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SOME ASYMPTOTIC PROPERTIES OF SOLUTIONS OF A NEUTRAL DELAY EQUATION WITH AN OSCILLATORY COEFFICIENT

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ABSTRACT. The authors consider the nonlinear neutral delay differential equation

$$[y(t)+P(t)y(g(t))]^{(n)}+Q(t)f(y(h(t)))=0$$

and obtain results on the asymptotic behavior of solutions. Some of the results require that $P(t)$ has arbitrarily large zeros or that $P(t)$ oscillates about -1.

1. **Introduction.** Recently the oscillatory and asymptotic behavior of the solutions of neutral delay differential equations (differential equations in which the highest order derivative of the unknown function appears both with and without delays) has been studied by a number of authors, for example see [1–15] and the references contained therein. Virtually all such results are for equations of the form

$$\text{(D)} \qquad \Big[y(t)+P(t)y\big(g(t)\big)\Big]^{(n)}+F\Big(t,y\big(h_1(t)\big),\dots,y\big(h_m(t)\big)\Big)=0$$

with the majority of them being for n either one or two, $g(t)=t-\tau$, and $h_i(t)=t-\sigma_i$ for $i=1,2,\dots,m$. Furthermore, the function P is almost always required to either be a constant function or have its image to be a subset of either $[\lambda_1,\ -1]$, $[-1,\ 0]$, or $[0,\ \lambda_2]$ for some constants λ_1 and λ_2. In particular, Ladas and Sficas [8] showed that if p and q are constants, $q>0$, and n is odd, then the point $p=-1$ is a bifurcation point for the behavior of the nonoscillatory solutions of the constant coefficient linear equation

$$(\mathrm{E}_c) \qquad [y(t)+py(t-\tau)]^{(n)}+qy(t-\sigma)=0.$$

More precisely, they showed that:

(i) If $p<-1$, the absolute value of any nonoscillatory solution of (E_c) tends to $+\infty$ as $t\longrightarrow\infty$;

(ii) If $p=-1$, every solution of (E_c) oscillates; and

(iii) If $p>-1$, any nonoscillatory solution of (E_c) tends to 0 as $t\longrightarrow\infty$.

Ladas and Sficas [8, Theorem 1b] also showed that if n is even and $p>-1$, then every nonoscillatory solution $y(t)$ of (E_c) satisfies

$$(*) \qquad y(t)\longrightarrow 0 \text{ as } t\longrightarrow\infty.$$

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They also conjectured that if n is even and $p = -1$, then $(*)$ holds. Examples given in [8] showed that for n even and $p < -1$ it is possible to have

$$(**) \qquad |y(t)| \longrightarrow \infty \text{ as } t \longrightarrow \infty$$

as well as $(*)$. This led them to conjecture that if n is even, $p < -1$, and $y(t) \not\equiv 0$ is a solution with $z(t) = y(t) + py(t-\tau)$ satisfying $z^{(i)}(t)z^{(i+1)}(t) < 0$ for $i = 1, 2, \ldots, n-1$, then $(*)$ holds. Xu [14] proved that both conjectures in [8] were true ([14, Theorems 1 and 2]). In [14, Theorem 5], Xu proved that if n is even, $p < -1$, $z^{(k)}(t)z^{(k-1)}(t) > 0$ for some k with $0 < k \leq n-1$, then $(**)$ holds. Thus, by combining Theorems 2 and 5 in [14], if n is even and $p < -1$, then any nonoscillatory solution of (E_c) satisfies either $(*)$ or $(**)$ (see Conjecture 1 in [2]). In another direction, and this is not applicable in the constant coefficient case, many of the results on the asymptotic behavior of nonoscillatory solutions allow for cases like $P(t) \geq 0$ or $P(t) \leq 0$. While some authors have proved some oscillation results which allow for $P(t)$ to oscillate, only the recent paper of Lu [12] on linear equations has results explicitly requiring $P(t)$ to have arbitrarily large zeros.

The discussion above gives rise to the following two questions. First, what effect does allowing the function $P(t)$ to oscillate about -1 and/or 0 have on the oscillatory and asymptotic behavior of solutions; and, second, does (ii) hold for nonlinear equations when $P(t) \equiv -1$?

Here we investigate these questions for the nonlinear equation

$$(\mathrm{E}) \qquad \left[y(t) + P(t)y\big(g(t)\big)\right]^{(n)} + Q(t)f\Big(y\big(h(t)\big)\Big) = 0$$

where: $n \geq 1$; $g, h, P, Q\colon [t_0, \infty) \longrightarrow R$ are continuous with $Q(t) \geq 0$, $g(t) \leq t$, $h(t) \leq t$, $g(t) \longrightarrow \infty$ and $h(t) \longrightarrow \infty$ as $t \longrightarrow \infty$; neither P nor Q is identically zero on any half line; and $f\colon R \longrightarrow R$ is continuous with $uf(u) > 0$ for $u \neq 0$. A number of our results will be for the case when $g(t) = t - \tau$, $\tau > 0$, *i.e.*, for the equation

$$(\mathrm{E}_\tau) \qquad [y(t) + P(t)y(t-\tau)]^{(n)} + Q(t)f\Big(y\big(h(t)\big)\Big) = 0.$$

For notational purposes we let

$$z(t) = y(t) + P(t)y(t-\tau) \text{ and } w(t) = y(t) + P(t)y\big(g(t)\big).$$

We will also ask that

$$(1) \qquad f(u) \text{ is bounded away from zero if } u \text{ is bounded away from zero,}$$

and that

$$(2) \qquad \int_{t_0}^{\infty} Q(s)\, ds = \infty.$$

Since we are only interested in the oscillatory and asymptotic properties of solutions, every solution $y(t)$ of either (E) or (E_τ) considered here is assumed to be continuable and nontrivial, *i.e.*, $y(t)$ is defined on $[t_y, \infty)$ for some $t_y \geq t_0$ and $\sup\{|y(t)| : t \geq t_1\} > 0$ for every $t_1 \geq t_y$. Such a solution will be called *oscillatory* if its set of zeros is unbounded from above and will be called *nonoscillatory* otherwise. We will say that either (E) or (E_τ) is oscillatory if all its nontrivial continuable solutions are oscillatory.

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2. **Oscillation and asymptotic behavior.** For use in the proof of our first result, we will say that $P(t)$ has *property (H)* if:

(H) for every $t_1 \geq t_0$ there exists $T \geq t_1$ with the property that for each fixed t in $[T, T+\tau]$ there is a nonnegative integer N_t such that $P(t+N_t\tau) = -P_t > -1$.

We will also need the following result which was proved as a part of Lemma 2 in [2].

LEMMA 1. *Suppose that, in addition to (1) and (2), there exists a constant $P_1 < 0$ such that $P_1 \leq P(t) \leq 0$. If n is even and $y(t)$ is a nonoscillatory solution of (E), then $w(t) < 0$ $\big(w(t) > 0\big)$ for $y(t) > 0$ $\big(y(t) < 0\big)$.*

Throughout this paper, proofs will be carried out only for the case when a nonoscillatory solution of the equation involved is eventually positive, since the corresponding proof for an eventually negative solution is similar in each case. We begin with a result for (E_τ) where $P(t)$ is allowed to oscillate about -1.

THEOREM 2. *Suppose that, in addition to (1)–(2), n is even and there exists a constant $P_2 < 0$ such that*

$$P_2 \leq P(t) < 0. \tag{3}$$

If $P(t)$ has property (H), then any nonoscillatory solution $y(t)$ of (E_τ) satisfies $y(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

PROOF. Let $y(t)$ be a nonoscillatory solution of (E_τ) such that $y(t-\tau) > 0$ and $y\big(h(t)\big) > 0$ for $t \geq t_1 \geq t_0$. By Lemma 1, there exists $t_2 \geq t_1$ such that $z(t) < 0$ for $t \geq t_2$; since $P(t)$ has property (H) and satisfies (3), there exists $T \geq t_2$ such that for each t in $[T, T+\tau]$, $0 > P(t+N_t\tau) = -P_t > -1$ for some positive integer N_t. Therefore, for each fixed t in $[T, T+\tau]$,

$$y(t+N_t\tau) + P(t+N_t\tau)y\big(t+(N_t-1)\tau\big) = z(t+N_t\tau) < 0,$$

and hence

$$y(t+N_t\tau) < P_t y\big(t+(N_t-1)\tau\big).$$

Iterating, we obtain

$$y\big(t+(N_t+j)\tau\big) < P_t^{j+1} y\big(t+(N_t-1)\tau\big) \longrightarrow 0$$

as $j \longrightarrow \infty$. Thus, for each fixed t in $[T, T+\tau]$, $y(t+k\tau) \longrightarrow 0$ as $k \longrightarrow \infty$ and therefore $y(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

If Q and f satisfy (1)–(2) and n is even, then the equation

$$\left[y(t) + \left(\sin t - \frac{3}{2}\right)y\left(t - \frac{\pi}{2}\right)\right]^{(n)} + Q(t)f\Big(y\big(h(t)\big)\Big) = 0 \tag{E_1}$$

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satisfies all the hypotheses of Theorem 2. To see that $P(t) = \sin t - \frac{3}{2}$ has property (H), let $T > 0$ be given and let t be any number in $[T, T + \frac{\pi}{2}]$. We must show that there exists a nonnegative integer N_t such that

$$P\Big(t + N_t\frac{\pi}{2}\Big) = \sin\Big(t + N_t\frac{\pi}{2}\Big) - \frac{3}{2} > -1$$

or,

$$\sin\Big(t + N_t\frac{\pi}{2}\Big) = \sin t \cos\Big(N_t\frac{\pi}{2}\Big) + \cos t \sin\Big(N_t\frac{\pi}{2}\Big) > \frac{1}{2}.$$

If $|\sin t| \le \frac{1}{2}$, then $|\cos t| > \frac{1}{2}$. In this case, it suffices to let $N_t = 1$ if $\cos t > \frac{1}{2}$ and $N_t = 3$ if $\cos t < -\frac{1}{2}$. For $|\sin t| > \frac{1}{2}$, let $N_t = 0$ if $\sin t > \frac{1}{2}$ and $N_t = 2$ if $\sin t < -\frac{1}{2}$.

Next we prove a result for (E) with $P(t)$ oscillating about 0.

THEOREM 3. *If n is even, (1)–(2) hold, the set of zeros of $P(t)$ is unbounded from above, and*

$$(3') \qquad P_1 \le P(t) \le 0,$$

then (E) is oscillatory.

PROOF. Assume that (E) has an eventually positive solution $y(t)$, say $y\big(g(t)\big) > 0$ and $y\big(h(t)\big) > 0$ for $t \ge t_1 \ge t_0$. Lemma 1 ensures the existence of $t_2 \ge t_1$ such that $w(t) < 0$ for $t \ge t_2$. Choose $T > t_2$ such that $P(T) = 0$. Then $y(T) = w(T) < 0$ contradicting $y(t) > 0$ on $[t_1, \infty)$.

REMARK. The conclusion of Theorem 2 was obtained for equation (E) in [2, Theorem 7] with $g(t)$ increasing and with condition (3) and property (H) replaced by

$$(4) \qquad -1 < P_1' \le P(t) \le 0.$$

Notice that, in this connection, if (4) holds and (E_τ) has a nonoscillatory solution, then Theorem 3 implies that (3) eventually holds. Thus the upper bound imposed on $P(t)$ in (3) is no more restrictive than that required by (4). Also, as a part of Theorem 1(b) in [5], it was proved that the nonoscillatory solutions of the linear equation

$$(E_\tau') \qquad [y(t) + P(t)y(t-\tau)]^{(n)} + Q(t)y(t-\sigma) = 0,$$

satisfy the conclusion of Theorem 2 provided $-1 < P_1' \le P(t) \le P_1'' < 0$ and $Q(t) > q > 0$. The same conclusion was obtained as a part of Theorem 1(b) in [8] for equation (E_c) with $-1 < p$ and $q > 0$. In Theorem 1 and a part of Theorem 4 in [14], Xu showed that, for n even, the nonoscillatory solutions of an equation of type (E_c) with several delays satisfy the conclusion of Theorem 2 for $p \ge -1$. Furthermore, he showed [14, Theorem 2] the same result for the eventually positive solutions of that equation when $p < -1$ and $z^{(i)}(t)z^{(i-1)}(t) < 0, i = 1, 2, \ldots, n$. Notice that none of these results apply to equation (E_1) since they all require either $P(t) > -1$ or $P(t) \equiv p$. In this sense Theorem 2 generalizes all of these results.

Xu [14, Theorem 6] also proved that for n even and $p \ge 0$, all solutions of equation (E_c) (with several delays) oscillate. Notice that this result and Theorem 3 overlap when $P(t) \equiv p = 0$.

Next we state a lemma that was proved as a part of Lemma 1 in [2].

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LEMMA 4. *Suppose that (1) and (2) hold and that $y(t)$ is an eventually positive (negative) solution of (E).*

(a) If $w(t) \longrightarrow 0$ as $t \longrightarrow \infty$, then $w^{(i)}(t)$ is monotonic, $w^{(i)}(t) \longrightarrow 0$ as $t \longrightarrow \infty$, and $w^{(i)}(t)w^{(i+1)}(t) < 0$ for $i = 0, 1, \ldots, n-1$.

(b) Let $w(t) \longrightarrow 0$ as $t \longrightarrow \infty$. If n is even, then $w(t) < 0$ $\big(w(t) > 0\big)$ for $y(t) > 0$ $\big(y(t) < 0\big)$. If n is odd, then $w(t) > 0$ $\big(w(t) < 0\big)$ for $y(t) > 0$ $\big(y(t) < 0\big)$.

The following additional conditions on the functions f, g, h, and Q will be used in various places in the remainder of the paper. Assume that:

$$f \text{ is increasing,} \tag{5}$$

$$f(u+v) \leq f(u) + f(v) \text{ if } u, v > 0, \tag{6}$$

$$f(u+v) \geq f(u) + f(v) \text{ if } u, v < 0, \tag{7}$$

$$\text{both } h' \text{ and } g' \text{ are integrable on } [t_0, b] \text{ for all } b > t_0, \tag{8}$$

and that there exist positive constants k_1, k_2, and q such that

$$k_1 \leq g'(t), \tag{9}$$

$$0 \leq h'(t) \leq k_2, \tag{10}$$

and

$$Q(t) \geq q. \tag{11}$$

We will also assume that for each positive constant k_3 there exists a positive constant A such that

$$|f(k_3 u)| \leq A|f(u)| \text{ for all } u. \tag{12}$$

LEMMA 5. *Suppose that, in addition to (5)–(12), there exists a positive constant P_3 such that*

$$|P(t)| \leq P_3. \tag{13}$$

(a) If $y(t)$ is an eventually positive solution of (E), then either

$$w^{(i)}(t) \longrightarrow -\infty \text{ as } t \longrightarrow \infty \tag{14}$$

for $i = 0, 1, \ldots, n-1$, or

$$w^{(i)}(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty \text{ with } w^{(i)}(t)w^{(i+1)}(t) < 0 \tag{15}$$

for $i = 0, 1, \ldots, n-1$.

(b) If $y(t)$ is an eventually negative solution of (E), then either

$$w^{(i)}(t) \longrightarrow \infty \text{ as } t \longrightarrow \infty \tag{16}$$

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for $i = 0, 1, \ldots, n-1$, *or (15) holds.*

PROOF. Let $y(t)$ be an eventually positive solution of (E) and let $t_1 \geq t_0$ be such that $y\big(g(t)\big) > 0$ and $y\big(h(t)\big) > 0$ on $[t_1, \infty)$. Then

$$w^{(n)}(t) = -Q(t)f\Big(y\big(h(t)\big)\Big) < 0, \tag{17}$$

and therefore $w^{(i)}(t)$ is monotonic for $i = 0, 1, \ldots, n-1$ and $w^{(n-1)}(t) \longrightarrow L < \infty$ as $t \longrightarrow \infty$. If $L = -\infty$, then clearly $w^{(i)}(t) \longrightarrow -\infty$ as $t \longrightarrow \infty$ for $i = 0, 1, \ldots, n-1$. If $L > -\infty$, then integrating (17) over $[t, T]$ and then letting $T \longrightarrow \infty$ yields

$$w^{(n-1)}(t) = L + \int_t^\infty Q(s)f\Big(y\big(h(s)\big)\Big)\, ds \geq L + q\int_t^\infty f\Big(y\big(h(s)\big)\Big)\, ds$$

by (11). Hence $f\Big(y\big(h(t)\big)\Big) \in L^1[t_0, \infty)$ and, in view of (8)–(10), it is easily seen that $f\big(y(t)\big)$ and $f\Big(y\big(g(t)\big)\Big)$ also belong to $L^1[t_0, \infty)$. In addition, by (13),

$$|w(t)| \leq y(t) + P_3 y\big(g(t)\big). \tag{18}$$

Since f is increasing, it follows from (6) and (12) that

$$f\big(|w(t)|\big) \leq f\big(y(t)\big) + Af\Big(y\big(g(t)\big)\Big), \tag{19}$$

for some $A > 0$, so $f\big(|w(t)|\big) \in L^1[t_0, \infty)$. Therefore, $\liminf_{t\to\infty} |w(t)| = 0$ and, since $w(t)$ is monotonic, $w(t) \longrightarrow 0$ as $t \longrightarrow \infty$. We then have, by Lemma 4, that (15) holds.

REMARK. While conditions (5)–(7) and (12) imposed on the function f in the hypotheses of Lemma 5 may seem to be somewhat artificial, we observe that $f(u) = u^\gamma$, where $\gamma \leq 1$ is the quotient of odd positive integers, satisfies these conditions.

THEOREM 6. *Let (5)–(13) hold. If n is even and $P(t)$ is not eventually negative, then (E) is oscillatory.*

PROOF. Asssume that (E) has an eventually positive solution $y(t)$. Then there exists $t_1 \geq t_0$ such that $y\big(g(t)\big) > 0$ and $y\big(h(t)\big) > 0$ for $t \geq t_1$. Since $P(t)$ is not eventually negative, (14) cannot hold. Thus, by Lemma 5, (15) holds. In particular, n even implies that $w(t)$ is negative and satisfies $w(t) \longrightarrow 0$ as $t \longrightarrow \infty$. But $w(t) < 0$ contradicts $P(t)$ not eventually negative.

REMARK. When $Q(t)$ and $P(t)$ satisty (11) and (13) respectively and $P(t)$ is not eventuually negative, equation (E'_τ) satisfies all hypotheses of Theorem 6. Consequently, Theorem 6 generalizes Theorem 7 in [5]. Theorem 6 also generalizes Theorem 2 of Lu [11], who considered the linear equation (E'_τ) with several delays. Whether or not the conclusion of Theorem 6 remains valid for nonlinear equations under hypotheses less restrictive than (5)–(13) is still an open question. In this connection, we note that it has been proved [2, Theorem 5], that (1)–(2) and $P(t)$ not eventually negative are sufficient to ensure that any solution $y(t)$ of (E) is either oscillatory or satisfies $\liminf_{t\to\infty} |y(t)| = 0$.

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Theorem 6 implies that the equation

$$(E_2) \qquad \left[y(t) + \frac{1}{2}(\sin t)y(t-1)\right]^{(n)} + (2+\cos t)\, y^{\frac{1}{3}}(t-2) = 0,$$

$t \geq 2$, is oscillatory for n even. Notice that while $P(t) = \frac{1}{2}\sin t$ has arbitrarily large zeros, Theorem 3 does not imply that (E_2) is oscillatory since $P(t)$ changes sign on every half line $[t, \infty)$, $t \geq t_0$, and hence (3′) is not satisfied.

The next result is for the case when n is odd.

THEOREM 7. *Suppose that (5)–(13) hold and that n is odd. If $P(t) \geq 0$, then any nonoscillatory solution $y(t)$ of (E) satisfies $y(t) \longrightarrow 0$ as $t \longrightarrow \infty$.*

PROOF. Let $y(t)$ be an eventually positive solution of (E), say $y\big(g(t)\big) > 0$ and $y\big(h(t)\big) > 0$ for $t \geq t_1 \geq t_0$. Now $P(t) \geq 0$ makes (14) impossible, so by Lemma 5, (15) holds. But then $y(t) \longrightarrow 0$ as $t \longrightarrow \infty$ since $y(t) \leq w(t)$.

The equation

$$(E_3) \qquad [y(t) + (1+\sin t)y(t-1)]^{(2k-1)} + (2+\cos t)y^{\frac{1}{3}}(t-2) = 0,$$

$k \geq 1$, satisfies all the hypotheses of Theorem 7.

REMARK. The conclusions of Theorems 6 and 7 were obtained in Theorems 11 and 12 respectively in [2] under the hypotheses that (11) holds and that there exists a positive constant A_1 such that

$$(20) \qquad |f(u)| \geq A_1|u| \text{ for all } u.$$

Notice that the theorems in [2] do not apply to equations (E_2) and (E_3) since the function $f(u) = u^{\frac{1}{3}}$ in both these examples fails to satisfy the inequality in (20) for $|u| > 1$. Thus, Theorems 6 and 7 in this paper are independent of Theorems 11 and 12 in [2]. Also, Theorem 7 generalizes a part of Theorem 1(b) in [5]. Xu [14, Theorem 4] proved a result like Theorem 7 for linear equations with constant coefficients and several delays under the assumption that $P(t) \equiv p > -1$.

Next, we obtain a boundedness result for the nonoscillatory solutions of (E_τ), with $P(t)$ allowed to change signs, by requiring a modified version of (20), namely, that there exist positive constants B and C such that

$$(21) \qquad |f(u)| \geq B|u| \text{ for } |u| < C.$$

THEOREM 8. *Suppose that (5)–(8), (10)–(12), and (21) hold and that there exists a constant P_4 such that*

$$(22) \qquad -1 \leq P(t) \leq P_4.$$

Then every nonoscillatory solution of (E_τ) is bounded.

PROOF. Let y(t) be a nonoscillatory solution of (E_τ) such that $y(t-\tau) > 0$ and $y\big(h(t)\big) > 0$ for $t \geq t_1 \geq t_0$. First observe that (9) is satisfied, since $g(t) = t - \tau$, and that

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(22) implies (13). Therefore, the hypotheses of Lemma 5 are satisfied, so either (14) or (15) holds. If (14) holds, we have, by (22), that $-y(t-\tau) \leq z(t) \longrightarrow -\infty$ as $t \longrightarrow \infty$, so $y(t) \longrightarrow \infty$ as $t \longrightarrow \infty$. But, again by (22), $y(t) = z(t) - P(t)y(t-\tau) \leq y(t-\tau)$ contradicting $y(t) \longrightarrow \infty$ as $t \longrightarrow \infty$. Hence, we conclude that (15) holds.

If n is odd, there exists $t_2 \geq t_1$ such that $0 < z(t) < C$ for $t \geq t_2$. Furthermore, from the proof of Lemma 5, we have $f\big(z(t)\big) \in L^1[t_0, \infty)$ which, together with (21), implies that $z(t) \in L^1[t_0, \infty)$. Now $z(t) \in L^1[t_0, \infty)$ and monotonic implies that the series $\sum_{\ell=1}^{\infty} z(t + \ell\tau)$ converges uniformly on $[T, T+\tau]$ for any $T \geq t_0$. Thus, there exists a positive integer N such that for all k, $\sum_{\ell=N+1}^{N+k} z(t+\ell\tau) < 1$ for every t in $[T, T+\tau]$. But

$$\sum_{\ell=N+1}^{N+k} z(t+\ell\tau) = y\big(t+(N+k)\tau\big) + P\big(t+(N+1)\tau\big)y(t+N\tau) + \sum_{j=2}^{k}\Big[1 + P\big(t+(N+j)\tau\big)\Big]y\big(t+(N+j-1)\tau\big)$$

which implies, in view of (22), that for all k

$$y\big(t+(N+k)\tau\big) < 1 + y(t+N\tau)$$

for each t in $[T, T+\tau]$. Hence $y(t)$ is bounded.

If n is even, then $z(t) < 0$ by (15) and so $P(t)$ is eventually negative. Now if $y(t)$ is unbounded, *i.e.*, $\limsup_{t\to\infty} y(t) = \infty$, then there is an increasing sequence $\{s_k\} \longrightarrow \infty$ as $k \longrightarrow \infty$ such that $P(t) < 0$ for $t \geq s_1$,

$$y(s_k) \longrightarrow \infty \text{ as } k \longrightarrow \infty, \text{ and } y(s_k) = \max\{y(t) : t_1 \leq t \leq s_k\}.$$

We then have

$$z(s_k) = y(s_k) + P(s_k)y\big(g(s_k)\big) \geq y(s_k) + P(s_k)y(s_k) = \big(1 + P(s_k)\big)y(s_k) \geq 0$$

which contradicts $z(t) < 0$. Thus $y(t)$ is bounded.

COROLLARY 9. *Suppose that (5)–(8), (10)–(12) and (21) hold. If n is odd and $P(t) \equiv -1$, then (E_τ) is oscillatory.*

PROOF. Assume that (E_τ) has a nonoscillatory solution $y(t)$. By Theorem 8, $y(t)$ is bounded. But under hypotheses implied by (5) and (11), Theorem 8 in [2] shows that $|y(t)| \longrightarrow \infty$ as $t \longrightarrow \infty$ contradicting the boundedness of $y(t)$.

REMARK. Notice that the equation

$$(E_4) \qquad [y(t) - y(t-\tau)]^{(2k-1)} + Q(t)y^{\gamma}\big(h(t)\big) = 0,$$

$k \geq 1$, satisfies all the hypotheses of Corollary 9 for any h and Q satisfying (8) and (10)–(11), and $\gamma \leq 1$ the quotient of odd positive integers. Hence, Corollary 9 generalizes Theorem 2(ii) in [8] and the single delay case of the corollary in [11]. Corollary 9 also gives an affirmative answer to the second question raised in the Introduction. As a part

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of Theorem 6 in [14], Xu extended the result of Ladas and Sficas [8, Theorem 2(ii)] to an equation of type (E_c) with several delays. Thus, Corollary 9 also generalizes Xu's result for the case of one delay.

In our final theorem we replace condition (22) in Theorem 8 by the more restrictive condition that there exists a constant P_5 such that

$$-1 < P_5 \leq P(t) \leq P_4 \tag{23}$$

and obtain the following sharper conclusion.

THEOREM 10. *If conditions (5)–(8), (10)–(12), (21), and (23) hold, then any non-oscillatory solution $y(t)$ of (E_τ) satisfies $y(t) \longrightarrow 0$ as $t \longrightarrow \infty$.*

PROOF. Let $y(t)$ be an eventually positive solution of (E_τ), say $y(t-\tau) > 0$ and $y\big(h(t)\big) > 0$ for $t \geq t_1 \geq t_0$. Then by Theorem 8, $y(t)$ is bounded. Let T be any number satisfying $T \geq t_1$ and suppose there exists t in $[T, T+\tau]$ such that $y(t+k\tau) \not\longrightarrow 0$ as $k \longrightarrow \infty$. Then there is a subsequence $\{k_j\}$ of $\{k\}$ such that $y(t+k_j\tau) \longrightarrow K > 0$ as $j \longrightarrow \infty$. Choose $\lambda > 0$ so that $0 < P_6 = \lambda - P_5 < 1$; then, since $z(t) \longrightarrow 0$ as $t \longrightarrow \infty$, there exists a positive integer N such that $z(t+N\tau) < \lambda y\big(t+(N-1)\tau\big)$. Hence we have

$$\begin{aligned} y(t+N\tau) &= z(t+N\tau) - P(t+N\tau)y\big(t+(N-1)\tau\big) \\ &< \lambda y\big(t+(N-1)\tau\big) - P_5 y\big(t+(N-1)\tau\big) \\ &= P_6 y\big(t+(N-1)\tau\big). \end{aligned}$$

Iterating, we obtain

$$y\big(t+(N+\ell)\tau\big) < (P_6)^{\ell+1} y\big(t+(N-1)\tau\big),$$

$\ell = 0, 1, 2, \ldots$. The last inequality implies that $y(t+k\tau) \longrightarrow 0$ as $k \longrightarrow \infty$ which is a contradiction. Thus, we conclude that for each fixed t in $[T, T+\tau], y(t+k\tau) \longrightarrow 0$ as $k \longrightarrow \infty$ and therefore $y(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

REMARK. Notice that while the conclusions of Theorems 7 and 10 are identical, the conditions imposed on the functions f, g and P in the two theorems are different so that neither theorem implies the other.

REMARK. As noted in the remark following Theorem 7, the conclusion of Theorem 10 was obtained for equation (E) in [2, Theorem 12] under the hypotheses that (11) and (20) hold and that (23) holds with $P_5 \geq 0$. Also, Theorem 10 generalizes a part of Theorem 1(b) in [5] and, for one delay, Theorem 1 in [11].

It follows from Theorem 10 that all nonoscillatory solutions of equation (E_2) tend to zero as $t \longrightarrow \infty$. Since $P(t) = \frac{1}{2}\sin t$ changes sign for arbitrarily large t, none of the results cited in the preceeding remark apply. Also, observe that Theorem 7 does not apply to (E_2) with n odd. In this regard, also notice that the equation

$$\Big[y(t) + (1+\sin t)y\Big(t - \frac{1}{2}\tanh t\Big)\Big]^{(n)} + (2+\cos t)y^{\frac{1}{3}}(t-2) = 0,$$

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$t \geq 2$, satisfies all the hypotheses of Theorem 7, but $g(t) = t - \frac{1}{2}\tanh t$ does not satisfy the hypotheses of Theorem 10 which requires a constant delay.

In conclusion, we would like to point out a couple of directions for further study. First of all, it would be interesting to see which of the results in this paper carry over to the case where $Q(t) \leq 0$. Secondly, any results on the behavior of solutions of forced neutral equations, whether analogous to the theorems in this paper or not, would be of interest.

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