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ABSOLUTE CONTINUITY FOR GROUP-VALUED MEASURES

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In this note we generalize the following classical theorem:

If μ and ν are finite real-valued measures such that $\nu(A)=0$ implies $\mu(A)=0$, then for every $\varepsilon>0$, there exists $\delta>0$ such that $\mu(A)<\varepsilon$ whenever $\nu(A)<\delta$.

The corresponding result is known to hold when μ has values in a locally convex space and ν is real-valued (Rickart [1, Theorem 1.3]). We give an extension to the case of group-valued measures, valid whenever the dominating measure ν has metrizable range.

Notation. In the following, $\mathbf{A}$ is a σ-ring of sets, Y and Z are commutative Hausdorff topological groups (written additively), and μ and ν are σ-additive functions on $\mathbf{A}$ to Y and Z, respectively.

1. Definitions.

1. A is ν-null iff $A \in \mathbf{A}$ and $\nu(E)=0$, whenever $A \supset E \in \mathbf{A}$.
2. μ is ν-continuous ($\mu \ll \nu$) iff $\mu(A)=0$, whenever A is ν-null.
3. μ is topologically ν-continuous ($\mu \ll_t \nu$) iff for every neighborhood V of 0 in Y, there exists a neighborhood W of 0 in Z such that $\mu(A) \in V$, whenever $A \in \mathbf{A}$ and $\nu(E) \in W$, for every E in $\mathbf{A}$ contained in A.

(Thus, $\mu \ll_t \nu$ means that μ is a continuous function on $\mathbf{A}$ to Y when $\mathbf{A}$ is given the uniform topology induced by ν (Cf. Sion [2]).

2. Theorem.

1. *If $\mu \ll_t \nu$, then $\mu \ll \nu$.*
2. *If $\mu \ll \nu$ and Z is metrizable, then $\mu \ll_t \nu$.*

Proof. The first statement is an immediate consequence of the definitions. On the other hand, Z is metrizable iff there exists a countable base $\{W_n : n \text{ in } \mathbb{N}\}$ for the neighborhoods of 0 in Z, consisting of closed sets. (The symbol $\mathbb{N}$ denotes the

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nonnegative integers.) By continuity of addition, we may assume that, for all n in $\mathbb{N}$, $W_{n+1}+W_{n+1} \subset W_n$, so that, by induction.

$$\sum_{i=n+1}^{m} W_i \subset W_n, \qquad \text{for all } n < m \text{ in } \mathbb{N}.$$

Now, suppose $\mu \ll \nu$ but μ is not topologically ν-continuous. Then, for some neighborhood V of 0 in Y, there exists a sequence A in $\mathbf{A}$ such that, for all n in $\mathbb{N}$,

$$\nu(E \cap A_n) \in W_n, \qquad \text{for all } E \text{ in } \mathbf{A}, \qquad \text{but } \mu(A_n) \notin V.$$

For each n, put $B_n = \bigcup_{i \geq n} A_n$ and $B'_n = B_n \backslash B_{n+1}$. As in the standard real-valued proof, the set $\bigcap_n B_n$ is ν-null. Indeed, if $n < m$ and $E \in \mathbf{A}$, we have

$$\sum_{i=n+1}^{m} \nu(E \cap B'_i) \in \sum_{i=n+1}^{m} W_i \subset W_n,$$

and hence $\nu(E \cap B_{n+1}) = \sum_{i=n+1}^{\infty} \nu(E \cap B'_i) \in W_n$. Thus, $\nu(E \cap \bigcap_n B_n) = \lim_n \nu(E \cap B_n) = 0$, for all E in $\mathbf{A}$, whence $\bigcap_n B_n$ is ν-null.

Now, since $\mu \ll \nu$, $\bigcap_n B_n$ is also μ-null. Therefore, for each E in $\mathbf{A}$,

$$(*) \quad \lim_n \mu(E \cap B_n) = \mu(E \cap \bigcap_n B_n) = 0.$$

Let V' be a neighborhood of 0 in Y such that $V' + V' \subset V$. Put $n_0 = 0$ and use $(*)$ to choose, by recursion, n_k in $\mathbb{N}$ such that

$$n_{k+1} > n_k \quad \text{and} \quad \mu(A_{n_k} \cap B_{n_{k+1}}) \in V', \qquad \text{for all } k \text{ in } \mathbb{N}.$$

For each k in $\mathbb{N}$, put $C_k = A_{n_k} \backslash B_{n_{k+1}}$. If $\mu(C_k)$ were in V', we would have $\mu(A_{n_k}) = \mu(A_{n_k} \cap B_{n_{k+1}}) + \mu(C_k) \in V' + V' \subset V$. Thus, $\mu(C_k)$ never belongs to V'. Yet, the C_k's are disjoint, so $\mu(C_k)$ tends to 0 by σ-additivity. This contradiction completes the proof.

3. Example. The condition that Z be metrizable cannot be eliminated.

Let $\mathbf{A}$ be the Lebesgue measurable sets of the unit interval I; let Z be the bounded real functions on I under pointwise convergence; and let $\nu(A)$ be the characteristic function of A, for each A in $\mathbf{A}$. Since $\varnothing$ is the only ν-null set, Lebesgue measure is ν-continuous. On the other hand, basic neighborhoods in Z are of the form $W = \{f \in Z : |f(x)| \leq \varepsilon \text{ for all } x \in T\}$, where T is a finite subset of I. For any such W, we have $\nu(E) \in W$, whenever $I \backslash T \supset E \in \mathbf{A}$, but $I \backslash T$ has Lebesgue measure 1. Thus, Lebesgue measure is not topologically ν-continuous.

Remark. The above definitions of absolute continuity make sense also for finitely additive functions. In this case they are not equivalent.

Lebesgue decompositions for these notions appear in (3).

References

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