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Injective modules and soluble groups satisfying the minimal condition for normal subgroups

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Let p be a prime and let Q be a centre-by-finite p'-group. It is shown that the ZQ-modules which satisfy the minimal condition on submodules and have p-groups as their underlying additive groups can be classified in terms of the irreducible Z_pQ-modules. If such a ZQ-module V is indecomposable it is either the ZQ-injective hull W' of an irreducible Z_pQ-module (viewed as a ZQ-module) or is the submodule $W[p^n]$ of such a W consisting of the elements $w \in W$ which satisfy $p^n w = 0$. This classification is used to classify certain abelian-by-nilpotent groups which satisfy Min-n, the minimal condition on normal subgroups. Among the groups to which our classification applies are all quasi-radicable metabelian groups with Min-n, and all metabelian groups which satisfy Min-n and have abelian Sylow p-subgroups for all p.

It is also shown that if Q is any countable locally finite p'-group and V is a ZQ-module whose additive group is a p-group, then V can be embedded in a ZQ-module $\overline{V}$ whose additive group is a minimal divisible group containing that of V. Some applications of this result are given.

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1. Introduction

A group G is said to be *quasi-radicable* if, for each integer $n > 0$, G is generated by the n-th powers of its elements. One of the main purposes of this paper is to classify quasi-radicable metabelian groups satisfying Min-n , the minimal condition on normal subgroups. However it turns out to be equally convenient to work with a somewhat larger class, namely the class $\underline{\underline{Z}}$ of all abelian-by-nilpotent groups which satisfy Min-n and in addition satisfy the condition

(Z) *If* $G \in \underline{\underline{Z}}$ *and* P *is a* p*-subgroup of* G *then* $G^{\underline{\underline{N}}} \cap P$ *is contained in the centre of* P .

Here $G^{\underline{\underline{N}}}$ denotes the uniquely determined normal subgroup K of G which is minimal subject to the condition that G/K is nilpotent; its existence is assured by Min-n . By a theorem of Baer [4] soluble groups satisfying Min-n , and hence $\underline{\underline{Z}}$-groups, are locally finite; another theorem of Baer [3] states that nilpotent groups with Min-n are centre-by-finite, so that $\underline{\underline{Z}}$-groups are metabelian-by-finite. $\underline{\underline{Z}}$ contains all metabelian groups which satisfy Min-n and have abelian Sylow p-subgroups for all primes p ; hence ([14], Corollary 3.3) it contains all quasi-radicable metabelian groups with Min-n .

Let $G \in \underline{\underline{Z}}$ and let $K = G^{\underline{\underline{N}}}$. Then G splits over K (Lemma 4.1)

$$G = KA , \quad K \cap A = 1 .$$

Here, by the remarks above, A is nilpotent and centre-by-finite; the results of Baer [3] also show that A satisfies Min , the minimal condition on subgroups. Now the condition (Z) ensures that the Sylow p-subgroup of A centralizes the Sylow p-subgroup K_p of K ; hence K_p is effectively a module for $A_{p'}$, the Sylow p'-subgroup of A , satisfying the minimal condition on submodules.

Our problem is thus closely related to that of classifying those modules over the integral group ring ZQ of a centre-by-finite p'-group Q which satisfy Min-Q , the minimal condition on Q-submodules, and have p-groups as their underlying additive groups. Modules with the latter property will be called *p-modules*. We shall deal with this problem in

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§2. Strictly speaking our results simply reduce the classification problem to that of classifying the irreducible Z_pQ-modules; in the case when Q is abelian this can be done quite easily (Lemma 2.5). We shall see that the indecomposable p-modules over ZQ with Min-Q arise naturally from the injective hulls of the irreducible ones, and our results in §2 lean heavily on the properties of the injective hull of a module as defined by Eckmann and Schopf [8] (see also [7], p. 384 *et seq.*, or [15]).

A problem which arises naturally from the work in §2 is the following: Let V be a ZG-module, where G is any group, and let $\bar{V}$ be a minimal divisible group containing the additive group V^+ of V, or in other words a Z-injective hull of V^+. Under what conditions can the ZG-structure of V be extended to $\bar{V}$?

This cannot invariably be done; for example, since a group of type C_{p^∞} has no automorphism of order p if p is odd, it cannot be done if V is a cyclic group of order p^2 which is a non-trivial module for a cyclic group of odd prime order p. In §3 we deduce easily from the results of §2 that the extension can be carried out provided V is a p-module and G is a countable locally finite p'-group (Theorem B1). This result is used to construct examples of soluble groups of any given derived length which satisfy Min-n and have a series of finite length in which all the factors are divisible abelian groups (Lemma 3.4). These groups are $P\underline{\underline{Q}}$-groups in the sense of [13].

Finally in §4 we deduce our classification of $\underline{\underline{Z}}$-groups from the results of §2.

We are indebted to Dr M.C.R. Butler who suggested the possibility of applying the theory of injective modules to the problems described above; this resulted in a considerable simplification of our previous work.

2. Injective p-modules for centre-by-finite p'-groups

We begin by recalling the basic facts about injective modules which we shall need (*cf.* [7], p. 384 *et seq.*, or [15]). Let R be a ring with 1 . By an R-module we shall always understand a right R-module on which

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1 acts as the identity map. An R-module X is called *injective* if whenever $U \leq W$ are R-submodules then every R-homomorphism of U into X can be extended to W. This is equivalent (but not immediately) to the requirement that X be a direct summand of every R-module which contains it. If V is an arbitrary R-module then an injective hull of V (in the category of R-modules) is an R-module $\overline{V}$ satisfying:-

(i) $\overline{V}$ is injective, and either

(ii) no proper submodule of $\overline{V}$ containing V is injective, or

(ii)' $\overline{V}$ is an essential extension of V.

Here a module W is said to be an *essential* (or related) extension of a submodule U if every non-trivial submodule of W meets U non-trivially. It was shown by Eckmann and Schopf [8] that every R-module V has an injective hull $\overline{V}$ which is unique in the sense that if V^* is another injective hull of V then there is an isomorphism from $\overline{V}$ to V^* extending the identity map on V.

We shall need the following fact:

LEMMA 2.1. *Let R be a ring with* 1, *let V be an R-module and let $\overline{V}$ be an injective hull of V. Suppose $V = \bigoplus_{\lambda\in\Lambda} V_\lambda$ where each V_λ is an R-submodule of V. If either*

(i) Λ is finite, or

(ii) R satisfies the maximal condition on right ideals,

then $\overline{V} = \bigoplus_{\lambda\in\Lambda} \overline{V}_\lambda$, where $\overline{V}_\lambda$ is an injective hull of V_λ.

Proof. We can embed V in a module $W = \bigoplus_{\lambda\in\Lambda} W_\lambda$, where each W_λ is an injective hull of V_λ; it will suffice to show that W is an injective hull of V. Now W is injective ([15], Theorems 5 and 6) and so we need only show that W is an essential extension of V. If this is not the case then there is a non-zero element w of W such that whenever $r \in R$ and $wr \in V$ then $wr = 0$. We may express w in the form $w = w_{\lambda_1} + \dots + w_{\lambda_k}$ $(0 \neq w_{\lambda_i} \in W_{\lambda_i}, \lambda_i \neq \lambda_j$ if $i \neq j)$ and suppose k is minimal with respect to w having the desired property.

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Now $0 \neq w_{\lambda_1} r \in V_{\lambda_1}$ for some $r \in R$. Consider $(w - w_{\lambda_1})r$. If it is zero then $wr = w_{\lambda_1} r$, a non-zero element of V. Hence $(w - w_{\lambda_1})r \neq 0$, and by the minimality of k we have $0 \neq (w - w_{\lambda_1})rs \in V$ for some $s \in R$. Hence $0 \neq wrs = (w - w_{\lambda_1})rs + w_{\lambda_1} rs \in V$, which is a contradiction.

Notice that Lemma 2.1 holds in particular when R is the integral group ring of a finite group.

Now it is not difficult to see that every injective R-module U is divisible in the sense that $Ud = U$ for every element d of R which is not a zero-divisor (*cf.* [11], Theorem 3.1). We shall call an R-module V Z-*divisible* if the additive group V^+ of V is a divisible group. We then have immediately

LEMMA 2.2. *Every injective* ZG-*module is* Z-*divisible.*

We shall now see that under certain circumstances the converse is true, and that sometimes the injective hull $\bar{V}$ of a ZG-module V has for its additive group a minimal divisible group containing V^+. This cannot happen when V is cyclic of order p^2 and G is cyclic of odd prime order p, as we have already remarked. Furthermore if G is any infinite group, D is any non-trivial divisible group, $\bar{D}$ is the base-group of the restricted wreath product $D \text{ wr } G$, and $V = [\bar{D}, G]$, then V is Z-divisible but is not ZG-injective since it is not complemented by a ZG-submodule of $\bar{D}$. Thus in this case the injective hull of V does not have for its additive group a minimal divisible group containing V^+.

If p denotes a prime (as it always will) and V is any abelian group we denote by $V[p^k]$ the set of elements $v \in V$ satisfying $p^k v = 0$ (where $k > 0$ is an integer). If V is in addition an R-module then $V[p^k]$ will be an R-submodule of V.

LEMMA 2.3. *Let* Q *be a centre-by-finite* p'-*group and let* V *be a* p-*module over* ZQ. *Suppose that either*

(i) Q *is finite, or*

(ii) V *satisfies* Min-Q.

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Let $\bar{V}$ *be an injective hull of* V . *Then*

(a) $\bar{V}$ *is a* p*-module and* $\bar{V}[p] = V[p]$,

(b) V *is injective if and only if* V *is* Z*-divisible.*

For the proof we shall require the following lemma, which is a straightforward consequence of a result of Kovács and Newman [12]. We shall call a module *monolithic* if the intersection of its non-zero submodules is again non-zero.

LEMMA 2.4. *Let* Q *be a centre-by-finite* p'*-group, let* V *be a* ZQ*-module and let* W *be a submodule of* V . *Suppose that* W *is a* p*-module and is the direct sum of finitely many monolithic submodules, and suppose further that* W *is a direct summand of* V *as an additive group. Then* W *is a direct summand of* V *as a* ZQ*-module.*

Proof of Lemma 2.3 *(a)*. In case *(i)* it is clear that every element of $V[p]$ lies in a finite submodule of $V[p]$. It then follows by Maschke's Theorem that $V[p]$ is generated by its irreducible submodules and so is the direct sum of a selection of them. In case *(ii)* we find that if $V \neq 0$ then $V[p]$ contains an irreducible submodule, and this is a direct summand of $V[p]$ by Lemma 2.4. By applying this argument to a complementary submodule and continuing in the same way, we find that $V[p]$ is in this case the direct sum of finitely many irreducible submodules. Now $\bar{V}$ is clearly an injective hull of $V[p]$ and so in either case Lemma 2.1 allows us to assume that V is irreducible.

We then have $V \leq \bar{V}[p]$. Since V is certainly a direct summand of the additive group of $\bar{V}[p]$, Lemma 2.4 shows that V is a direct summand of the ZQ-module $\bar{V}[p]$. But $\bar{V}$ is an essential extension of V and so $V = \bar{V}[p]$.

It follows that the submodule $\bar{V}_p$ formed by the p-elements of $\bar{V}$ is monolithic with V as its unique minimal submodule. Now by Lemma 2.2 $\bar{V}_p$ is Z-divisible and so it is a direct summand of the additive group of $\bar{V}$. Consequently, by Lemma 2.4 again, $\bar{V}_p$ is a direct summand of $\bar{V}$ as ZQ-module. It then follows that $\bar{V}_p = \bar{V}$, completing the proof.

(b) If V is injective then it is Z-divisible by Lemma 2.2.

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Conversely suppose that V is Z-divisible, and let $\overline{V}$ be an injective hull of V. By *(a)* the additive group of $\overline{V}$ is a minimal divisible group containing that of V, and so $V = \overline{V}$. Hence V is injective.

We conclude this section by describing the structure of p-modules with Min-Q over ZQ, where Q is a centre-by-finite p'-group. Let $\{V_\lambda : \lambda \in \Lambda\}$ be a complete set of representatives for the isomorphism types of irreducible Z_pQ-modules. We view the V_λ as ZQ-modules and denote by $\overline{V}_\lambda$ a ZQ-injective hull of V_λ. Let $V_\lambda(n)$ denote the submodule of $\overline{V}_\lambda$ formed by the elements v satisfying $p^n v = 0$ $(n = 0, 1, \ldots)$, and put $V_\lambda(\infty) = \overline{V}_\lambda$. Then $V_\lambda(n)$ is determined up to isomorphism by λ and n; notice also that by Lemma 2.3 $V_\lambda(n+1)/V_\lambda(n) \cong V_\lambda = V_\lambda(1)$, which is irreducible. It follows from this that the $V_\lambda(n)$ $(n = 0, 1, \ldots, \infty)$ are the only submodules of $V_\lambda(\infty)$.

THEOREM A. *Let* Q *be a centre-by-finite* p'*-group and let* V *be a* p*-module over* ZQ. *Then* V *satisfies* Min-Q *if and only if* V *is a direct sum of finitely many submodules each isomorphic to some* $V_\lambda(n)$ $(1 \le n \le \infty)$.

If V *satisfies* Min-Q *and* V *is expressed in two ways as the direct sum of indecomposable submodules, then there is an automorphism of* V *mapping the first decomposition onto the second.*

Since $V_\lambda(n)$ is isomorphic to $V_\mu(m)$ if and only if $\lambda = \mu$ and $m = n$ it follows that the p-modules over ZQ with Min-Q are classified by the functions of finite support from the set of pairs (λ, n) $(\lambda \in \Lambda, \ 1 \le n \le \infty)$ to the non-negative integers.

Proof of Theorem A. From our remarks preceding the statement of Theorem A it follows that each $V_\lambda(n)$ satisfies Min-Q; hence any finite direct sum of such modules also satisfies Min-Q.

Conversely suppose that V satisfies Min-Q. If V is not expressible as stated then among the submodules of V which are not so expressible there is a minimal one. It thus suffices for the proof to

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assume that, while every proper submodule of V is expressible in the manner stated, V itself is not, and to obtain a contradiction. This assumption implies that V is indecomposable.

Let W be the maximal divisible subgroup of V and suppose first that $W \neq 0$. Then W is a submodule of V, and it follows from Lemma 2.3 that W is injective. Consequently W is a direct summand of V and so $V = W$. However Lemma 2.4 shows that $W[p]$ is the direct sum of finitely many irreducibles, and Lemma 2.1 then shows that W is the direct sum of the injective hulls of these irreducibles. Therefore V is a direct sum of submodules of type $V_\lambda(\infty)$, which is a contradiction. We therefore have that $W = 0$.

By the minimal condition the chain $V \geq pV \geq p^2V \geq \ldots$ must become stationary after finitely many steps. Since V contains no non-trivial divisible subgroup it follows that $p^nV = 0$ for some n, which we suppose chosen as small as possible. Then $p^{n-1}V \neq 0$ and so $p^{n-1}V$ contains an irreducible submodule U. There is an isomorphism of U onto some V_λ, and this may be extended to a homomorphism ϕ of V into the injective module $V_\lambda(\infty)$. Now $p^{n-1}(V\phi) = (p^{n-1}V)\phi \neq 0$, and so $V\phi$ has exponent p^n precisely. Hence $V\phi = V_\lambda(n)$. Let K be the kernel of ϕ. Then V/K, as an additive group, is the direct sum of cyclic groups of the same order p^n. Such a group is free in the category of abelian groups of exponent dividing p^n; hence K is a direct summand of V as an additive group. Also, by the minimality of V, K is the direct sum of finitely many submodules of the type $V_\lambda(n)$. Since these are all monolithic, Lemma 2.4 shows that K is a direct summand of V as a module. This contradicts the indecomposability of V and establishes the result.

To establish the final statement if suffices, by a well-known version of the Krull-Schmidt Theorem due to Azumaya [1], to show that in the endomorphism ring of each $V_\lambda(n)$, $n \geq 1$, the sum of two non-units is a non-unit. Since $V_\lambda(n)$ has no proper submodule isomorphic to itself,

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every such non-unit has a non-trivial kernel, which must contain the unique minimal submodule of $V_\lambda(n)$. This makes it clear that the sum of two such non-units is a non-unit, as claimed.

As we have remarked, the irreducible Z_pQ-modules are quite readily obtained when Q is abelian. In fact let Q be any periodic abelian group, which need not even be a p'-group for these purposes, and let k be an algebraic closure of Z_p. Let θ be a homomorphism of Q into the multiplicative group k^* of non-zero elements of k. Then since the elements of $Q\theta$ are all roots of unity, it follows that the additive group L_θ generated by $Q\theta$ is in fact a field. Let K_θ be the Z_pQ-module whose underlying vector space is L_θ with the Q-action given by

$$vg = v.g\theta \quad (v \in V \ , \ g \in Q) \ .$$

Since $Q\theta$ generates L_θ additively any Q-submodule of K_θ is invariant under multiplication by any element of L_θ; consequently K_θ is irreducible. We have the following result, which is no doubt well known.

LEMMA 2.5. *With the above notation*

(i) every irreducible Z_pQ*-module is isomorphic to some* K_θ *;*

(ii) $K_\theta \cong K_\phi$ *if and only if* $L_\theta = L_\phi$ *and* $\theta = \phi\rho$ *for some element* ρ *of the Galois group of* L_θ *over* Z_p .

Proof *(i)*. Let V be an irreducible Z_pQ-module and let $E = \mathrm{End}_Q V$, which is a division algebra over Z_p by Schur's Lemma. Since Q is abelian, if $g \in Q$ the map $g\tau$ given by

$$v(g\tau) = vg \quad (v \in V)$$

is an element of E and τ maps Q homomorphically into the centre Z of E, which is a field. Let L be the additive subgroup of E generated by $Q\tau$. Then since the elements of $Q\tau$ are roots of unity L is a subring of Z which is algebraic over Z_p, and so it is a field.

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is an L-module and since $Q\tau \leq L$ we must have $\dim_L V = 1$. So $V = vL$ for some $v \in V$. Choose a monomorphism $\bar{\psi}$ of L into k and define

$$(vl)\psi = l\bar{\psi} \quad (l \in L) .$$

This gives a well-defined additive isomorphism of V onto the field $L\bar{\psi}$. Let $\theta = \tau\bar{\psi}$. Then θ maps Q homomorphically into the multiplicative group k^* and $L\bar{\psi}$ is additively generated by $Q\theta$. We now verify that the map ψ is an isomorphism of V onto K_θ . In fact if $u \in V$ and $g \in Q$ we have $u = vl$ for some uniquely determined $l \in L$, and

$$(ug)\psi = [vL(g\tau)]\psi = (l.g\tau)\bar{\psi} = l\bar{\psi}.g\tau\bar{\psi} = u\psi.g\theta ,$$

as required.

(ii) Suppose first that $L_\theta = L_\phi$ and $\theta = \phi\rho$ with ρ an element of the Galois group of L_θ over Z_p . Then ρ is an additive automorphism of L_θ , and if $x \in L_\theta$ and $g \in Q$ we have

$$(x.g\phi)\rho = x\rho.g\phi\rho = x\rho.g\theta$$

so that ρ maps K_ϕ isomorphically onto K_θ .

Suppose conversely that $K_\phi \cong K_\theta$. Then since the kernel of ϕ is the kernel of the representation of Q determined by K_ϕ we must have that θ and ϕ have the same kernel. Therefore $Q\theta \cong Q\phi$. Now k^* is a direct product of groups of type C_{q^∞} , one for each prime $q \neq p$; it follows that no two distinct subgroups of k^* are isomorphic. Hence $Q\theta = Q\phi$ and so $L_\theta = L_\phi$. Let ρ be any isomorphism of K_ϕ onto K_θ . Then ρ is an additive automorphism of L_θ and, since multiplication by any non-zero element of L_θ determines an automorphism of K_θ , we may choose ρ so that $1\rho = 1$. Then for $x \in K_\phi$ and $g \in Q$ we have

$$(x.g\phi)\rho = x\rho.g\theta .$$

Putting $x = 1$ gives $\phi\rho = \theta$ and so $(x.g\phi)\rho = x\rho.g\phi\rho$, or $(x.y)\rho = x\rho.y\rho$ for $x \in L_\theta$ and $y \in Q\theta$. Since $Q\theta$ generates L_θ additively it follows that ρ preserves multiplication, and so belongs to

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the Galois group of L_θ over Z_p .

3. Embedding in Z-divisible modules

It follows in particular from the existence of the injective hull, that every R-module can be embedded in an injective R-module. In fact this was first proved by Baer [2]. Lemma 2.2 then gives

LEMMA 3.1. *Every* ZG*-module can be embedded in a* Z*-divisible* ZG*-module.*

It is natural to ask under what circumstances a ZG-module V may be embedded in a Z-divisible ZG-module $\bar{V}$ whose additive group is a minimal divisible group containing that of V . This is not always possible, as we remarked in §1. Now the following facts are immediate from Lemma 2.3:

LEMMA 3.2. *Let* Q *be a centre-by-finite* p'*-group and let* V *be a* p*-module over* ZQ . *Let* $\bar{V}$ *be a minimal divisible group containing the additive group of* V , *and suppose that either*

(i) Q *is finite, or*

(ii) V *satisfies* Min-Q .

Then

(a) $\bar{V}$ *admits a* ZQ*-module structure extending that on* V ;

(b) if V_1, V_2 *are* ZQ*-modules containing* V *in such a manner that their additive groups are minimal divisible groups containing that of* V , *then the identity map on* V *extends to an isomorphism of* V_1 *onto* V_2 .

We shall now show that *(a)* holds in considerably greater generality:

THEOREM B1. *Let* Q *be a countable locally finite* p'*-group, let* V *be a* p*-module over* ZQ , *and let* $\bar{V}$ *be a minimal divisible group containing the additive group of* V . *Then* $\bar{V}$ *admits a* ZQ*-module structure extending that of* V .

In this generality, however, *(b)* of Lemma 3.2 may break down, and the resulting ZQ-module $\bar{V}$ is not always even determined up to isomorphism by V . We shall not pursue this point at present, but hope to return to it in a later publication. We have no idea whether the restriction of

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countability is necessary.

Proof. Write $Q = \bigcup_{i=0}^{\infty} Q_i$, where

$$1 = Q_0 \leq Q_1 \leq \dots$$

is a tower of finite subgroups of Q . We shall construct for each $n \geq 0$ a map $f_n : \overline{V} \times ZQ_n \to \overline{V}$ which makes $\overline{V}$ into a ZQ_n-module and is such that

(1) $$(v, r)f_n = vr \quad (v \in V , \quad r \in ZQ_n) .$$

We shall also arrange that each f_{n+1} extends f_n . These maps will then determine a map from $\overline{V} \times ZQ$ to $\overline{V}$ which makes $\overline{V}$ into a ZQ-module in the required manner.

Now f_0 can certainly be obtained (and is in fact uniquely determined). Suppose that for some $n \geq 0$, f_n has been constructed. It follows from Lemma 3.2 that there is a Z-divisible ZQ_{n+1}-module W containing the restricted module $V_{Q_{n+1}}$ in such a manner that the additive group of W is a minimal divisible group containing the additive group of $V_{Q_{n+1}}$ (or the additive group of V , which is the same thing). It further follows from Lemma 3.2 that the identity map on V can be extended to a ZQ_n-isomorphism ϕ of the ZQ_n-module $(\overline{V}, f_n)$ onto W_{Q_n} . The mapping $f_{n+1} : \overline{V} \times ZQ_{n+1} \to \overline{V}$ defined by

$$(v, r)f_{n+1} = (v\phi . r)\phi^{-1} \quad (v \in V , \quad r \in ZQ_{n+1})$$

then makes $\overline{V}$ into a ZQ_{n+1}-module, extends f_n , and satisfies (1) with n replaced by $n + 1$. Thus the maps f_n can be constructed and the result is established.

THEOREM B2. *Let Q be a countable locally finite p'-group and let V be a p-module over ZQ satisfying* Min-Q . *Then V can be embedded in a Z-divisible p-module over ZQ which satisfies* Min-Q .

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Proof. Let $\bar{V}$ be a ZQ-module containing V in such a manner that the additive group of $\bar{V}$ is a minimal divisible group containing that of V. The existence of such a $\bar{V}$ is given by Theorem B1. Then $\bar{V}[p] = V[p]$ and Theorem B2 follows from the following lemma:

LEMMA 3.3. *Let V be an abelian p-group admitting a set Ω of distributive operators. Then V satisfies* Min-Ω, *the minimal condition on Ω-subgroups, if and only if $V[p]$ satisfies* Min-Ω.

Proof. It is clear that if V satisfies Min-Ω so does $V[p]$. Conversely assume that $V[p]$ satisfies Min-Ω and let

$$V_1 \geq V_2 \geq \dots$$

be a descending chain of Ω-subgroups of V. Consider the Ω-subgroups

$$U_{i,m} = p^m\left(V_i \cap V\left[p^{m+1}\right]\right) \quad (i = 1, 2, \dots ; \ m = 0, 1, \dots)$$

of $V[p]$. Clearly $U_{i,m} \geq U_{i+1,m}$ and $U_{i,m} \geq U_{i,m+1}$. Therefore $U_{i,m} \geq U_{j,n}$ if $j \geq i$ and $n \geq m$, and since $V[p]$ satisfies Min-Ω we can choose i and m so that

(2) $$U_{i,m} = U_{j,n} \quad \text{for } j \geq i \text{ and } n \geq m .$$

Now the map $v \to p^k v$ determines an embedding of $V\left[p^{k+1}\right]/V\left[p^k\right]$ in $V[p]$ and so each $V\left[p^{k+1}\right]/V\left[p^k\right]$ satisfies Min-Ω. Therefore $V\left[p^k\right]$ satisfies Min-Ω and we may suppose i chosen in (2) so that in addition

$$V_i \cap V\left[p^{m+1}\right] = V_j \cap V\left[p^{m+1}\right] \quad \text{whenever } j \geq i .$$

We now show by induction on n that

(3) $$V_i \cap V\left[p^n\right] = V_j \cap V\left[p^n\right] \quad \text{for all } j \geq i \text{ and } n \geq m+1 .$$

Indeed suppose (3) holds for some $n \geq m+1$ and let $v \in V_i \cap V\left[p^{n+1}\right]$. Then by (2) $p^n v \in U_{i,n} = U_{j,n}$ and so $p^n v = p^n w$ for some $w \in V_j \cap V\left[p^{n+1}\right]$. Therefore $p^n(v-w) = 0$ and $v - w \in V_i \cap V\left[p^n\right] = V_j \cap V\left[p^n\right]$. Hence $v \in V_j$; as required.

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It follows from (3), and the fact that $V = \bigcup_{n \geq m+1} V[p^n]$, that $V_i = V_j$ for all $j \geq i$, whence we have that V has Min-Ω .

In [13] a group possessing a series of finite length in which the factors are periodic divisible abelian groups was called a $\underline{\underline{PQ}}$-group. We extend Example 4 of [14] as follows:-

LEMMA 3.4. *The class of* $\underline{\underline{PQ}}$*-groups satisfying* Min-n *contains groups of any prescribed derived length.*

Proof. The construction is similar to that of [14]. Suppose that we have constructed, for some integer $n \geq 1$, a $\underline{\underline{PQ}}$-group G_n which satisfies Min-n , is a π-group for some finite set π of primes, and is in addition monolithic with monolith M . As G_1 we may take a group of type C_{q^∞} , where q is a prime. Let x be an element of prime order $p \notin \pi$, and let X be the base group of $\langle x \rangle$ wr $G_n = W$. There is a chief series of W through X . If M centralized every factor below X in this series it would centralize X itself since X is a p-group and $p \notin \pi$. Consequently M fails to centralize some such factor, which then furnishes a faithful irreducible p-module V for G_n . It follows from Theorem B1 and Lemma 3.3 that there is a p-module $\bar{V}$ for G_n which satisfies Min-G_n and is such that $V = \bar{V}[p]$. Let G_{n+1} be the semidirect product $\bar{V}G_n$. Then G_{n+1} is a $\underline{\underline{PQ}}$ group with Min-n and is monolithic with monolith V . It follows easily that G_{n+1} has derived length $n + 1$ exactly.

4. Classification of $\underline{\underline{Z}}$-groups

Our aim in this section is to classify, up to isomorphism, groups in the class $\underline{\underline{Z}}$, the class introduced in §1. We shall classify these groups in terms of nilpotent centre-by-finite groups with Min and irreducible modules for such groups. A result which will be of fundamental importance for our classification is the following:

LEMMA 4.1. *Let* $G \in \underline{\underline{Z}}$. *Then* G *splits over* $G^{\underline{\underline{N}}}$ *and the*

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complements to $G^{\underline{\underline{N}}}$ *are conjugate in* G . $G/G^{\underline{\underline{N}}}$ *is centre-by-finite.*

Our proof of Lemma 4.1 will depend on properties of the class $\underline{\underline{U}}$ introduced in [9]. We recall that a locally finite group X belongs to $\underline{\underline{U}}$ if and only if X has a series of finite length with locally nilpotent factors and every subgroup of X has conjugate Sylow (that is, maximal) π-subgroups for all sets π of primes.

LEMMA 4.2. *Let* G *be a soluble group satisfying* Min-n . *Suppose that* G *contains a normal subgroup of finite index which is nilpotent-by-locally nilpotent. Then* $G \in \underline{\underline{U}}$.

Proof. By Baer's Theorem [4] G is locally finite. Since the condition Min-n is inherited by normal subgroups of finite index [16], and since the class $\underline{\underline{U}}$ is closed under extensions by finite soluble groups ([10], Lemma 6.6) we may assume that G contains a normal nilpotent subgroup H such that G/H is locally nilpotent. Then G/H is a locally nilpotent group satisfying Min-n . Such groups satisfy the minimal condition on all subgroups ([6], Corollary 4.6) and so are countable and abelian-by-finite. Therefore, arguing as before, we may suppose that G/H is abelian.

Let X be any subgroup of G and let K be the Sylow π-subgroup of H . Then $XK/K \cong X/X \cap K$ and, since $X \cap K$ is a normal π-subgroup of X , the conjugacy of the Sylow π-subgroups of XK/K implies the conjugacy of those of X . We may therefore assume that $K = 1$.

Let S and T be Sylow π-subgroups of X . We shall now show that S and T are conjugate in X by induction on the nilpotency class c of H . If $c = 0$ then $H = 1$, X is abelian, and $S = T$. Assume then that $c > 0$ and let Z be the centre of H . We may assume by induction that X contains an element x such that $\langle S^x Z/Z, TZ/Z \rangle$ is a π-group, U/Z say. Now U/Z must be countable and so $U = ZW$ for some π-subgroup W of U (see for instance [10], Lemma 2.1). Let L be any subgroup of W and let F be any finite subgroup of L . Then $C_Z(F) = Z \cap C_G(HF)$, which is normal in G since G/H is abelian. Therefore by Min-n we can choose F so that $C_Z(F)$ is minimal among the centralizers in Z of the finite subgroups of L . Then clearly

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$C_Z(F) = C_Z(L)$. It follows from Lemma 4.3 of [10] that every countable subgroup of U containing W has conjugate Sylow π-subgroups, and hence from Theorem B of [10] that every countable subgroup whatsoever of U has conjugate Sylow π-subgroups. Therefore S^x and T are conjugate in the group they generate, which establishes the lemma.

Proof of Lemma 4.1. We have by Lemma 4.2 that $G \in \underline{\underline{U}}$. Let K be the uniquely determined normal subgroup of G which is minimal subject to the condition that G/K is locally nilpotent. Then K is abelian. By [9], Theorem 4.12, G splits over K and the complements to K in G are conjugate - in fact they are the basis normalizers of G . We shall show that $K = G^{\underline{\underline{N}}}$. Now it is clear from (Z) (in §1) that every p-subgroup of G is nilpotent. Since G/K satisfies Min , as we have seen, and is therefore countable, every p-subgroup of G/K is the image of a p-subgroup of G ([10], Lemma 2.1). Therefore the Sylow p-subgroups of G/K are nilpotent and so G/K is nilpotent. Hence $K = G^{\underline{\underline{N}}}$.

Finally, since it is a nilpotent group satisfying Min-n , G/K is centre-by-finite by a theorem of Baer [3].

Lemmas 4.1 and 4.2 are generalizations of Theorem 3.5 and 5.6 of [14].

Now it follows from Lemma 4.1 that in trying to classify groups in the class $\underline{\underline{Z}}$ it is sufficient to restrict ourselves to considering those groups $G \in \underline{\underline{Z}}$ such that $G/G^{\underline{\underline{N}}}$ is isomorphic to a given nilpotent centre-by-finite group A satisfying the minimal condition.

Let $A_{p'}$ be the Sylow p'-subgroup of A and let $\{V_\lambda ; \lambda \in \Lambda_p\}$ be a complete set of representatives for the isomorphism classes of non-trivial irreducible $Z_pA_{p'}$-modules. We assume the sets Λ_p to be pairwise disjoint, as we may. Notice that if A is actually abelian then the V_λ may be constructed by the method of Lemma 2.5. For $n = 1, 2, \ldots, \infty$ let $V_\lambda(n)$ denote the $A_{p'}$-module obtained from V_λ as described before Theorem A, and view each $V_\lambda(n)$ as an A-module by

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allowing A_p to act trivially. Let $\Lambda = \bigcup_p \Lambda_p$ and let $\mathsf{X} = \mathsf{X}(A)$ denote the set of all external direct sums of finitely many modules $V_\lambda(n)$ $(n = 1, 2, \ldots, \infty\ ;\ \lambda \in \Lambda)$. We admit a zero module to X as the direct sum of the empty set.

An equivalence relation is introduced on X as follows. First, if $X \in \mathsf{X}$ and $\alpha \in \mathrm{Aut}A$, let X^α be the A-module which has X as its underlying additive group and which has the A-action defined by

$$(x, a) \to x(a\alpha) \quad (x \in X\ ,\ \ a \in A)\ .$$

Now if X and Y are elements of X we define $X \sim Y$ to mean that $X \cong Y^\alpha$ for some $\alpha \in \mathrm{Aut}A$; we shall say that X is an *automorphism conjugate* of Y . The relation of automorphism conjugacy is easily seen to be an equivalence relation on X .

Finally for each $X \in \mathsf{X}$ let XA denote the semidirect product of X by A , that is the group consisting of all pairs (x, a) , where $x \in X$, $a \in A$, with the multiplication $(x, a)(x', a') = \left(x + x'a^{-1}, aa'\right)$. When appropriate we shall identify X with a subgroup of XA in the usual manner. We now have

THEOREM C. *With the above notation, if* $X \in \mathsf{X}$ *then* $H = XA \in \underline{Z}$, $H^{\underline{\underline{N}}} = X$ *and* $H/H^{\underline{\underline{N}}} \cong A$. *If* $G \in \underline{Z}$ *and* $G/G^{\underline{\underline{N}}} \cong A$ *then* $G \cong XA$ *for some* $X \in \mathsf{X}$. *If* $X, Y \in \mathsf{X}$ *then* $XA \cong YA$ *if and only if* $X \sim Y$.

Thus there is a natural one-to-one correspondence between the isomorphism classes of groups G in $\underline{Z}$ with $G/G^{\underline{\underline{N}}} \cong A$, and the automorphism conjugacy classes of elements of X . We shall have more to say about the relation of automorphism conjugacy after proving Theorem C.

Notice that, with the notation of Theorem C, H will be quasi-radicable if and only if A is quasi-radicable. For if A is quasi-radicable and $n > 1$ then the subgroup generated by the n-th powers of elements of H contains A and so, being normal in H , contains $[X, A] = \left[H^{\underline{\underline{N}}}, A\right] = H^{\underline{\underline{N}}} = X$. Since A is in any case centre-by-finite it follows that H is quasi-radicable if and only if A is abelian and radicable (that is, every element has an n-th root for all $n > 1$).

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Proof of Theorem C. If V_λ is any irreducible module in $\mathfrak{X}$ then since V_λ is non-trivial and irreducible the submodule $[V_\lambda, A]$ additively generated by the elements $v - va$ $(v \in V_\lambda\ ,\ a \in A)$ must be the whole of V_λ. It then follows easily, since $V_\lambda(n+1)/V_\lambda(n) \cong V_\lambda$ if n is finite, that $[V_\lambda(m), A] = V_\lambda(m)$ for any $m = 1, 2, \ldots, \infty$. Hence $[X, A] = X$ for any $X \in \mathfrak{X}$. Since $XA/X \cong A$, which is nilpotent, this means that $X = (XA)^{\underline{\underline{N}}}$.

Now let $G \in \underline{\underline{Z}}$ and suppose that $G/G^{\underline{\underline{N}}} \cong A$. If $G^{\underline{\underline{N}}} = 1$ then $G \cong A$ and taking X to be the zero module there is nothing to prove. Thus we may assume that $G^{\underline{\underline{N}}} \neq 1$. Then by Lemma 4.1 we have, if $K = G^{\underline{\underline{N}}}$

$$G = K\bar{A}\ ,\quad K \cap \bar{A} = 1 \tag{4}$$

for some $\bar{A} \cong A$. G is locally finite and the condition (Z) ensures that the Sylow p-subgroup $\bar{A}_p$ of $\bar{A}$ centralizes the Sylow p-subgroup K_p of the abelian group K. Therefore if we view K_p as an $\bar{A}_{p'}$-module in the natural way it satisfies Min-$\bar{A}_{p'}$. By Lemma 4.1 $\bar{A}_{p'}$ is centre-by-finite and so by Theorem A K_p is a direct sum of finitely many submodules of the type $W(n)$, where W is some irreducible $Z_p\bar{A}_{p'}$-module. Now since K is a non-trivial abelian group it follows from (4) that $K = [K, \bar{A}]$. Hence $K_p = [K_p, \bar{A}_{p'}]$ and

$$W(n) = [W(n), \bar{A}_{p'}]\ . \tag{5}$$

Consequently W must not be a trivial module. Otherwise, since W determines $W(n)$ up to isomorphism $W(n)$ would be trivial, in contradiction to (5).

Let $a \to \bar{a}$ be an isomorphism of A onto $\bar{A}$. We view K as an A-module by defining

$$xa = \bar{a}^{-1}x\bar{a}\quad (x \in K\ ,\ a \in A)\ .$$

It now follows from the remarks just made that K is isomorphic to some module $X \in \mathfrak{X}$. Let ψ be an A-isomorphism of X onto K. Then the

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map $(x, a) \to x\psi.\bar{a}$ maps XA isomorphically onto G .

Finally let $X, Y \in \mathfrak{X}$. If $X \sim Y$ then $X \cong Y^{\alpha}$ for some $\alpha \in \operatorname{Aut}A$. Let ψ be an A-isomorphism of X onto Y^{α} . Then the map $(x, a) \to (x\psi, a\alpha)$ maps XA isomorphically onto YA . On the other hand suppose that $XA \cong YA$ and let ϕ be an isomorphism of XA onto YA . We have already seen that $X = (XA)^{\underline{\underline{N}}}$ and $Y = (YA)^{\underline{\underline{N}}}$ and so ϕ maps X onto Y . Since Y is abelian the complements to it in YA are conjugate under the automorphism group of YA ; in fact in this case, by Lemma 4.1, they are conjugate in YA itself. We may therefore assume that ϕ maps the elements of the form $(0, a)$ in XA to the elements of similar form in YA . Thus ϕ has the form $(x, a) \to (x\psi, a\alpha)$ where ψ is an additive isomorphism of X onto Y and α is a bijection of A onto itself. It is then easy to verify that $\alpha \in \operatorname{Aut}A$ and ψ determines an isomorphism of X onto Y^{α} . Therefore $X \sim Y$, which completes the proof of Theorem C.

We notice, for example, that if A is a non-trivial locally cyclic p'-group then there is, up to automorphism conjugacy, exactly one faithful irreducible Z_pA-module. For by Lemma 2.5 every such module has the form K_θ for some monomorphism θ of A into an algebraic closure k of Z_p . The existence of such a θ follows since A is a locally cyclic p'-group, and clearly $K_\phi = K_\theta^{\phi\theta^{-1}}$. Consequently there is, up to isomorphism, exactly one quasi-radicable metabelian group with Min-n of the form NA , where N is a normal p-subgroup faithfully and irreducibly transformed by a given non-trivial radicable locally cyclic p'-group A . Such groups were first constructed by Čarin [5].

Let $V_\lambda(n)^m$ denote that member of $\dot{\mathfrak{X}}$ which is given as the direct sum of m copies of $V_\lambda(n)$ $(m \geq 1)$. Then, since each member X of $\mathfrak{X}$ is given together with a direct decomposition, X determines uniquely a set

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$$S(X) = \left\{V_{\lambda_1}\left(n_1\right)^{m_1}, \ldots, V_{\lambda_k}\left(n_k\right)^{m_k}\right\}$$

where $k \geq 0$ and the pairs $\left(\lambda_i, n_i\right)$ are all distinct. Now if elements X and Y of X are isomorphic then their p-components are isomorphic as $A_{p'}$-modules for each prime p ; Theorem A then shows that this happens if and only if $S(X) = S(Y)$.

LEMMA 4.3. *Let* $X, Y \in \mathrm{X}$ *and suppose*

$$S(X) = \left\{V_{\lambda_1}\left(n_1\right)^{m_1}, \ldots, V_{\lambda_k}\left(n_k\right)^{m_k}\right\}$$

and

$$S(Y) = \left\{V_{\mu_1}\left(s_1\right)^{t_1}, \ldots, V_{\mu_l}\left(s_l\right)^{t_l}\right\} .$$

Then $X \sim Y$ *if and only if*

(i) $k = l$,

(ii) *there exists an automorphism* α *of* A *and a permutation* σ *of* $\{1, 2, \ldots, k\}$ *such that* $V_{\lambda_i} \cong V^{\alpha}_{\mu_{i\sigma}}$, $n_i = s_{i\sigma}$, $m_i = t_{i\sigma}$ *for* $1 \leq i \leq k$.

Proof. Suppose first that $X \sim Y$. Then $X \cong Y^{\alpha}$ for some $\alpha \in \mathrm{Aut}A$ and, since $(U \oplus W)^{\alpha} = U^{\alpha} \oplus W^{\alpha}$ for any A-modules U and W , we have

$$(6)\quad V_{\lambda_1}\left(n_1\right)^{m_1} \oplus \ldots \oplus V_{\lambda_k}\left(n_k\right)^{m_k} \cong \left(V_{\mu_1}\left(s_1\right)^{\alpha}\right)^{t_1} \oplus \ldots \oplus \left(V_{\mu_l}\left(s_l\right)^{\alpha}\right)^{t_l} .$$

Since the modules $V_{\lambda_i}\left(n_i\right)$ and $V_{\mu_j}\left(s_j\right)^{\alpha}$ are indecomposable it follows from Theorem A that there is a one-to-one correspondence between the summands of this form on the two sides of (6) such that corresponding summands are isomorphic. Now $V_{\mu_i}\left(s_i\right)^{\alpha} \cong V_{\mu_j}\left(s_j\right)^{\alpha}$ if and only if $i = j$; consequently to each i with $1 \leq i \leq k$ there is a uniquely determined

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integer $i\sigma$ with $1 \leq i\sigma \leq l$, such that $V_{\lambda_i}(n_i) \cong V_{\mu_{i\sigma}}(s_{i\sigma})^{\alpha}$. It then follows that $n_i = s_{i\sigma}$, $m_i = t_{i\sigma}$, $k = l$, and that σ is a permutation of $\{1, 2, \ldots, k\}$. Clearly $V_{\lambda_i} \cong V^{\alpha}_{\mu_{i\sigma}}$.

Conversely suppose that *(i)* and *(ii)* hold. Then by Theorem A $V_{\mu_{i\sigma}}(s_{i\sigma})^{\alpha}$ must be isomorphic to some module of the form $V_{\lambda}(n)$, and consideration of the minimal submodules shows that the module required must be $V_{\lambda_i}(n_i)$. We then easily obtain (6) and hence that $X \cong Y^{\alpha}$, as required.

Let us call an abelian p-group *homogeneous* if it is either homocyclic or divisible. As an application of Theorem C and Lemma 4.3 we prove

COROLLARY 4.4. *Let* G_1 *and* G_2 *belong to* $\underline{Z}$. *For* $i = 1, 2$ *let* $K_i = (G_i)^{\underline{N}}$, *let* N_i *be the product of the minimal normal subgroups of* G_i *contained in* K_i, *and let* A_i *be a complement for* K_i *in* G_i. *Suppose that, for each prime* p, *the* p*-component of* K_i *is homogeneous of exponent* $p^{n_i(p)}$. *Then* $G_1 \cong G_2$ *if and only if*

(i) $n_1(p) = n_2(p)$ *for each prime* p,

(ii) $N_1A_1 \cong N_2A_2$.

Proof. If $G_1 \cong G_2$ then any isomorphism from G_1 to G_2 maps K_1 onto K_2, and as in the argument of Theorem C there exists an isomorphism which maps A_1 onto A_2. Such an isomorphism maps N_1A_1 isomorphically onto N_2A_2. Thus the necessity of the conditions is clear.

To see the sufficiency we notice first that by Theorem C the minimal normal subgroups of G_i in K_i are non-central. Hence $N_i = [N_i, A_i]$, and so $N_i = (N_iA_i)^{\underline{N}}$. Therefore it follows from *(ii)* that $A_1 \cong A_2$. Theorem C now allows us to assume that $G_i = X_iA$, where A is a non-trivial nilpotent centre-by-finite group with Min and $X_i \in \mathfrak{X} = \mathfrak{X}(A)$

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$(i = 1, 2)$. Since the p-components of X_1 are homogeneous,

$$S(X_1) = \left\{V_{\lambda_1}(s_1)^{t_1}, \ldots, V_{\lambda_k}(s_k)^{t_k}\right\}$$

where $\lambda_1, \ldots, \lambda_k$ are all distinct and $s_i = n_1(p)$ if V_{λ_i} is a p-module. The subgroup N_1A_1 of G_1 corresponds naturally to Y_1A_1 where $S(Y_1) = \left\{V_{\lambda_1}(1)^{t_1}, \ldots, V_{\lambda_k}(1)^{t_k}\right\}$. Thus k is the number of distinct isomorphism types of A-modules in Y_1 . It now follows from *(ii)* that $S(X_2) = \left\{V_{\mu_1}(u_1)^{w_1}, \ldots, V_{\mu_k}(u_k)^{w_k}\right\}$, where $\mu_1, \ldots, \mu_k$ are all distinct. Since $Y_1A \cong Y_2A$, Theorem C gives $Y_1 \sim Y_2$. Hence by Lemma 4.3 there is an automorphism α of A and a permutation σ of $\{1, 2, \ldots, k\}$ such that $V_{\lambda_i} \cong V^{\alpha}_{\mu_{i\sigma}}$ and $t_i = w_{i\sigma}$. Now $s_i = n_1(p)$ and $u_{i\sigma} = n_2(p)$, where p is the prime such that both V_{λ_i} and $V^{\alpha}_{\mu_{i\sigma}}$ are p-modules. Hence by *(i)* $s_i = u_{i\sigma}$. Lemma 4.3 now shows that $X_1 \sim X_2$, and Theorem C gives $X_1A \cong X_2A$, which completes the proof.

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