COMPACTIFICATIONS OF TOTALLY BOUNDED QUASI-UNIFORM SPACES

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1. Introduction. The notation and terminology of this paper coincide with that of reference [**4**], except that here the term, compactification, refers to a T_1-space. It is known that a completely regular totally bounded Hausdorff quasi-uniform space $(X, \mathcal{V})$ has a Hausdorff compactification if and only if $\mathcal{V}$ contains a uniformity compatible with $\mathcal{T}(\mathcal{V})$ [**4**, Theorem 3.47]. The use of regular filters by E. M. Alfsen and J. E. Fenstad [**1**] and O. Njåstad [**5**], suggests a construction of a compactification, which differs markedly from the construction obtained in [**4**]. We use this construction to show that a totally bounded T_1 quasi-uniform space has a compactification if and only if it is point symmetric. While it is pleasant to have a characterization that obtains for all T_1-spaces, the present construction has several further attributes. Unlike the compactification obtained in [**4**], the compactification given here preserves both total boundedness and uniform weight, and coincides with the uniform completion when the quasi-uniformity under consideration is a uniformity. Moreover, any quasi-uniformly continuous map from the underlying quasi-uniform space of the compactification onto any totally bounded compact T_1-space has a quasi-uniformly continuous extension to the compactification. If $\mathcal{U}$ is the Pervin quasi-uniformity of a T_1-space X, the compactification $(\check{X}, \mathcal{T}(\check{\mathcal{U}}))$ we obtain is the Wallman compactification of $(X, \mathcal{T}(\mathcal{U}))$. It follows that our construction need not provide a Hausdorff compactification, even when such a compactification exists; but we obtain a sufficient condition in order that our compactification be a Hausdorff space and note that this condition is satisfied by all uniform spaces and all normal equinormal quasi-uniform spaces. Finally, we note that our construction is reminiscent of the completion obtained by Á. Császár for an arbitrary quasi-uniform space [**2**, Section 3]; in particular our Theorem 3.7 is comparable with the result of [**2**, Theorem 3.5].

2. Preliminary results. For the sake of completeness, we begin by citing some definitions given in reference [**4**]. A quasi-uniform space $(X, \mathcal{U})$ is *point symmetric* provided that for each $U \in \mathcal{U}$ and $x \in X$ there is a symmetric $V \in \mathcal{U}$ such that $V(x) \subset U(x)$. It is useful to observe that $\mathcal{U}$ is point symmetric if and only if $\mathcal{T}(\mathcal{U}) \subset \mathcal{T}(\mathcal{U}^{-1})$. Evidently, if $\mathcal{U}$ contains a uniformity compatible with $\mathcal{T}(\mathcal{U})$, then $\mathcal{U}$ is point symmetric; the converse is false even for completely regular quasi-uniform spaces. Every compact T_1-space is point symmetric; and, since every quasi-uniform subspace of a point-symmetric quasi-uniform space is point symmetric, point symmetry is a necessary condition for a quasi-uniform space to have a compactification.

If $(X, \mathcal{U})$ is a quasi-uniform space, $\mathcal{U}^*$ denotes the coarsest uniformity that contains $\mathcal{U}$ and, for each $x \in X$, η_x^* denotes the $\mathcal{T}(\mathcal{U}^*)$-neighborhood filter of x. A filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ is a *Cauchy filter* provided that for each $U \in \mathcal{U}$ there is an

$x \in X$ such that $U(x) \in \mathcal{F}$. A quasi-uniform space $(X, \mathcal{U})$ is *totally bounded* provided that for each $U \in \mathcal{U}$ there is a finite cover $\mathcal{C}$ of X so that $C \times C \subset U$ for each $C \in \mathcal{C}$. Equivalently, $(X, \mathcal{U})$ is totally bounded provided that every ultrafilter over X is a $\mathcal{U}^*$-Cauchy filter. If A and B are subsets of a set X, $T(A, B)$ denotes $X \times X - A \times B$. If $\mathcal{U}$ is a totally bounded quasi-uniformity, $\mathcal{S} = \{T(A, B)$: for some $U \in \mathcal{U}$, $A \times B \cap U = \emptyset\}$ is a subbase for $\mathcal{U}$. Each $\mathcal{U}^*$-Cauchy filter $\mathcal{F}$ contains exactly one minimal $\mathcal{U}^*$-Cauchy filter, namely the filter that has as a base $\{U(F): U$ is a symmetric member of $\mathcal{U}^*$ and $F \in \mathcal{F}\}$.

Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $\tilde{X}$ denotes the set of all minimal $\mathcal{U}^*$-Cauchy filters on X, for each $U \in \mathcal{U}$, $\tilde{U} = \{(\mathcal{F}, \mathcal{G}) \in X \times X$: there is an $F \in \mathcal{F}$ and a $G \in \mathcal{G}$ so that $F \times G \subset U\}$ and $\tilde{\mathcal{U}}$ denotes the quasi-uniformity on $\tilde{X}$ for which $\{\tilde{U}: U \in \mathcal{U}\}$ is a base. The pair $(\tilde{X}, \tilde{\mathcal{U}})$ is called the *bicompletion* of $(X, \mathcal{U})$. Since $(\tilde{\mathcal{U}})^* = (\mathcal{U}^*)^\sim$, we always write $\tilde{\mathcal{U}}^*$ to denote this uniformity. It is a complete uniformity, and $(X, \mathcal{U})$ is quasi-unimorphic to a $\mathcal{T}(\tilde{\mathcal{U}}^*)$-dense subspace of $(\tilde{X}, \tilde{\mathcal{U}})$.

In the study of quasi-uniform spaces, the bicompletion of a quasi-uniform space is the natural analogue of the completion of a uniform space; and, since the bicompletion $(\tilde{X}, \tilde{\mathcal{U}})$ of a quasi-uniform space $(X, \mathcal{U})$ is compact if the quasi-uniform space is totally bounded, the bicompletion appears to provide the natural compactification of a totally bounded quasi-uniform space. Our first result rules out this red herring.

Proposition 2.1. *Let $(X, \mathcal{U})$ be a totally bounded T_1 quasi-uniform space. Then $\mathcal{T}(\tilde{\mathcal{U}})$ is a T_1 topology if and only if $\mathcal{U}$ is a uniformity.*

Proof. If $\mathcal{U}$ is a uniformity, $\tilde{\mathcal{U}}$ is the usual completion, which is well known to be a Hausdorff uniformity.

Now suppose that $\mathcal{T}(\tilde{\mathcal{U}})$ is a T_1 topology. Both $\mathcal{T}(\tilde{\mathcal{U}})$ and $\mathcal{T}(\tilde{\mathcal{U}}^{-1})$ are coarser than $\mathcal{T}(\tilde{\mathcal{U}}^*)$, which is compact. Thus $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{U}}^{-1}$ are point symmetric and $\mathcal{T}(\tilde{\mathcal{U}}) = \mathcal{T}(\tilde{\mathcal{U}}^{-1}) = \mathcal{T}(\tilde{\mathcal{U}}^*)$. Since $\tilde{\mathcal{U}}$ is a T_1 quasi-uniformity, $\bigcap \tilde{\mathcal{U}} = \Delta_{\tilde{X}}$ and it follows that $\tilde{\mathcal{U}}$ consists of all the $T(\tilde{\mathcal{U}}) \times T(\tilde{\mathcal{U}})$-neighborhoods of $\Delta_{\tilde{X}}$ [**4**, Theorem 1.20]. Evidently $\tilde{\mathcal{U}}$ and hence $\mathcal{U}$ is a uniformity. ■

A filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ is a *regular filter* provided that for each $F \in \mathcal{F}$ there exists an $E \in \mathcal{F}$ and a $U \in \mathcal{U}$ such that $U^{-1}(E) \subset F$. Note that in case $(X, \mathcal{U})$ is a uniform space the definition of a regular filter given here coincides with the definition of a regular filter given by Alfsen and Fenstad [**1**]. For any filter $\mathcal{F}$ on $(X, \mathcal{U})$, $\mathcal{F}^r$ denotes the filter for which $\{V^{-1}(F): V \in \mathcal{U}, F \in \mathcal{F}\}$ is a base. We omit the proof of the following proposition, since comparable results are obtained in reference [**1**].

Proposition 2.2. *Let $\mathcal{F}$ be a filter on a quasi-uniform space $(X, \mathcal{U})$.*

(a) *$\mathcal{F}$ and $\mathcal{F}^r$ have the same set of cluster points.*

(b) *Every regular filter is contained in a maximal regular filter.*

(c) *A regular filter $\mathcal{F}$ is a maximal regular filter if and only if either $X - A$ or B belongs to $\mathcal{F}$ whenever $U^{-1}(A) \subset B$.*

3. Construction of a compactification. The first result of this section demonstrates the importance of total boundedness in the forthcoming construction.

LEMMA 3.1. *Let* $(X, \mathcal{U})$ *be a quasi-uniform space. Every regular* $\mathcal{U}^*$*-Cauchy filter is a maximal regular filter, and if* $(X, \mathcal{U})$ *is totally bounded every maximal regular filter is a* $\mathcal{U}^*$*-Cauchy filter.*

Proof. Suppose that $\mathcal{F}$ is a regular $\mathcal{U}^*$-Cauchy filter on X, let A and B be subsets of X and let U be an entourage in $\mathcal{U}$ such that $U^{-1}(A) \subset B$. Let $F \in \mathcal{F}$ such that $F \times F \subset U$. If $F \cap A \neq \emptyset$, then $F \subset U^{-1}(A) \subset B$ so that $B \in \mathcal{F}$. If $F \cap A = \emptyset$, then $F \subset X - A$ and $X - A \in \mathcal{F}$. It follows from Proposition 2.2(c) that $\mathcal{F}$ is a maximal regular filter.

Now suppose that $\mathcal{U}$ is totally bounded and that $\mathcal{F}$ is a maximal regular filter. Then $\mathcal{S} = \{T(A, B): \text{for some } U \in \mathcal{U},\ A \times B \cap U = \emptyset\}$ is a subbase for $\mathcal{U}$. Let $T(A, B) \in \mathcal{S}$. Then $T(B, A)(B) \subset X - A$ so that either $X - A$ or $X - B$ belongs to $\mathcal{F}$. Since $(X - A) \times (X - A) \cup (X - B) \times (X - B) \subset T(A, B)$, we have shown that $\mathcal{F}$ is a $\mathcal{U}^*$-Cauchy filter. ■

PROPOSITION 3.2. *Let* $(X, \mathcal{U})$ *be a totally bounded quasi-uniform space and let* $\mathcal{F}$ *be a maximal regular filter on* X. *Then for each* $U \in \mathcal{U}$ *and* $F \in \mathcal{F}$, *there exists a* $x \in F$ *such that* $U(x) \cap U^{-1}(x) \in \mathcal{F}$.

Proof. Let $U \in \mathcal{U}$ and $F \in \mathcal{F}$. By the preceding lemma, $\mathcal{F}$ is a $\mathcal{U}^*$-Cauchy filter and so there is a $G \in \mathcal{F}$ such that $G \times G \subset U$. Let $x \in F \cap G$; then $U(x) \cap U^{-1}(x) \in \mathcal{F}$. ■

PROPOSITION 3.3. *Let* $(X, \mathcal{U})$ *be a totally bounded quasi-uniform space. Then every maximal regular filter is a minimal* $\mathcal{U}^*$*-Cauchy filter.*

Proof. Let $\mathcal{F}$ be a maximal regular filter. By the preceding lemma, $\mathcal{F}$ is a $\mathcal{U}^*$-Cauchy filter so that by [**4**, Proposition 3.30] it suffices to show that $\mathcal{B} = \{U(F): U \text{ is a symmetric entourage in } \mathcal{U}^* \text{ and } F \in \mathcal{F}\}$ is a base for $\mathcal{F}$. Let $F \in \mathcal{F}$. There is a $U \in \mathcal{U}$ and an $E \in \mathcal{F}$ such that $U^{-1}(E) \subset F$. Evidently, $U \cap U^{-1} \in \mathcal{B}$ and $U \cap U^{-1}(E) \subset F$. ■

PROPOSITION 3.4. *Let* $(X, \mathcal{U})$ *be a point-symmetric quasi-uniform space. Then, for each* $x \in X$, $\eta^*(x)$ *is a maximal regular filter.*

Proof. Let $x \in X$. Since $(X, \mathcal{U})$ is point symmetric, $\{U^{-1}(x): U \in \mathcal{U}\}$ is a base for $\eta^*(x)$. Let $U \in \mathcal{U}$ and let $V \subset U$ such that $V^2 \subset U$. Then $V^{-1}(V^{-1}(x)) \subset U^{-1}(x)$ and so $\eta^*(x)$ is a regular filter. The result follows from Lemma 3.1. ■

THEOREM 3.5. *Let* $(X, \mathcal{U})$ *be a point-symmetric totally bounded* T_1 *quasi-uniform space. Then* $(X, \mathcal{U})$ *has a totally bounded compactification* $(\check{X}, \check{\mathcal{U}})$ *that is a subspace of the bicompletion of* $(X, \mathcal{U})$. *Moreover, if* $\mathcal{U}$ *is a uniformity,* $(\check{X}, \check{\mathcal{U}})$ *is the uniform completion of* $(X, \mathcal{U})$.

Proof. Let $\check{X}$ denote the set of all maximal regular filters on X. By Proposition 3.3, $\check{X} \subset \tilde{X}$. For each $U \in \mathcal{U}$ let $\check{U} = \tilde{U} \cap \check{X} \times \check{X}$ and let $\check{\mathcal{U}} = \tilde{\mathcal{U}} \mid \check{X} \times \check{X}$. Since $(\tilde{X}, \tilde{\mathcal{U}})$ is totally bounded, so is $(\check{X}, \check{\mathcal{U}})$.

To show that $(\check{X}, \check{\mathcal{U}})$ is a T_1 space, let $\mathcal{F}$ and $\mathcal{S}$ be two members of X and suppose that $(\mathcal{F}, \mathcal{G}) \in \bigcap \check{\mathcal{U}}$. Since $\mathcal{F}$ and $\mathcal{G}$ are maximal regular filters, there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \cap G = \emptyset$. As $\mathcal{G}$ is a regular filter, there exist $U \in \mathcal{U}$ and $G_1 \in \mathcal{G}$ such

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that $U^{-1}(G_1) \subset G$. Since $(\mathcal{F}, \mathcal{G}) \in \check{\mathcal{U}}$, there exist $F_2 \in \mathcal{F}$ and $G_2 \in \mathcal{G}$ such that $F_2 \times G_2 \subset U$. Let $x \in F \cap F_2$ and $y \in G_1 \cap G_2$. Then $x \in F \cap U^{-1}(y) \subset F \cap U^{-1}(G_1) \subset F \cap G = \emptyset$—a contradiction.

The map $i: X \to \check{X}$ defined by $i(x) = \eta^*(x)$ is a quasi-uniform embedding and, by Proposition 3.4, $i(X) \subset \check{X}$. Furthermore $i(X)$ is a dense subspace of $(\tilde{X}, \mathcal{T}(\tilde{U}^*))$ and therefore $i(X)$ is a dense subset of $(\check{X}, \mathcal{T}(\check{\mathcal{U}}^*))$.

We show that $(\check{X}, \check{\mathcal{U}})$ is compact. By Proposition 2.2(a), it suffices to show that every regular filter on $\check{X}$ has a cluster point. Let $\mathcal{M}$ be a regular filter on $\check{X}$. Since $i(X)$ is a $T(\mathcal{U}^{-1})$-dense subset of X, $\{i^{-1}(M): M \in \mathcal{M}\}$ is a base for a filter $\mathcal{F}$ on X. It is a routine matter to show that $\mathcal{F}$ is a regular filter. Let $\mathcal{G}$ be a maximal regular filter containing $\mathcal{F}$. We show that $\mathcal{G}$, as a point of $\check{X}$, is a $\mathcal{T}(\check{\mathcal{U}})$-cluster point of $\mathcal{M}$. Let $U \in \mathcal{U}$, $V \in \mathcal{U}$ such that $V^2 \subset U$ and let $M \in \mathcal{M}$. Since $i^{-1}(M) \in \mathcal{G}$, by Proposition 3.2 there exists an x in $i^{-1}(M)$ such that $V^{-1}(x) \in \mathcal{G}$. As $V(x) \in \eta^*(x)$ and $V^{-1}(x) \times V(x) \subset U$, $\eta^*(x) \in U(\mathcal{G}) \cap M$.

Finally, if $\mathcal{U}$ is a uniformity, $(\check{X}, \check{\mathcal{U}})$ coincides with the standard completion of a uniform space by means of regular Cauchy filters [**1**, Page 101]. ■

The following corollary is a curious consequence of the preceding theorem and Proposition 2.1.

Corollary. *Let $(X, \mathcal{U})$ be a totally bounded point-symmetric T_1 space. Then $\mathcal{U}$ is a uniformity if and only if every minimal $\mathcal{U}^*$-Cauchy filter is a maximal regular filter.*

In general, a totally bounded quasi-uniform space may have many totally bounded compactifications; indeed, if $\mathcal{P}$ denotes the Pervin quasi-uniformity of a Tychonoff space X and $\hat{\mathcal{P}}$ denotes the Pervin quasi-uniformity of any Hausdorff compactification $\hat{X}$ of X, then $(\hat{X}, \hat{\mathcal{P}})$ is a totally bounded compactification of $(X, \mathcal{P})$. [**3**, Proposition, Page 203]. The remaining results indicate the well-behaviour of the compactification selected by the construction of Theorem 3.5.

Proposition 3.6. *Let $(X, \mathcal{U})$ be a point-symmetric T_1 quasi-uniform space, let $\mathcal{F}$ be a maximal regular filter on X, and let x be a cluster point of $\mathcal{F}$. Then $\mathcal{F} = \eta^*(x)$.*

Proof. Since $\eta^*(x)$ is a regular filter, it suffices to show that $\mathcal{F} \subset \eta^*(x)$. Let $\mathcal{B} = \{F \in \mathcal{F}: F = \bar{F}\}$. Then $x \in \bigcap \mathcal{B}$ and $\mathcal{B}$ is a base for $\mathcal{F}$. Let $U \in \mathcal{U}$ and $B \in \mathcal{B}$. Then $U \cap U^{-1}(x) \subset U^{-1}(B)$ and so $U^{-1}(B) \in \eta^*(x)$. Thus $\mathcal{F} = \mathcal{F}^r \subset \eta^*(x)$. ■

Corollary. *If $(X, \mathcal{U})$ is a compact totally bounded T_1 quasi-uniform space, $X = \check{X}$.*

Theorem 3.7. *Let $(X, \mathcal{U})$ be a totally bounded point-symmetric T_1 quasi-uniform space, let $(Y, \mathcal{V})$ be a totally bounded compact T_1 quasi-uniform space, and let $f: (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a quasi-uniformly continuous map. If f maps X onto Y, or $\mathcal{V}$ is a uniformity, then f has a quasi-uniformly continuous extension $\check{f}: (\check{X}, \check{\mathcal{U}}) \to (Y, \mathcal{V})$.*

Proof. By [**4**, Theorem 3.29], there is a $\tilde{\mathcal{U}}$-$\tilde{\mathcal{V}}$ quasi-uniformly continuous map $g: X \to Y$ defined for each minimal $\mathcal{U}^*$-Cauchy filter $\mathcal{F}$ by $g(\mathcal{F}) = \mathcal{T}(\tilde{\mathcal{V}}^*)$-limit fil$\{f(F): F \in \mathcal{F}\}$. Let $\check{f} = g \mid \check{X}$. If $\mathcal{V}$ is a uniformity, $(Y, \mathcal{V}) = (\tilde{Y}, \tilde{\mathcal{V}})$ and we are finished.

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Now suppose that f maps X onto Y and let $\mathcal{F} \in X$. Then, as is easily verified, $\{f(F): F \in \mathcal{F}\}$ is a base for a maximal $\mathcal{V}$-regular filter $\mathcal{H}$; we show that $\mathcal{H}$, considered as a point of $\breve{Y}$, is $\breve{f}(\mathcal{F})$. Since $\mathcal{T}(\tilde{\mathcal{V}}^*)$ is a Hasudorff topology, it suffices to show that $\mathcal{H}$ is a $\mathcal{T}(\tilde{\mathcal{V}}^*)$-cluster point of $\mathcal{H}$. To this end, let $F \in \mathcal{F}$, let $V \in \mathcal{V}$, and let $W \in V$ so that $W^2 \subset V$. By Proposition 3.2, there is a $y \in f(F)$ so that $W(y) \cap W^{-1}(y) \in \mathcal{H}$. Since $W^{-1}(y) \in \eta^*(y)$ and $W(y) \times W^{-1}(y) \subset V^{-1}$, $\eta^*(y) \in \tilde{V}^{-1} \times (\mathcal{H}) \cap f(F)$. Similarly, we see that $\eta^*(y) \in \tilde{V}(\mathcal{H}) \cap f(F)$, and so $\mathcal{H}$ is a $\mathcal{T}(\tilde{\mathcal{V}}^*)$-cluster point of $\mathcal{H}$. By the previous corollary, $Y = \breve{Y}$, and so $\breve{f}$ maps into Y as required. ■

Any continuous map between two topological spaces is a quasi-uniformly continuous map between the two corresponding Pervin quasi-uniform spaces. The extension property established in the previous theorem suggests, therefore, that the compactification $(\breve{X}, \breve{\mathcal{P}})$ might be of particular interest. In considering this compactification, we use the following standard notation: For any subset A of a set X, A^* denotes $\{\mathcal{F}: \mathcal{F}$ is a maximal closed filter on X and $A \in \mathcal{F}\}$ and $S(A) = T(A, X - A)$. A subbase for the Pervin quasi-uniformity of a topological space $(X, \mathcal{T})$ is $\{S(A): A \in \mathcal{T}\}$ and a base for the topology of the Wallman compactification of a T_1-space $(X, \mathcal{T})$ is $\{G^*: G \in \mathcal{T}\}$.

THEOREM 3.8. *Let X be a T_1-space and let $\mathcal{P}$ be the Pervin quasi-uniformity for X. Then $(\breve{X}, \mathcal{T}(\breve{\mathcal{P}}))$ is the Wallman compactification of X.*

Proof. We take $\hat{X}$ to be the collection of all filters on X that are maximal with respect to the property of having a closed base; since a filter has a closed base if and only if it is $\mathcal{P}$-regular, $\breve{X} = \hat{X}$.

Let $\mathcal{P}$ be the Pervin quasi-uniformity for $\hat{X}$. To see that $\breve{\mathcal{P}} \subset \hat{\mathcal{P}}$, let E be a closed subset of X and let $U = S(X - E)$. We show that $\breve{U} = S(\breve{X} - E^*)$. Let $(\mathcal{F}, \mathcal{G}) \in \breve{U}$. If $\mathcal{F} \in E^*$, it is obvious that $(\mathcal{F}, \mathcal{G}) \in S(X - E^*)$. If $\mathcal{F} \notin E^*$, there exists an $F \in \mathcal{F}$ and a $G \in \mathcal{G}$ such that $F \times G \subset U$ and $F \cap E = \varnothing$. It follows that $G \subset X - E$ so that $\mathcal{G} \notin E^*$; hence $(\mathcal{F}, \mathcal{G}) \in S(\breve{X} - E^*)$. Now suppose that $(\mathcal{F}, \mathcal{G}) \in S(\breve{X} - E^*)$. If $\mathcal{F} \in E^*$, then $E \in \mathcal{F}$ and $X \in \mathcal{G}$ so that $(\mathcal{F}, \mathcal{G}) \in \breve{U}$. If $\mathcal{F} \notin E^*$, then $X - E \in \mathcal{F} \cap \mathcal{G}$ so that $(\mathcal{F}, \mathcal{G}) \in \breve{U}$. Thus $\mathcal{T}(\breve{\mathcal{P}})$ is coarser than the topology of the Wallman compactification of X.

To see that $\mathcal{T}(\hat{\mathcal{P}}) \subset \mathcal{T}(\mathcal{P})$, let G be a $\mathcal{T}(\hat{\mathcal{P}})$-open set, let $\mathcal{F} \in G$ and let $E = \breve{X} - G$. Then $E = \bigcap \{E^*_\alpha: \alpha \in A\}$ where, for each $\alpha \in A$, E_α is a closed subset of X. There exists $\alpha \in A$ so that $\mathcal{F} \notin E^*_\alpha$. Since $S(\breve{X} - E^*_\alpha)$ is an entourage of $\hat{\mathcal{P}}$, $V = S(\breve{V} - E^*_\alpha) \cap X \times X$ is an entourage of the Pervin quasi-uniformity on X. It suffices to show that $\breve{V}(\mathcal{F}) \subset G$. Suppose that $\mathcal{H} \in \breve{V}(\mathcal{F}) \cap E$. There exist $F \in \mathcal{F}$ and $H \in \mathcal{H}$ so that $F \times H \subset V = X \times X - (X - E_\alpha \times E_\alpha)$. Since $\mathcal{F} \notin E^*_\alpha$, we assume, without loss of generality, that $F \subset X - E_\alpha$. Thus $H \cap E_\alpha = \varnothing$; and, since $\mathcal{H} \in E \subset E^*_\alpha$, we have a contradiction. ■

Our final result establishes a sufficient condition in order that $(\breve{X}, \breve{\mathcal{U}})$ be a Hausdorff compactification. This condition is easily seen to be satisfied by a T_1 totally bounded quasi-uniform space that is either normal and equinormal or a uniform space.

We say that a relation V on a set X *separates* subsets A and B of X provided that $V(A) \cap V(B) = \varnothing$. A quasi-uniform space $(X, \mathcal{U})$ satisfies property * provided that any two subsets of X that are separated by a member of $\mathcal{U}^{-1}$ are also separated by a member of $\mathcal{U}$.

PROPOSITION 3.9. *Let* $(X, \mathcal{U})$ *be a point-symmetric totally bounded* T_1 *quasi-uniform space satisfying property* *. *Then* $(\check{X}, \check{\mathcal{U}})$ *is a Hausdorff compactification of* $(X, \mathcal{U})$.

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be two members of $\check{X}$. There is an $A \in \mathcal{F}$, and $B \in \mathcal{G}$, and a $U \in \mathcal{U}$ so that $U^{-1}(A) \cap U^{-1}(B) = \varnothing$. By hypothesis there is a $V \in \mathcal{U}$ with $V(A) \cap V(B) = \varnothing$. Let $W \in \mathcal{U}$ with $W^2 \subset V$. We assert that $\check{W}(F) \cap \check{W}(G) = \varnothing$. Suppose that $\mathcal{H} \in \check{W}(\mathcal{F}) \cap \check{W}(\mathcal{G})$. Then there is an $F \in \mathcal{F}$, a $G \in \mathcal{G}$, and an $H \in \mathcal{H}$ such that $F \times H \subset W$, $G \times H \subset W$, and $H \times H \subset W$. Thus $F \times G \subset W \circ W \circ W^{-1} \subset V \circ V^{-1}$. Since there exists $(p, q) \in (F \times G) \cap (A \times B)$, there is an $r \in X$ such that $(p, r) \in V$ and $(r, q) \in V^{-1}$; hence $r \in V(p) \cap V(q) \subset V(A) \cap V(B)$—a contradiction. ■

According to Theorem 3.47 of reference **[4]**, a totally bounded Tychonoff space $(X, \mathcal{U})$ has a Hausdorff compactification if and only if $\mathcal{U}$ contains a uniformity compatible with $\mathcal{T}(\mathcal{U})$. Thus any point-symmetric totally bounded Tychonoff quasi-uniformity $\mathcal{U}$ satisfying property * contains a uniformity compatible with $\mathcal{T}(\mathcal{U})$. If X is a Tychonoff space that is not normal, then $(X, \mathcal{P})$ has a Hausdorff compactification, but $(\check{X}, \check{\mathcal{P}})$ is the Wallman compactification, which fails to be a Hausdorff space. Thus a quasi-uniformity $\mathcal{U}$ may contain a uniformity compatible with $\mathcal{T}(\mathcal{U})$ and still fail to satisfy property *. The problem of determining necessary and sufficient conditions in order that $(\check{X}, \check{\mathcal{U}})$ be a Hausdorff compactification is still open; indeed, even property * has not yet been ruled out as such a condition.

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