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ITÔ'S THEOREM AND MONOMIAL BRAUER CHARACTERS II

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Abstract

Let G be a finite solvable group and let p be a prime. We prove that the intersection of the kernels of irreducible monomial p-Brauer characters of G with degrees divisible by p is p-closed.

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All groups are finite throughout this note. A group G is said to be p-closed for the prime p if G has a normal Sylow p-subgroup. In [5], Pang and Lu proved that when G is solvable and there is a prime p so that p does not divide the degree of any monomial irreducible character, then G is p-closed. In our paper [1], we mistakenly stated that they also proved the converse. When G is a nonabelian p-group, it is p-closed and has at least one monomial irreducible character whose degree is divisible by p. Thus, not only did Pang and Lu not prove the converse; in fact, the converse is not true. We note that Pang and Lu's theorem can be viewed as a generalisation of the normality part of Itô's theorem.

In [6, Theorem 1.1], Pang and Lu proved a further generalisation of Itô's theorem. In particular, when G is solvable and p is a prime, they defined M to be the intersection of the kernels of the irreducible monomial characters of G with degrees divisible by p. When no such character exists, M is defined to be G. By [6, Theorem 1.1], M is p-closed. The example from the previous paragraph shows that the converse need not be true.

For Brauer characters of p-solvable groups, Itô recovered the normality of Sylow subgroups (see [3, Theorem 13.1(b) and (c)]). We generalised this result in [1] for solvable groups by only using the monomial p-Brauer characters. Following the idea of Pang and Lu, we now let $\mathrm{IBr}_{m,p}(G)$ be the set of irreducible monomial p-Brauer characters of G whose degrees are divisible by p and we define $\mathcal{M}$ to be the intersection

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of the kernels of Brauer characters in $\mathrm{IBr}_{m,p}(G)$. When $\mathrm{IBr}_{m,p}(G)$ is an empty set, we set $\mathcal{M} = G$. The following is our main result.

Theorem 1. *If G is a solvable group and p is a prime divisor of $|G|$, then $\mathcal{M}$ is p-closed.*

Proof. We work by induction on $|G|$. We may assume that $\mathcal{M} > 1$. Let N be a minimal normal subgroup of G contained in $\mathcal{M}$. Since

$$\bigcap_{\varphi \in \mathrm{IBr}_{m,p}(G/N)} \ker \varphi = \mathcal{M}/N,$$

it follows by induction that $\mathcal{M}/N$ is p-closed. Let P be a Sylow p-subgroup of $\mathcal{M}$. Then PN/N is a normal Sylow p-subgroup of $\mathcal{M}/N$. If N is a p-group, then $PN = P$ and so P is normal in $\mathcal{M}$. Thus, we may assume that N is an abelian q-group for some prime $q \neq p$.

Write $H = PN$ and observe that H is a normal subgroup of G. By the Frattini argument,

$$G = H\mathbf{N}_G(P) = NP\mathbf{N}_G(P) = N\mathbf{N}_G(P).$$

Observe that $N \cap \mathbf{N}_G(P)$ is normal in $N\mathbf{N}_G(P) = G$. Applying the minimality of N, either $N \leq \mathbf{N}_G(P)$ or $N \cap \mathbf{N}_G(P) = 1$. If $N \leq \mathbf{N}_G(P)$, then $G = \mathbf{N}_G(P)$ and P is normal in $\mathcal{M}$, as desired.

Now assume that $N \cap \mathbf{N}_G(P) = 1$. Let $1_N \neq \lambda \in \mathrm{IBr}(N) = \mathrm{Irr}(N)$ and write $T = \mathrm{I}_G(\lambda)$ for the inertia group of λ in G. Since N is complemented in G, we see that N is complemented in T. Using [2, Problem 6.18], it follows that λ extends to $\nu \in \mathrm{Irr}(T)$. Let μ be the restriction of ν to the p-regular elements of T. We see that $\mu \in \mathrm{IBr}(T)$ and $\mu_N = \lambda$. Applying the Clifford correspondence for Brauer characters [4, Theorem 8.9] gives $\varphi = \mu^G \in \mathrm{IBr}(G)$. This implies that φ is monomial with degree $|G : T|$. If p divides $\varphi(1) = |G : T|$, then $N \leq \ker \varphi$ as $\varphi \in \mathrm{IBr}_{m,p}(G)$ and so $N \leq \ker \mu$ as $\varphi = \mu^G$. This yields $N \leq \ker(\mu_N) = \ker \lambda$ and we deduce that $\lambda = 1_N$, which is a contradiction to the choice of λ.

Consequently, p does not divide $\varphi(1)$. Hence, there exists some Sylow p-subgroup of G that is contained in T. Since $P \in \mathrm{Syl}_p(H)$ and $PN = H \lhd G$, we may, without loss of generality, assume that $P \leq T$. For all elements $x \in P$ and $n \in N$, we have $\lambda(n) = \lambda^x(n) = \lambda(xnx^{-1})$. Since λ is linear, this yields $\lambda(xnx^{-1}n^{-1}) = 1$. Because λ is arbitrary, it follows that

$$[P, N] \leq \bigcap_{\lambda \in \mathrm{IBr}(N)} \ker \lambda = 1.$$

We conclude that N centralises P. This implies that P is a characteristic subgroup of $H = PN$ and, therefore, P is normal in $\mathcal{M}$, as desired. □

Now we obtain the main result of [1] as a corollary.

Corollary 2. *Let G be a solvable group and p be a prime divisor of $|G|$. Then G is p-closed if and only if p does not divide $\varphi(1)$ for every monomial Brauer character $\varphi \in \mathrm{IBr}(G)$.*

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Proof. Note that if p does not divide the degree of every monomial irreducible Brauer character, then $\mathrm{IBr}_{m,p}(G) = \emptyset$ and so $\mathcal{M} = G$. By Theorem 1, G is p-closed. Conversely, as we noted in [1], if G is p-closed, then p does not divide the degree of any irreducible Brauer character. □

We also obtain a corollary in terms of the quotients of the group.

Corollary 3. *Let G be a solvable group and p be a prime divisor of $|G|$. Then G is p-closed if and only if $G/\ker\varphi$ is p-closed for every Brauer character $\varphi \in \mathrm{IBr}_{m,p}(G)$.*

Proof. Let P be a Sylow p-subgroup of G. Suppose first that G is p-closed. We see that P is normal in G and $P \subseteq \ker\varphi$ for every Brauer character $\varphi \in \mathrm{IBr}(G)$. Therefore, $G/\ker\varphi$ is p-closed. (In this case, the Sylow p-subgroup of $G/\ker\varphi$ is trivial.)

Conversely, suppose that $G/\ker\varphi$ is p-closed for every Brauer character φ in $\mathrm{IBr}_{m,p}(G)$. Let φ be any Brauer character in $\mathrm{IBr}_{m,p}(G)$. By hypothesis, $G/\ker\varphi$ is p-closed, so we may assume that $\ker\varphi > 1$. Since $G/\ker\varphi$ is p-closed, it follows that $P\ker\varphi \subseteq \ker\varphi$ and so $P \subseteq \ker\varphi$. Therefore, $P \subseteq \mathcal{M}$. By Theorem 1, P is a normal subgroup of $\mathcal{M}$. We see that P is characteristic in $\mathcal{M}$, which is normal in G, and we conclude that P is normal in G, as desired. □

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References

[1] X. Chen and M. Lewis, 'Itô's theorem and monomial Brauer characters', *Bull. Aust. Math. Soc.*, doi:10.1017/S0004972717000399.
[2] I. M. Isaacs, *Character Theory of Finite Groups* (Academic Press, New York, 1976).
[3] O. Manz and T. R. Wolf, *Representations of Solvable Groups* (Cambridge University Press, Cambridge, 1993).
[4] G. Navarro, *Characters and Blocks of Finite Groups* (Cambridge University Press, Cambridge, 1998).
[5] L. Pang and J. Lu, 'Finite groups and degrees of irreducible monomial characters', *J. Algebra Appl.* **15** (2016), 1650073 (4 pp).
[6] L. Pang and J. Lu, 'Finite groups and degrees of irreducible monomial characters II', *J. Algebra Appl.* **15** (2017), 1750231 (5 pp).

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