boundary components and* $X/\langle\sigma_2\rangle$ *is a surface with* $g-2[g/2]+1$ *boundary components and* $[g/2]$ *handles.*

(b) $\Psi(n, H)=1$, *if* $H\neq\langle\sigma_1\rangle$ *or* $H\neq\langle\sigma_2\rangle$ *and* $n>2$ *is even.*

(c) $\Psi(n, H)=2[g/2]+[(g+1)/2]+2$, *if n is odd.*

THEOREM 2. *If* $H=\langle\sigma\rangle$ *has order two,* σ *is orientation preserving, and* $n>2$ *is even, then*

(a) $\Psi(n, H)=2$, *if* σ *has zero or one fixed point.*

(b) $\Psi(n, H)=1$, *if* σ *has more than one fixed point.*

REMARK. Theorem A and Theorem 1(b) together imply that $R(X, H)$ is a real algebraic variety if H satisfies the hypotheses of Theorem 1(b).

§2. In this section we prove Theorems 1 and 2. We first need a lemma.

LEMMA. *There are* $2[g/2]+[(g+1)/2]+2$ $\text{Diff}^+(X)$ *conjugacy classes of cyclic subgroups H of order two if the generator of X is orientation reversing.*

Proof. The conjugacy class of H is determined by the topological type of X/H ([1], pp. 57–58). It now follows from Theorem 3.6 of [7] that the number of conjugacy classes of H is $x+1$, where x is the number of triples (r, s, t) with $r=0, 1, 2$, $r\leq s$, $s+2t=g$. The lemma now follows by a simple counting argument.

Proof of Theorem 1. We first consider (a) and (b). We let $H_1=\langle\sigma_1\rangle$ and $H_2=\langle\sigma_2\rangle$. Then it follows by Theorem 3.6 of [7] that a conjugate of σ_1 induces the same action on $H_1(X, \mathbb{Z})$ as σ_2. Thus by Theorem A $\Psi(n, H_1)\geq 2$ and $\Psi(n, H_2)\geq 2$, for all $n\geq 3$.

Now suppose σ and τ are two orientation reversing maps which induce M_1 and M_2 on $H_1(X, \mathbb{Z})$, respectively, and suppose $\{\sigma, \tau\}\neq\{\sigma_1, \sigma_2\}$. We investigate whether there is a symplectic matrix A such that $AM_1A^{-1}=M_2 \bmod n$. By pp. 221–222 [7] we may assume that

$$M_1=\left[\begin{array}{c|c} I_g & \begin{matrix} I_r & 0 \\ & F_t \end{matrix} \\ \hline 0 & -I_g \end{array}\right]$$

$$M_2=\left[\begin{array}{c|c} I_g & \begin{matrix} I_u & 0 \\ & F_v \end{matrix} \\ \hline 0 & -I_g \end{array}\right]$$

https://doi.org/10.4153/CMB-1978-061-0 Published online by Cambridge University Press

where I_k denotes the $k\times k$ identity and

$$F_k=\begin{bmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & & 0 \quad 1 \\ & & & & & & & 1 \quad 0 \end{bmatrix}, \qquad 2k\times 2k.$$

We first consider the case in which n is even. If $AM_1A^{-1}=M_2 \bmod n$ then we must also have $AM_1A^{-1}=M_2 \bmod 2$. However by results in [7] pp. 221–222 this is impossible if $M_1\neq M_2$. This proves (a) and (b).

We now consider the case in which n is odd. We claim that we may always find a symplectic matrix A such that $AM_1=M_2A \bmod n$. We let

$$A=\begin{bmatrix} I_g & B \\ 0 & I_g \end{bmatrix}$$

so that equation $AM_1=M_2A$ reduces to

$$2B=\begin{bmatrix} I_r & & \\ & 0 & \\ & & F_t \end{bmatrix}-\begin{bmatrix} I_u & & \\ & 0 & \\ & & F_v \end{bmatrix} \bmod n.$$

It is easy to check that this equation will always have a solution in some symmetric matrix B. This implies (c).

Proof of Theorem 2. We first define some matrices. Let

$$M(r,s,t)=\left[\begin{array}{ccc|ccc} I_r & & & & & \\ & -I_s & & & 0 & \\ & & F_t & & & \\ \hline & & & I_r & & \\ & 0 & & & -I_s & \\ & & & & & F_t \end{array}\right],$$

where F_t is defined in the proof of Theorem 1. Let

$$L(r,s,t)=\left[\begin{array}{ccc|ccc} I_r & & & 0 & & 0 \\ & -I_s & & & & \\ & & H_t & 0 & & G_t \\ \hline & & & I_r & & \\ & 0 & & & -I_s & \\ & & & & & H_t \end{array}\right]$$

https://doi.org/10.4153/CMB-1978-061-0 Published online by Cambridge University Press

where

$$H_t = \begin{bmatrix} -1 & & & & & & \\ & 1 & & & & & \\ & & -1 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & 1 \end{bmatrix} \quad 2t \times 2t$$

and

$$G_t = \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{bmatrix} \quad 2t \times 2t$$

Let

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \in Sp(2, \mathbb{Z})$$

$$R_t = \begin{bmatrix} R & & & & & \\ & R & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & R \end{bmatrix} \quad 4t \times 4t$$

and

$$M = \left[\begin{array}{cc|cc} I_{r+s} & & 0 & \\ & R_t & & \\ \hline 0 & & I_{r+s} & \\ & & & R_t \end{array}\right]$$

We remark that there is a canonical homology basis of X such that with respect to this basis σ induces the matrix $M(r, s, t)$, where $r=0$ and $s>1$ or $r=1$ and $s=0$. By multiplying we see that $MM(r, s, t)M^{-1}=L(r, s, t)$. This implies that with respect to a suitable canonical homology basis σ induces $L(r, s, t)$.

To prove Theorem 2 we first show that if $t \neq w$ then there is no matrix K in $Sp(g, \mathbb{Z})$ such that $KL(r, s, t) = L(u, v, w)K \bmod 2$. We assume that there is such a matrix K and obtain a contradiction. Thus we must have

$$K\left[\begin{array}{c|c} I_g & \begin{matrix} 0 & \\ & F_t \end{matrix} \\ \hline 0 & I_g \end{array}\right] = \left[\begin{array}{c|c} I_g & \begin{matrix} 0 & \\ & F_w \end{matrix} \\ \hline 0 & I_g \end{array}\right] K \bmod 2.$$

We write

$$K = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A, B, C and D are $g \times g$. Upon multiplying and equating terms mod 2, we see that

(1) $$A\begin{bmatrix} 0 & 0 \\ 0 & F_t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & F_w \end{bmatrix} D \bmod 2$$

(2) $$\begin{bmatrix} 0 & 0 \\ 0 & F_w \end{bmatrix} C = 0 \bmod 2$$

and

(3) $$C\begin{bmatrix} 0 & 0 \\ 0 & F_t \end{bmatrix} = 0 \bmod 2.$$

Equations (2) and (3) imply that

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} \bmod 2,$$

where C_1 is $n-2w \times n-2t$. Equation (1) implies that

$$A = \begin{bmatrix} A_1 & 0 \\ A_3 & A_4 \end{bmatrix} \bmod 2$$

and

$$D = \begin{bmatrix} D_1 & D_2 \\ 0 & D_4 \end{bmatrix} \bmod 2,$$

where A_4 is $2w \times 2t$, D_4 is $2t \times 2w$, etc. Denote the transpose of a matrix L by ${}^tL$. Then the symplectic condition that $A\,{}^tD - B\,{}^tC = I_{2g}$ implies that $A_4\,{}^tD_4 = I_{2w}$. If $w > t$ then $A_4\,{}^tD_4$ can have rank at most $2t$, a contradiction. Thus $w \leq t$. Similarly $t \leq w$ so that $t = w$.

To finish the proof we remark that if $KL(r, s, t) = L(u, v, w)K \bmod n$, where n is even, then $KL(r, s, t) = L(u, v, w)K \bmod 2$. Also the condition $t = w$ implies $r + s = u + v$. If σ is fixed point free then $r = 1$ and $s = 0$. This implies that $u + v = 1$ so that either $u = 0$, $v = 1$ or $u = 1$, $v = 0$. Thus $\Psi(n, H) = 2$. If σ has one fixed point then $r=0$ and $s=1$. Again $u+v=1$ and as before $\Psi(n, H)=2$. If σ has more than one fixed point, then $r=0$ and $s>1$. If $u=1$ then $v=0$ and it is impossible that $u+v=r+s$. If $u=0$ then $v>1$ and $u+v=r+S$ implies $v=s$ thus $\Psi(n, H)=1$. This completes the proof.

REMARK 1. I do not know what $\Psi(n, H)$ is if n is odd and H is generated by an orientation preserving map of order two.

REMARK 2. If $H=\langle\sigma\rangle$ and σ has fixed points and prime order $p>2$, then by looking at the formula in [5] and the matrices in [4], it is easy to see that there are non-conjugate groups H' which induces the same or conjugate matrices on $H_1(X, \mathbb{Z})$. Thus $\Psi(n, H)>1$.

REFERENCES

1. N. L. Alling and N. Greenleaf, *Foundations of the theory of Klein surfaces*, Lecture Notes in Math., vol. **219,** Springer-Verlag, New York, 1971.

2. C. J. Earle, *On the moduli of closed Riemann surfaces with symmetries*, Advances in the Theory of Riemann surfaces, Ann. of Math, studies no. **66,** Princeton University Press 1971, pp. 119–130.

3. C. J. Earle, *A fact about matrices*, preprint.

4. J. Gilman, *A matrix representation for automorphisms of compact Riemann surfaces*, Linear Algebra and its applications, **17** (1977), pp. 139–147.

5. J. Gilman, *On conjugacy classes in the Teichmüller modular group*, Mich. Math. J., **23** (1976), pp. 53–63.

6. Kato, Takao, *Analytic self mappings inducing the identity on* $H_1(W, \mathbb{Z}/m\mathbb{Z})$, to appear.

7. R. Zarrow, *A canonical form for symmetric and skew-symmetric extended symplectic modular matrices with applications to Riemann surface theory*, Trans. Amer. Math. Soc., **204** (1975), pp. 207–227.

DEPARTMENT OF MATHEMATICAL SCIENCES
NORTHERN ILLINOIS UNIVERSITY
DEKALB, ILLINOIS 60115