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MONOMORPHISMS OF SEMIGROUPS OF LOCAL DENDRITES

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1. Introduction. When we speak of the semigroup of a topological space X, we mean $S(X)$ the semigroup of all continuous selfmaps of X. Let h be a homeomorphism from a topological space X onto a topological space Y. It is immediate that the mapping which sends $f \in S(X)$ into $h \circ f \circ h^{-1}$ is an isomorphism from the semigroup of X onto the semigroup of Y. More generally, let h be a continuous function from X into Y and k a continuous function from Y into X such that $k \circ h$ is the identity map on X. One easily verifies that the mapping which sends f into $h \circ f \circ k$ is a monomorphism from $S(X)$ into $S(Y)$. Now for "most" spaces X and Y, every isomorphism from $S(X)$ onto $S(Y)$ is induced by a homeomorphism from X onto Y. Indeed, a number of the early papers dealing with $S(X)$ were devoted to establishing this fact. See [**4**], Chapter 1, Section 2 for a discussion and references. This is in considerable contrast, however, to the situation for monomorphisms from $S(X)$ into $S(Y)$. There are many instances of monomorphisms which are not of the type described previously. For example, let X be any space with more than one point and for $f \in S(X)$ define $\varphi(f) \in S(X^2)$ by

$$(\varphi(f))(x, y) = (f(x), f(y)).$$

One easily checks that φ is a monomorphism from $S(X)$ into $S(X^2)$ but it can be shown (we discuss this further in Section 3), that it is not induced by two functions in the manner described above. On the other hand, the mapping ψ defined by

$$(\psi(f))(x, y) = (f(x), f(x))$$

is a monomorphism from $S(X)$ into $S(X^2)$ which is induced by two continuous functions. Specifically, $\psi(f) = h \circ f \circ k$ where h and k are given by $h(x) = (x, x)$ and $k(x, y) = x$.

Our purpose here is to present a class of spaces and then find conditions on pairs X and Y from the class so that every monomorphism from $S(X)$ into $S(Y)$ is induced by two continuous functions. The spaces are the local dendrites with finite branch numbers. These spaces arose naturally in the solution of a seemingly unrelated problem [**7**]. The precise formulation of

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the Monomorphism Theorem is the main result of this paper. It is formally stated in Section 2 which is devoted almost entirely to its proof. It can be regarded, in some sense as being the dual of one of the main results in [**9**]. This and some other observations are discussed further in Section 3.

2. The monomorphism theorem.

Definition (2.1). A monomorphism φ from $S(X)$ into $S(Y)$ is said to be *natural* if there exist continuous functions h from X into Y and k from Y into X such that $k \circ h$ is the identity on X and $\varphi(f) = h \circ f \circ k$ for each $f \in S(X)$.

We need to review some terminology. We recall that a dendrite is any Peano continuum which contains no simple closed curves and a local dendrite is a Peano continuum with the property that each point is contained in a neighborhood which is a dendrite. For general information on dendrites, we suggest that either [**3**] or [**12**] be consulted. A discussion of local dendrites can be found in [**3**]. Let X be any local dendrite and let $x \in X$. Choose any neighborhood D of X which is a dendrite. We showed in [**7**] that the number of components of $D - \{x\}$ does not depend on D and we define the rank of x in X (denoted Rank (x, X)) to be this number. We note that what we call rank here was called local rank in [**7**]. It is known that for any local dendrite X and any point $x \in X$, Rank $(x, X) \leqq \aleph_0$. A point x is an endpoint if Rank $(x, X) = 1$, a local cutpoint if Rank $(x, X) > 1$ and a branch point if Rank $(x, X) > 2$. The *branch number* of a local dendrite X is denoted by $\mathrm{Brn}(X)$ and is the sum of all the ranks of the branch points of X. Evidently, $\mathrm{Brn}(X)$ is finite, if and only if X has only finitely many branch points and each branch point has finite rank. In any event, $\mathrm{Brn}(X) \leqq \aleph_0$ for any local dendrite X. We are now in a position to state the

MONOMORPHISM THEOREM. *Let X and Y be local dendrites with finite branch numbers and suppose that for each copy Z of X contained in Y,* $\mathrm{Rank}(x, Z) = \mathrm{Rank}(x, Y)$ *for each branch point x of Z. Then every monomorphism from $S(X)$ into $S(Y)$ is natural.*

We accomplish the proof through a sequence of lemmas the first of which is

LEMMA (2.3). *Let Y be any topological space whatsoever and let $L(Y)$ be a left zero subsemigroup of $S(Y)$. Then the ranges of all functions in $L(Y)$ are mutually homeomorphic.*

Proof. Let v and w be any two elements of $L(Y)$. Since $v \circ w \circ v = v$ and $w \circ v \circ w = w$, it readily follows that $v \circ w$ restricted to Ran v (the range of v) is the identity and similarly, $w \circ v$ restricted to Ran w is the identity. From this, it follows that v maps Ran w homeomorphically onto Ran v and, of course, w maps Ran v homeomorphically onto Ran w.

LEMMA (2.4). *Let* f, $v \in S(Y)$ *with* v *idempotent and suppose that* $\mathrm{Ran}\, f \subset \mathrm{Ran}\, v$ *and* $v \circ f = v$. *Then* $f = v$.

Proof. Since v is idempotent, it is the identity on its range and thus

$$f(x) = v(f(x)) = v(x) \quad \text{for all } x \in Y.$$

LEMMA (2.5). *Let* Y *be a local dendrite with finite branch number and let* $\mathcal{S}$ *be an uncountable collection of mutually homeomorphic nondegenerate subcontinua of* Y *such that* $E \not\subset F$ *for each pair* E, $F \in \mathcal{S}$. *Then* $\mathrm{bd}\, V \cap \mathrm{int}\, W \neq 0$ *for some pair* V, $W \in \mathcal{S}$ *where* bd *and* int *denote respectively boundary and interior* (*with respect to* Y).

Proof. We first dispose of the case where $\mathcal{S}$ consists of arcs. Let E denote the collection of endpoints of the arcs in $\mathcal{S}$. Then E is uncountable and, since Y is separable, E has uncountably many condensation points [**2**, p. 251]. Choose such a point e which is not a branch point of Y. Then every neighborhood of e contains uncountably many points of E and it follows from this that there exist arcs A, $B \in \mathcal{S}$ such that

$$\mathrm{bd}\, A \cap \mathrm{int}\, B \neq \emptyset.$$

Now we go to the case where no subspace in $\mathcal{S}$ is an arc. Since Y is a local dendrite, it contains at most finitely many simple closed curves [**3**, p. 304] and since Y has finite branch number, it also contains only finitely many branch points. The same is true for any subcontinuum. Since no $A \in \mathcal{S}$ is an arc, it follows from Lemma (2.10) of [**11**] that each such A must contain at least one simple closed curve or one branch point. It readily follows (since the subcontinua in $\mathcal{S}$ are mutually homeomorphic) that there is an uncountable subcollection $\mathcal{S}_1$ of $\mathcal{S}$ such that all the subcontinua of $\mathcal{S}_1$ contain precisely the same simple closed curves and the same branch points. Let

$$\mathcal{S}_1 = \{D_\alpha : \alpha \in \Lambda\},$$

choose any $D_1 \in \mathcal{S}_1$ and consider the subspaces $D_1 - D_\alpha$, $\alpha \in \Lambda$. Let A be a component of $D_1 - D_\alpha$. Since D_1 and D_α contain precisely the same branch points and simple closed curves, it follows that $\mathrm{cl}\, A$ is an arc with endpoints a and b where $\{b\} = \mathrm{cl}\, A - A$. Since a is an endpoint of D_1, it follows from Lemma (3.6) of [**7**] that $D_1 - D_\alpha$ consists of only finitely many components. Denote them by $\{A_{\alpha_j}\}_{j=1}^{n_\alpha}$. Then each $\mathrm{cl}\, A_{\alpha_j}$ is an arc and in keeping with our previous notation, we let a_{α_j} and b_{α_j} denote the endpoints of $\mathrm{cl}\, A_{\alpha_j}$ where

$$\mathrm{cl}\, A_{\alpha_j} - A_{\alpha_j} = \{b_{\alpha_j}\}.$$

If some b_{α_j} is not a branch point of Y then the conclusion follows immediately for in this case

$$b_{\alpha_j} \in \mathrm{bd}\, D_\alpha \cap \mathrm{int}\, D_1.$$

We assume, therefore, that

(2.5.1) each b_{α_j} is a branch point of Y.

Denote by B the collection of branch points of Y which are not branch points of any of the D_α. The set B is, of course, finite and we let $\{B_i\}_{i=1}^N$ denote the collection of all nonempty subsets of B. Our assumption (2.5.1) means simply that for each α,

$$\mathrm{cl}(D_1 - D_\alpha) - (D_1 - D_\alpha)$$

is one of the sets B_i. Consequently, there is an uncountable subset Λ_1 of Λ such that for all $\alpha \in \Lambda_1$, the sets

$$\mathrm{cl}(D_1 - D_\alpha) - (D_1 - D_\alpha)$$

consist of precisely the same points. We want to show next that

(2.5.2) $\quad D_1 - D_\alpha = D_1 - D_\beta \quad$ for all $\alpha, \beta \in \Lambda_1$.

It is immediate that $D_1 - D_\alpha$ and $D_1 - D_\beta$ must have the same number of components. Let $\{A_i\}_{i=1}^M$ be the components of the former and $\{E_i\}_{i=1}^M$ the components of the latter. Consider A_1 and let

$$\mathrm{cl}\, A_1 - A_1 = \{b\}.$$

Then

$$\mathrm{cl}\, E_i - E_i = \{b\} \quad \text{for some } i.$$

Let a denote the remaining endpoint of cl A_1 and d the remaining endpoint of cl E_i. The assumption $a \neq d$ implies that cl A_1 contains a branch point of D_1. But this point would then have to be a branch point of D_α (and D_β as well) and this is a contradiction since no point of cl A_1 can be a branch point of D_α. Thus $b = d$ and we conclude that a and b are the endpoints of both A_1 and E_i. The further assumption that the arcs cl A_1 and cl E_i do not coincide leads quickly to the contradiction that D_1 contains a simple closed curve which does not belong to either D_α or D_β. Thus, we must have cl A_1 = cl E_i and hence $A_1 = E_i$. We have now shown that every component of $D_1 - D_\alpha$ is also a component of $D_1 - D_\beta$. Similarly, each component of $D_1 - D_\beta$ is also a component of $D_1 - D_\alpha$ and (2.5.2) has been verified.

The next step is to consider all subspaces of the form $D_\alpha - D_1$ where $\alpha \in \Lambda_1$. Just as in our previous considerations, the components of each $D_\alpha - D_1$ are finite in number and we denote them by $\{H_{\alpha_i}\}_{i=1}^{M_\alpha}$. Each cl H_{α_i} is an arc and we let

$$\mathrm{cl}\, H_{\alpha_i} - H_{\alpha_i} = \{p_{\alpha_i}\}.$$

Again, if any p_{α_i} is not a branch point of Y, we are finished so we assume, as before, that

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(2.5.3) each p_{α_i} is a branch point of Y.

The arguments used previously carry over intact to show that (2.5.3) implies that there exists an uncountable subset Λ_2 of Λ_1 such that

(2.5.4) $\quad D_\alpha - D_1 = D_\beta - D_1 \quad$ for all $\alpha, \beta \in \Lambda_2$.

We are just about finished for all we need to do is choose two distinct α and β in Λ_2 and (2.5.2) and (2.5.4) assure us that

$$D_\alpha - D_1 = D_\beta - D_1$$

and at the same time

$$D_1 - D_\alpha = D_1 - D_\beta.$$

This implies $D_\alpha = D_\beta$ which is a contradiction since the subspaces $\{D_\alpha : \alpha \in \Lambda_2\}$ are all distinct. Consequently, either (2.5.1) is false or (2.5.3) is false. In either case we have the desired conclusion and the lemma has been proved.

Now we are ready to prove a lemma which is particularly crucial to our considerations here.

LEMMA (2.6.) *Let Y be a local dendrite with finite branch number and let $L(Y)$ be an uncountable left zero subsemigroup of $S(Y)$. Then each function in $L(Y)$ is a constant function.*

Proof. Suppose some function in $L(Y)$ is not constant. Then according to Lemma (2.3), no function in $L(Y)$ is constant and Lemmas (2.3), (2.4) and (2.5) together imply that there exist distinct functions v and w in $L(Y)$ such that

$$\text{bd } V \cap \text{int } W \neq \emptyset$$

where $V = \text{Ran } v$ and $W = \text{Ran } w$. Choose any point

$$p \in \text{bd } V \cap \text{int } W.$$

Since v is idempotent, we have $v(p) = p$ which implies

(2.6.1) $\quad p \in \text{int } W \cap v^{-1}(\text{int } W) \cap \text{cl}(Y - V)$.

This, in turn, implies that

(2.6.2) $\quad \text{int } W \cap v^{-1}(\text{int } W) \cap (Y - V) \neq \emptyset$,

and we choose any point q in the latter set. We then have

(2.6.3) $\quad q \in W - V$

and

(2.6.4) $\quad v(q) \in W$.

Now (2.6.3) implies that $v(q) \neq q = w(q)$ while (2.6.4), together with the fact that w is idempotent, implies that $w(v(q)) = v(q)$. This contradicts the fact that $w \circ v = w$ and we conclude that $L(Y)$ does, indeed, consist of constant functions.

We are now ready to complete the

Proof of the monomorphism theorem. Let φ be a monomorphism from $S(X)$ into $S(Y)$. Let $L(X)$ denote the collection of constant functions of X. Then $L(X)$ is a left zero subsemigroup of $S(X)$ and thus $\varphi[L(X)]$ is an uncountable left zero subsemigroup of $S(Y)$. It follows from Lemma (2.6) that for each $x \in X$, there exists a unique y in Y such that $\varphi\langle x\rangle = \langle y\rangle$ where $\langle x\rangle$ and $\langle y\rangle$ denote the constant functions which map everything into the points x and y respectively. We define a function h from X into Y by $h(x) = y$. We note that

(MT 1) $\quad \varphi\langle x\rangle = \langle h(x)\rangle \quad$ for each $x \in X$

and also that h is injective since φ is. Now take any $x \in X$ and $f \in S(X)$ and use (MT 1) to get

$$\begin{aligned}\langle (\varphi(f))(h(x))\rangle &= \varphi(f) \circ \langle h(x)\rangle \\ &= \varphi(f) \circ \varphi\langle x\rangle = \varphi(f \circ \langle x\rangle) = \varphi\langle f(x)\rangle \\ &= \langle h(f(x))\rangle.\end{aligned}$$

That is, the constant functions $\langle (\varphi(f))(h(x))\rangle$ and $\langle h(f(x))\rangle$ coincide, which means that

$$(\varphi(f))(h(x)) = h(f(x)).$$

We have shown that

(MT 2) $\quad (\varphi(f)) \circ h = h \circ f \quad$ for each $f \in S(X)$.

It follows readily from (MT 2) that

(MT 3) $\quad h(f^{-1}(x)) = h(X) \cap (\varphi(f))^{-1}(h(x))$
for each $x \in X$ and $f \in S(X)$.

It is an easy matter to verify that

$$\{f^{-1}(z): z \in Z, f \in S(Z)\}$$

is a basis for the closed subsets of Z whenever Z is any completely regular Hausdorff space which contains an arc. This fact together with (MT 3) implies that h^{-1} is a continuous function from $h(X)$ onto X. To show that h is continuous, let $\{x_n\}_{n=1}^{\infty}$ be a sequence of distinct points in X which converge to $p \in X$ where p is distinct from all the points x_n. We will show that the sequence $\{h(x_n)\}_{n=1}^{\infty}$ converges to $h(p)$. Since h is injective, $h(X)$ is an uncountable subset of Y which inherits a second countable topology from Y. Consequently, $h(X)$ has a condensation point q [**2**, p. 251] so we can choose an infinite sequence of points $\{y_n\}_{n=1}^{\infty}$ all different from each other and from q such that $\lim y_n = q$. Then

$$K = [\cup\{y_n\}_{n=1}^{\infty}] \cup \{q\}$$

is a compact subspace of $h(X)$ and since h^{-1} is continuous on $h(X)$, its restriction to K is a homeomorphism. Thus, $h^{-1}[K]$ is a compact subspace of X which contains precisely one limit point $h^{-1}(q)$. Since X is a local dendrite, there exists a neighborhood D of $h^{-1}(q)$ which is a dendrite and hence there exists a positive integer N such that

$$h^{-1}(y_n) \in D \quad \text{for } n \geqq N.$$

Define a map f from

$$H = [\cup\{h^{-1}(y_n)\}_{n=N}^{\infty}] \cup \{h^{-1}(q)\}$$

onto

$$W = [\cup\{x_n\}_{n=N}^{\infty}] \cup \{p\}$$

by

$$f(h^{-1}(y_n)) = x_n \quad \text{and} \quad f(h^{-1}(q)) = p.$$

The function f is a continuous map from a closed subspace (namely H) of X into the dendrite D. Since dendrites are absolute retracts [**1**, p. 138], f has a continuous extension to a function which maps all of X into D. We do not expect confusion to result so we use the symbol f to denote the extension as well. From (MT 2) we get

$$h(x_n) = h(f(h^{-1}(y_n))) = (\varphi(f))(y_n)$$

for $n \geqq N$ and similarly,

$$h(p) = h(f(h^{-1}(q))) = (\varphi(f))(q).$$

Since $\lim y_n = q$ and $\varphi(f) \in S(Y)$, we have

$$\lim(\varphi(f))(y_n) = (\varphi(f))(q)$$

and this establishes the fact that

(MT 4) h is a homeomorphism from X into Y.

Next, we let $\varphi(i) = v$ where i is the identity of $S(X)$ and we further let $V = \text{Ran } v$. It follows from (MT 2) that $v \circ h = h$ which, in turn, implies that $h(X) \subset V$. We will show that, in fact,

(MT 5) $h(X) = V$.

Suppose to the contrary. Since h is a homeomorphism, $h(X)$ is a local dendrite properly contained in V which is also a local dendrite. Since V is

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connected, it follows that there exists a point p such that

(MT 6) $p \in h(X) \cap \mathrm{cl}_V(V - h(X))$.

We assert that

(MT 7) p is not a branch point of $h(X)$.

Suppose it is. Then

$$\mathrm{Rank}(p, h(X)) = \mathrm{Rank}(p, Y).$$

This means that there exists an open subset G of Y such that $p \in G \subset h(X)$. But since $p \in \mathrm{cl}_V(V - h(X))$ we also have

$$G \cap (V - h(X)) \neq \emptyset$$

which is a contradiction so that (MT 7) is valid. Thus,

$$\mathrm{Rank}(p, h(X)) \leqq 2.$$

We remark that while p is not a branch point of $h(X)$, it may well be a branch point of Y. At any rate, there exists a neighborhood D of p in $h(X)$ which is a dendrite such that

(MT 8) $\mathrm{Rank}(p, D) \leqq 2$.

Moreover, since Y has only finitely many branch points, D can be chosen with sufficiently small diameter so that the only branch point of Y is the point p itself. We choose such a D which, by Lemma (2.10) of **[11]** must necessarily be an arc and we denote its endpoints by a and b. It follows that

(MT 9) $D - \{a, b, p\}$ is an open subset of Y.

We have two cases to consider depending upon the rank of p in D.

Case 1. $\mathrm{Rank}(p, D) = 2$.

Then $D - \{a, b, p\}$ is the disjoint union of two open subsets A and B of Y where

$$A \cup B \cup \{p\} = D - \{a, b\}.$$

The set

$$h^{-1}(D) - \{h^{-1}(a), h^{-1}(b)\}$$
$$= h^{-1}(A) \cup h^{-1}(B) \cup \{h^{-1}(p)\}$$

is an open subset of X and we take f to be any homeomorphism from X onto X such that

(MT 10) $f(h^{-1}(p)) \in h^{-1}(A)$.

For example, one could define f to be the identity on the complement of $h^{-1}(D) - \{h^{-1}(a), h^{-1}(b)\}$ and then define it on $h^{-1}(D) - \{h^{-1}(a), k^{-1}(b)\}$ in such a manner that it "shifts" the point $h^{-1}(p)$ into $h^{-1}(A)$

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(we could just as well have chosen $h^{-1}(B)$). We then use both (MT 2) and (MT 10) to get

$$\begin{aligned}\varphi(f)(p) &= \varphi(f)(h(h^{-1}(p))) \\ &= h(f(h^{-1}(p))) \in h(h^{-1}(A)) = A\end{aligned}$$

and since A is open in Y, there exists an open subset G of Y containing p such that

$$\varphi(f)(G) \subset A.$$

Because of (MT 6) there exists a point q such that

(MT 11) $q \in G \cap (V - h(X))$

and we have

(MT 12) $\varphi(f)(q) \in A \subset h(X)$.

We use (MT 12) and the fact that $q \in V$ to get

$$\begin{aligned}q &= v(q) = \varphi(i)(q) = \varphi(f^{-1} \circ f)(q) \\ &= \varphi(f^{-1})(\varphi(f)(q)) \in \varphi(f^{-1})(h(X)).\end{aligned}$$

Thus,

$$q = \varphi(f^{-1})(h(x)) \quad \text{for some } x \in X.$$

But it then follows from (MT 2) that

$$q = \varphi(f^{-1})(h(x)) = h(f^{-1}(x)) \in h(X)$$

which contradicts (MT 11). Hence, in the case Rank$(p, D) = 2$ we have obtained a contradiction. We will next show that

Case 2. Rank$(p, D) = 1$ is contradictory as well. Here, p is an endpoint of D and we denote the remaining endpoint by a. In this case, we take f to be the identity map on $X - h^{-1}(D)$ and define it on $h^{-1}(D)$ in such a manner that it fixes $h^{-1}(a)$ and maps $h^{-1}(D)$ homeomorphically onto a proper subarc of $h^{-1}(D)$. In particular, $f(h^{-1}(p))$ will be a cutpoint of $h^{-1}(D)$. We note that, among other things, f is a homeomorphism from X onto a proper subspace of X. Now define a function g by

$$\begin{aligned}g(x) &= f^{-1}(x) \quad \text{for } x \in \operatorname{Ran} f \quad \text{and} \\ g(x) &= h^{-1}(p) \quad \text{for } x \in X - \operatorname{Ran} f.\end{aligned}$$

Then $g \in S(X)$ and $g \circ f = i$. From this point on, the procedure is similar to that used in Case 1. We have

$$\begin{aligned}\varphi(f)(p) &= \varphi(f)(h(h^{-1}(p))) \\ &= h(f(h^{-1}(p))) \in h(h^{-1}(D) - \{h^{-1}(a), h^{-1}(p)\} \\ &= D - \{a, p\}.\end{aligned}$$

Since $D - \{a, p\}$ is open in Y, there exists an open subset G of Y containing p such that

(MT 13) $\varphi(f)(G) \subset D - \{a, b\} \subset h(X)$.

Again (MT 6) assures us that there is a point q such that

(MT 14) $q \in G \cap (V - h(X))$.

Just as before, we get

$$q = v(q) = \varphi(i)(q) = \varphi(g \circ f)(q)$$
$$= \varphi(g)(\varphi(f)(q)) \in \varphi(g)(h(X))$$

which means

$$q = \varphi(g)(h(x)) \quad \text{for some } x \in X.$$

But this implies that $q = h(g(x)) \in h(X)$ which contradicts (MT 14).

We have now shown that in any case, the assumption that $h(X) \neq V$ results in a contradiction so we conclude that (MT 5) is valid. This enables us to define a continuous function k from Y onto X. Specifically, we take $k = h^{-1} \circ v$. Since $h(X) = V$, it is immediate that $k \circ h = i$. Finally, we take any $f \in S(X)$ and we use (MT 2) once again to get

$$\varphi(f) = \varphi(f \circ i) = \varphi(f) \circ \varphi(i)$$
$$= \varphi(f) \circ v = \varphi(f) \circ h \circ h^{-1} \circ v$$
$$= h \circ f \circ h^{-1} \circ v = h \circ f \circ k.$$

Thus, φ is a natural monomorphism and the proof of the Monomorphism Theorem has now been completed.

3. Miscellaneous observations and closing remarks. We first discuss in a bit more detail an assertion made in the introduction. We took any space X with more than one point and defined a monomorphism φ from $S(X)$ into $S(X^2)$ by

(3.1.1) $\quad (\varphi(f))(x, y) = (f(x), f(y))$.

We asserted that φ is not a natural monomorphism. Suppose to the contrary that it is. In other words suppose there exist continuous functions h and k from X into Y and Y into X ($Y = X^2$) respectively such that

(3.1.2) $\quad k \circ h$ is the identity on X

and

(3.1.3) $\quad \varphi(f) = h \circ f \circ k \quad$ for all $f \in S(X)$.

Let i_X and i_Y denote the identity maps on X and Y respectively. It follows from (3.1.1) that $\varphi(i_X) = i_Y$. This, together with (3.1.3) gives

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(3.1.4) $i_Y = \varphi(i_X) = h \circ i_X \circ k = h \circ k.$

(3.1.2) and (3.1.4) together imply that h is a homeomorphism from X onto X^2. However, from (3.1.1) and (3.1.3) we get

$$h(x) = \langle h(x) \rangle(x, x) = (\varphi\langle x \rangle)(x, x)$$
$$= (\langle x \rangle(x), \langle x \rangle(x)) = (x, x)$$

for each $x \in X$. This is a contradiction since X has more than one point and consequently h cannot possibly map X onto X^2.

We also mentioned in the introduction that the Monomorphism Theorem is, in some sense, a dual to a result we obtained in [**9**]. The Monomorphism Theorem tells us that for certain semigroups $S(X)$ and $S(Y)$, every monomorphism φ from $S(X)$ into $S(Y)$ is given by $\varphi(f) = h \circ f \circ k$ where

(3.2.1) h is a homeomorphism from X into Y,
(3.2.2) k is a continuous function from Y onto X and
(3.2.3) $k \circ h$ is the identity on X.

In Theorem A of [**9**], we showed that for certain $S(X)$ and $S(Y)$, every epimorphism φ from $S(X)$ onto $S(Y)$ is of the form $\varphi(f) = h \circ f \circ k$ where, in this case,

(3.2.1)′ h is a continuous function from X onto Y,
(3.2.2)′ k is a homeomorphism from Y into X and
(3.2.3)′ $h \circ k$ is the identity on Y.

There are a number of contrasts regarding monomorphisms and epimorphisms. For one thing, conditions (3.2.1), (3.2.2) and (3.2.3) are sufficient to insure that the map φ defined by $\varphi(f) = h \circ f \circ k$ is a monomorphism from $S(X)$ into $S(Y)$ while (3.2.1)′, (3.2.2)′ and (3.2.3)′ are not sufficient to insure that the corresponding map φ will be an epimorphism. φ will certainly map $S(X)$ onto $S(Y)$ but it need not be a homomorphism. The mapping φ will be a homomorphism if, in addition, h is constant on components of X and both x and $k(h(x))$ lie in the same component for each $x \in X$ (see Theorem B of [**9**]).

As for other contrasts, monomorphisms are abundant while epimorphisms are rare. Moreover, if an epimorphism does exist from $S(X)$ onto $S(Y)$ it is generally induced by two continuous functions while there are many pairs of semigroups where some monomorphisms are induced by continuous functions (i.e., are natural monomorphisms) while others are not.

There are results already in the literature, notably Theorem (3.4) of [**10**], which describe other pairs of spaces X and Y such that every monomorphism from $S(X)$ into $S(Y)$ is natural. In that result, one of the conditions on X is that it be quasi-homogeneous. This means that for each nonempty open subset G of X and each point $p \in X$ there exist a pair of continuous self-maps f and g of X such that $g(p) \in G$ and $f \circ g$ is the identity on X. A little reflection will convince one that local dendrites fail badly at being quasi-homogeneous. In fact, one can show that the only quasi-homogeneous local dendrites are the arc and the simple closed curve.

Finally, a few remarks about local dendrites (particularly those with finite branch number) are in order. It turns out that the answers to various natural questions about $S(X)$ involve local dendrites. We have already seen one such example in this paper. For another example, in **[5]** and **[6]** we posed the problem of characterizing those Peano continua X such that $S(X)$ has only finitely many regular $\mathscr{D}$-classes. This class of spaces turns out to be precisely the local dendrites with finite branch numbers [7]. For yet another example, we showed in **[8]** that if X is any local dendrite with finite branch number, then Green's $\mathscr{D}$ and $\mathscr{J}$ relations coincide on the regular elements of $S(X)$.

References

1. K. Borsuk, *Theory of retracts*, Polska Akademia Nauk Monografie Matatyczne (Polish Scientific Publishers, Warszawa, 1967).

2. K. Kuratowski, *Topology*, Vol. I (Academic Press, New York and London, 1966).

3. ——— *Topology*, Vol. II (Academic Press, New York and London, 1968).

4. K. D. Magill, Jr., *A survey of semigroups of continuous self-maps*, Semigroup Forum *11* (1975/76), 189-282.

5. ——— *Some open problems and directions for further research in semigroups of continuous selfmaps*, Universal Algebra and Applications, Banach Center Pub. *9* (PWN-Polish Sci. Pub., Warsaw, 1980).

6. ——— *Recent results and open problems in semigroups of continuous selfmaps*, (Russian) Uspekhi Mat. Nauk (35) *3* (1980), 72-78; (English Translation) Russian Math Survey (35) *3* (1980), 77-91.

7. ——— *Semigroups with only finitely many regular $\mathscr{D}$-classes*, Semigroup Forum *25* (1982), 361-377.

8. ——— *Semigroups in which $\mathscr{D}$ and $\mathscr{J}$ coincide for regular elements*, Semigroup Forum *25* (1982), 383-385.

9. K. D. Magill, P. R. Misra and U. B. Tewari, *Epimorphisms from $S(X)$ onto $S(Y)$* (to appear).

10. K. D. Magill and S. Subbiah, *Embedding $S(X)$ into $S(Y)$*, Dissertiones Math. *120* (1974), 1-47.

11. ——— *Regular $\mathscr{J}$-classes of semigroups of continua*, Semigroup Forum *22* (1981), 159-179.

12. G. Whyburn, *Analytic topology* (AMS Colloquium Publication, New York, 1942).

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