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COMPUTATION OF NONSQUARE CONSTANTS OF ORLICZ SPACES

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Abstract

In this paper, we present the computation of exact value of nonsquare constants for some types of Orlicz sequence and function spaces. Main results: Let $\Phi(u)$ be an N-function, $\phi(t)$ be the right derivative of $\Phi(u)$, then we have

(i) if $\phi(t)$ is concave, then $1/\alpha'_\Phi \le J(l^{(\Phi)}) \le 1/\bar{\alpha}_\Phi$, $J(L^{(\Phi)}[0,\infty)) = 1/\bar{\alpha}_\Phi$;
(ii) if $\phi(t)$ is convex, then $2\beta'_\Phi \le J(l^{(\Phi)}) \le 2\bar{\beta}_\Phi$, $J(L^{(\Phi)}[0,\infty)) = 2\bar{\beta}_\Phi$.

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1. Introduction

The concept of nonsquareness is an important geometric property of Banach spaces which expose the intrinsic construction of a space according to the 'shape' of the unit ball of the spaces. The computation of nonsquare constants in Orlicz spaces has attracted the interest of many researchers and a considerable number of papers on this topic have appeared. However there has been little achievement of it since Gao and Lau [3] studied the value for Banach spaces. This paper is devoted to deriving exact estimates of nonsquare constants of Orlicz spaces which are easy to use in concrete applications.

Let X be a Banach space; $S(X) = \{x : \|x\| = 1, x \in X\}$ denotes the unit sphere of X. The nonsquare constants in the sense of James $J(X)$ and in the sense of Schaffer $g(X)$ are defined as:

(1) $$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in S(X)\},$$

(2) $$g(X) = \inf\{\max(\|x+y\|, \|x-y\|) : x, y \in S(X)\}.$$

Clearly, if $\dim X \geq 2$, then $1 \leq g(X) \leq \sqrt{2} \leq J(X) \leq 2$. Ji and Wang [5] asserted that

$$g(X)J(X) = 2 \tag{3}$$

for $\dim X \geq 2$. It was proved (Chen [1]) that $J(X) = 2$ if X fails to be reflexive. However, practical calculation for $J(X)$ when X is reflexive except L^p and l^p remains unsolved. In this paper, we extend the results of several authors (for instance, Ren [9], Ji and Wang [5], Ji and Zhan [6]) and deal with the computation of $J(X)$ when X is Orlicz function space $L^{(\Phi)}[0, \infty)$ and a sequence space $l^{(\Phi)}$ equipped with the Luxemburg norm.

Let $\Phi(u) = \int_0^{|u|} \phi(t)\,dt$ be an N-function, that is, $\phi(t)$ is right continuous, $\phi(0) = 0$, and $\phi(t) \nearrow \infty$ as $t \nearrow \infty$. The above two spaces are defined as follows:

$$L^{(\Phi)}[0, \infty) = \left\{x : \rho_\Phi(\lambda x) = \int_{[0,\infty)} \Phi(\lambda|x(t)|)\,dt < \infty \text{ for some } \lambda > 0\right\},$$

$$l^{(\Phi)} = \left\{x = \{x(i)\} : \rho_\Phi(\lambda x) = \sum_{n=1}^{\infty} \Phi(\lambda|x(i)|) < \infty \text{ for some } \lambda > 0\right\}.$$

The Luxemburg norm is expressed as

$$\|x\|_{(\Phi)} = \inf\{c > 0 : \rho_\Phi(x/c) \leq 1\}.$$

We say that $\Phi \in \Delta_2(0)$ (or Δ_2), if there exist $u_0 > 0$ and $k > 2$ such that $\Phi(2u) \leq k\Phi(u)$ for $0 \leq u \leq u_0$ (or for $u \geq 0$). Later, we will frequently use Semenove indices of $\Phi(u)$:

$$\alpha_\Phi = \liminf_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \beta_\Phi = \limsup_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \tag{4}$$

$$\alpha_\Phi^0 = \liminf_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \beta_\Phi^0 = \limsup_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \tag{5}$$

$$\bar{\alpha}_\Phi = \inf_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \bar{\beta}_\Phi = \sup_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}. \tag{6}$$

We extend the definition of the indices for the sequential usage:

$$\tilde{\alpha}_\Phi = \inf\left\{\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 \leq u \leq \frac{1}{2}\right\}, \quad \tilde{\beta}_\Phi = \sup\left\{\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 \leq u \leq \frac{1}{2}\right\}; \tag{7}$$

$$\begin{aligned} \alpha'_\Phi &= \inf\left\{\frac{\Phi^{-1}(1/(2k))}{\Phi^{-1}(1/k)} : k = 1, 2, \dots\right\}, \\ \beta'_\Phi &= \sup\left\{\frac{\Phi^{-1}(1/(2k))}{\Phi^{-1}(1/k)} : k = 1, 2, \dots\right\}. \end{aligned} \tag{8}$$

2. Lower bounds of $J(l^{(\Phi)})$ and $J(L^{(\Phi)}[0,\infty))$

We first estimate the lower bounds for $l^{(\Phi)}$ and $L^{(\Phi)}[0,\infty)$. The idea is refined from Ren [9]. We improve it so that the lower bounds may meet the upper ones and we obtain the exact values.

THEOREM 2.1. *Let $\Phi(u)$ be an N-function. Then the nonsquare constants of $l^{(\Phi)}$ and $L^{(\Phi)}[0,\infty)$, in the sense of James, satisfy*

$$\max\left(1/\alpha'_\Phi, 2\beta'_\Phi\right) \le J(l^{(\Phi)}) \quad \text{and} \tag{9}$$

$$\max\left(1/\bar{\alpha}_\Phi, 2\bar{\beta}_\Phi\right) \le J(L^{(\Phi)}[0,\infty)). \tag{10}$$

PROOF. To prove (9), we first show that

$$1/\alpha'_\Phi \le J(l^{(\Phi)}). \tag{11}$$

For any natural number k, put

$$x = (\overbrace{\Phi^{-1}(1/k),\dots,\Phi^{-1}(1/k)}^{k}, 0, 0, \dots),$$

$$y = (\overbrace{0,\dots,0}^{k}, \overbrace{\Phi^{-1}(1/k),\dots,\Phi^{-1}(1/k)}^{k}, 0, 0, \dots).$$

Then we have $\rho_\Phi(x) = \rho_\Phi(y) = 1$, $\|x\|_{(\Phi)} = \|y\|_{(\Phi)} = 1$ and

$$\|x-y\|_{(\Phi)} = \|x+y\|_{(\Phi)} = \frac{\Phi^{-1}(1/k)}{\Phi^{-1}(1/(2k))}.$$

Therefore,

$$\min(\|x-y\|_{(\Phi)}, \|x+y\|_{(\Phi)}) \ge \frac{\Phi^{-1}(1/k)}{\Phi^{-1}(1/(2k))} \quad (k=1,2,\dots).$$

Inequality (11) is proved.

Secondly, we prove that

$$2\beta'_\Phi \le J(l^{(\Phi)}). \tag{12}$$

Given a natural number k, put

$$x = (\overbrace{\Phi^{-1}(1/(2k)),\dots,\Phi^{-1}(1/(2k))}^{k}, \overbrace{\Phi^{-1}(1/(2k)),\dots,\Phi^{-1}(1/(2k))}^{k}, 0, \dots),$$

$$y = (\overbrace{\Phi^{-1}(1/(2k)),\dots,\Phi^{-1}(1/(2k))}^{k}, \overbrace{-\Phi^{-1}(1/(2k)),\dots,-\Phi^{-1}(1/(2k))}^{k}, 0, \dots).$$

Then $\|x\|_{(\Phi)} = \|y\|_{(\Phi)} = 1$ since $\rho_\Phi(x) = \rho_\Phi(y) = 1$, and

$$\|x - y\|_{(\Phi)} = \|x + y\|_{(\Phi)} = \frac{2\Phi^{-1}(1/(2k))}{\Phi^{-1}(1/k)}.$$

Therefore,

$$\min(\|x - y\|_{(\Phi)}, \|x + y\|_{(\Phi)}) \geq \frac{2\Phi^{-1}(1/(2k))}{\Phi^{-1}(1/k)} \quad (k = 1, 2, \dots)$$

and we obtain (12). Finally (9) follows from (11) and (12).

To prove (10), we first show

$$(13) \qquad 1/\bar{\alpha}_\Phi \leq J(L^{(\Phi)}[0, \infty)).$$

Take a real number $u \in (0, \infty)$, choose G_1 and G_2 in $[0, \infty)$ such that $G_1 \cap G_2 = \emptyset$ and $\mu(G_1) = \mu(G_2) = 1/2u$. Put $x(t) = \Phi^{-1}(2u)\chi_{G_1}(t)$ and $y(t) = \Phi^{-1}(2u)\chi_{G_2}(t)$, where χ_{G_1} is the characteristic function of G_1. Note that

$$\|\chi_{G_1}\|_{(\Phi)} = \|\chi_{G_2}\|_{(\Phi)} = \frac{1}{\Phi^{-1}(1/(\mu(G_1)))} = \frac{1}{\Phi^{-1}(2u)}.$$

We have $\|x\|_{(\Phi)} = \|y\|_{(\Phi)} = 1$ and

$$\|x - y\|_{(\Phi)} = \|x + y\|_{(\Phi)} = \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)}.$$

Take the supremum over $u \in (0, \infty)$. Since the function $G_\Phi(u) = \Phi^{-1}(u)/\Phi^{-1}(2u)$ is right continuous at 0 and takes value on $[1/2, 1]$, we deduce that

$$J(L^{(\Phi)}[0, \infty)) \geq \sup_{u \in (0,\infty)} \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)} = \sup_{u \in [0,\infty)} \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)} = \frac{1}{\bar{\alpha}_\Phi}.$$

Finally, we show

$$(14) \qquad 2\bar{\beta}_\Phi \leq J(L^{(\Phi)}[0, \infty)).$$

For every real number $v > 0$, choose E_1, E_2 in $[0, \infty)$ such that $E_1 \cap E_2 = \emptyset$ and $\mu(E_1) = \mu(E_2) = 1/2v$. Put

$$x(t) = \Phi^{-1}(v)[\chi_{E_1}(t) + \chi_{E_2}(t)] \quad \text{and} \quad y(t) = \Phi^{-1}(v)[\chi_{E_1}(t) - \chi_{E_2}(t)].$$

Then $\|x\|_{(\Phi)} = \|y\|_{(\Phi)} = 1$ and

$$\|x - y\|_{(\Phi)} = \|x + y\|_{(\Phi)} = \frac{2\Phi^{-1}(v)}{\Phi^{-1}(2v)}.$$

Take the supremum over $v \in (0, \infty)$ (the function $2\Phi^{-1}(v)/\Phi^{-1}(2v)$ is right continuous at 0 and takes value on $[1, 2]$) we also have $J(L^{(\Phi)}[0, \infty)) \geq 2\bar{\beta}_\Phi$. Hence (10) follows from (13) and (14). □

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3. Upper bounds of $J(l^{(\Phi)})$ and $J(L^{(\Phi)}[0,\infty))$

Upper bounds for Orlicz spaces remained unsolved (see [1, 9]) until Ji and Wang ([5, Theorem 3]) and Ji and Zhan ([6, Theorem 2]) offered the following equivalent presentation of $J(l^{(\Phi)})$ and $J(L^{(\Phi)}[0,\infty))$:

Assume $\Phi \in \Delta_2(0)$, then ([6])

(i) if $\phi(t)$ is a concave function, then

$$J(l^{(\Phi)}) = \sup\{k_x > 0 : \rho_\Phi(x/k_x) = 1/2,\ \rho_\Phi(x) = 1\}; \tag{15}$$

(ii) if $\phi(t)$ is convex, then

$$g(l^{(\Phi)}) = \inf\{k_x > 0 : \rho_\Phi(x/k_x) = 1/2,\ \rho_\Phi(x) = 1\}. \tag{16}$$

Suppose Φ satisfies the Δ_2-conditions for all u, we have [5]

(i) if $\phi(t)$ is a concave function, then

$$g(L^{(\Phi)}[0,\infty)) = \inf\{k_x > 0 : \rho_\Phi(2x/k_x) = 2,\ \rho_\Phi(x) = 1\}; \tag{17}$$

(ii) if $\phi(t)$ is convex, then

$$J(L^{(\Phi)}[0,\infty)) = \sup\{k_x > 0 : \rho_\Phi(2x/k_x) = 2,\ \rho_\Phi(x) = 1\}. \tag{18}$$

Now we extend these results and get the upper bounds.

THEOREM 3.1. *Suppose $\phi(t)$ is the right derivative of $\Phi(u)$, we have*

(i) *if $\phi(u)$ is concave, then*

$$J(l^{(\Phi)}) \le 1/\tilde{\alpha}_\Phi; \tag{19}$$

$$J(L^{(\Phi)}[0,\infty)) \le 1/\tilde{\alpha}_\Phi; \tag{20}$$

(ii) *if $\phi(u)$ is convex, then*

$$J(l^{(\Phi)}) \le 2\tilde{\beta}_\Phi, \tag{21}$$

$$J(L^{(\Phi)}[0,\infty)) \le 2\bar{\beta}_\Phi. \tag{22}$$

PROOF. For the sequence spaces, if $\Phi \notin \Delta_2(0)$, which is equivalent to $\beta^0_\Phi = 1$, then $l^{(\Phi)}$ is nonreflexive and hence $J(l^{(\Phi)}) = 2$ according to the results in Chen [1] or Hudzik [4]. Since $\phi(t)$ is concave implies $\Phi \in \Delta_2(0)$ (see Krasnoselskiĭ and Rutickiĭ [7, page 26]), we only need to check (21) when $\phi(t)$ is convex, but this is trivial since $J(l^{(\Phi)}) = 2 = 2\beta^0_\Phi = 2\tilde{\beta}_\Phi$. Similarly we check that (20) and (22) hold when $\Phi \notin \Delta_2$.

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Therefore it suffices for us to prove (19) and (21) for $\Phi \in \Delta_2(0)$ and (20) and (22) for $\Phi \in \Delta_2$.

To show (19) when $\Phi \notin \Delta_2(0)$, note that for $x = \{x(i)\} \in l^{(\Phi)}$, $\rho_\Phi(x(i)) = \sum_{n=1}^{\infty} \Phi(|x(i)|) = 1$ we have $u_i = \Phi(|x(i)|) \leq 1$ for $i \geq 1$. Define $G_\Phi(u) = \Phi^{-1}(u)/\Phi^{-1}(2u)$, then $u = \Phi[G_\Phi(u)\Phi^{-1}(2u)]$. Put $u_i = \Phi(|x(i)|)/2$, then $|x(i)| = \Phi^{-1}(2u_i)$ and

$$\frac{1}{2}\Phi(|x(i)|) = \Phi\left[G_\Phi\left(\frac{1}{2}\Phi(|x(i)|)\right)|x(i)|\right]. \tag{23}$$

Therefore, when $0 \leq u_i = \Phi(|x(i)|)/2 \leq 1/2$, we have

$$\tilde{\alpha}_\Phi \leq \frac{\Phi^{-1}(u_i)}{\Phi^{-1}(2u_i)} = G_\Phi(u_i) = G_\Phi[\Phi(|x(i)|)/2],$$

and hence, according to (23),

$$\rho_\Phi\left(\tilde{\alpha}_\Phi \cdot x\right) \leq \sum_{n=1}^{\infty} \Phi\left\{[G_\Phi(u_i)] \cdot |x(i)|\right\} = \frac{1}{2}\sum_{n=1}^{\infty} \Phi(|x(i)|) = \frac{1}{2}.$$

Thus we have $J(l^{(\Phi)}) \leq 1/\tilde{\alpha}_\Phi$ when $\phi(u)$ is concave by (15).

Analogously we prove $g(l^{(\Phi)}) \geq 1/\tilde{\beta}_\Phi$ by (16) when $\phi(u)$ is convex. From (3) we have $J(l^{(\Phi)}) \leq 2\tilde{\beta}_\Phi$.

Finally, we prove (20) for $\Phi(u) \in \Delta_2$, which is equal to

$$g(L^{(\Phi)}[0, \infty)) \geq 2\bar{\alpha}_\Phi \tag{24}$$

when $\phi(t)$ is concave in view of (3) and (17).

Let $H_\Phi(u) = \Phi^{-1}(2u)/\Phi^{-1}(u)$, then $\Phi^{-1}(2u) = H_\Phi(u)\Phi^{-1}(u)$. Put $x = \Phi^{-1}(u)$, then $u = \Phi(x)$ and $2\Phi(x) = \Phi[H_\Phi(\Phi(x))x]$. Therefore, when $u = \Phi(x(t)) \geq 0$ we have

$$\rho_\Phi\left(\frac{2x(t)}{2\bar{\alpha}_\Phi}\right) = \rho_\Phi\left(\frac{x(t)}{\bar{\alpha}_\Phi}\right) \geq \rho_\Phi\left(\frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)}x(t)\right)$$
$$= \rho_\Phi[H_\Phi(u)x(t)] = 2\rho_\Phi(x(t)) = 2$$

for $\rho_\Phi(x(t)) = 1$. It follows that (24) holds and hence (20) holds.

One can prove (22) similarly by (18). The proof is finished. □

4. Examples for computation

With the above bounds for $J(l^{(\Phi)})$ and $J(L^{(\Phi)}[0, \infty))$, we immediately obtain satisfactory estimates which are easy to compute.

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THEOREM 4.1. *Let $\Phi(u)$ be an N-function, $\phi(t)$ be the right derivative of $\Phi(u)$. We have*

(i) *if $\phi(t)$ is concave, then $1/\alpha'_\Phi \le J(l^{(\Phi)}) \le 1/\tilde{\alpha}_\Phi$ and $J(L^{(\Phi)}[0,\infty)) = 1/\bar{\alpha}_\Phi$;*
(ii) *if $\phi(t)$ is convex, then $2\beta'_\Phi \le J(l^{(\Phi)}) \le 2\tilde{\beta}_\Phi$ and $J(L^{(\Phi)}[0,\infty)) = 2\bar{\beta}_\Phi$.*

EXAMPLE 1. For $p > 1$, we have $J(L^p) = J(l^p) = \max(2^{1/p}, 2^{1-1/p})$, $(1<p<\infty)$. In fact, let $\Phi = |u|^p$, then $\alpha'_\Phi = \beta'_\Phi = \tilde{\alpha}_\Phi = \bar{\alpha}_\Phi = \bar{\beta}_\Phi = \tilde{\beta}_\Phi = 2^{-1/p}$. Obviously, if $1 < p \le 2$ then $\phi(t) = pt^{p-1}$ is concave, and if $2 \le p < \infty$ then $\phi(t)$ is convex. By Theorem 4.1 we get:

- if $1 < p \le 2$, then $J(L^p) = J(l^p) = 2^{1/p}$;
- if $2 \le p < \infty$, then $J(L^p) = J(l^p) = 2^{1-1/p}$.

REMARK 1. If the index function $G_\Phi(u) = \Phi^{-1}(u)/\Phi^{-1}(2u)$ is decreasing or increasing on an interval, then the indices α_Φ and β_Φ take the values at either end of it. The author [12] found that if $F_\Phi(t) = t\phi(t)/\Phi(t)$ is increasing (decreasing) on $(0, \Phi^{-1}(u_0)]$ then $G_\Phi(u)$ is also increasing (decreasing) on $(0, u_0/2]$, respectively. Rao and Ren [8] found the interrelation between Semenove and Simonenko indices:

$$2^{-1/A_\Phi} \le \alpha_\Phi \le \beta_\Phi \le 2^{-1/B_\Phi}, \quad 2^{-1/A^0_\Phi} \le \alpha^0_\Phi \le \beta^0_\Phi \le 2^{-1/B^0_\Phi},$$

where

$$A_\Phi = \liminf_{t\to\infty} \frac{t\phi(t)}{\Phi(t)}, \qquad B_\Phi = \limsup_{t\to\infty} \frac{t\phi(t)}{\Phi(t)};$$
$$A^0_\Phi = \liminf_{t\to 0} \frac{t\phi(t)}{\Phi(t)}, \qquad B^0_\Phi = \limsup_{t\to 0} \frac{t\phi(t)}{\Phi(t)}.$$

Therefore, when the index function $F_\Phi(t)$ is monotonic, the limits $C_\Phi = \lim_{t\to\infty} F_\Phi(t)$ and $C^0_\Phi = \lim_{t\to 0} F_\Phi(t)$ must exist and we have

$$(25) \qquad \alpha_\Phi = \beta_\Phi = \lim_{u\to\infty} G_\Phi(u) = 2^{-1/C_\Phi}, \quad \alpha^0_\Phi = \beta^0_\Phi = \lim_{u\to 0} G_\Phi(u) = 2^{-1/C^0_\Phi}.$$

This makes it easier to calculate the indices in Theorem 4.1.

EXAMPLE 2. Let a pair of complementary N-functions be

$$M(u) = e^{|u|} - |u| - 1 \quad \text{and} \quad N(v) = (1+|v|)\ln(1+|v|) - |v|.$$

Then $p(t) = M'(t) = e^t - 1$ is convex and $q(s) = N'(s) = \ln(1+s)$ is concave on $[0, +\infty)$. It is easy to check that the index function $F_M(t) = t(e^t-1)/(e^t - t - 1)$ is increasing and $F_N(t) = t\ln(1+t)/[(1+t)\ln(1+t) - t]$ is decreasing on $[0,\infty)$. In view of Remark 1, the index function G_M is accordingly increasing on $[0,\infty)$, with

G_N decreasing on $[0, \infty)$. Therefore $\tilde{\beta}_M$ and $\tilde{\alpha}_N$ both take their value at the right end of $[0, 1]$, that is,

$$\tilde{\beta}_M = \frac{M^{-1}(1/2)}{M^{-1}(1)} \approx 0.74828; \quad \tilde{\alpha}_N = \frac{N^{-1}(1/2)}{N^{-1}(1)} \approx 0.67250.$$

From Theorem 4.1 we have

$$J(l^{(M)}) = 2\tilde{\beta}_M = 2\beta'_M = \frac{2M^{-1}(1/2)}{M^{-1}(1)} \approx 1.49656;$$
$$J(l^{(N)}) = \frac{1}{\tilde{\alpha}_N} = \frac{1}{\alpha'_N} = \frac{N^{-1}(1)}{N^{-1}(1/2)} \approx 1.48699.$$

Since $C_M = \lim_{t\to\infty} F_M(t) = \infty$, $C_N = \lim_{t\to\infty} F_N(t) = 1$, we have

$$\alpha_M = \beta_M = 2^{-1/C_M} = 1, \quad \alpha_N = \beta_N = 2^{-1/C_N} = 1/2$$

by (25). Then from Theorem 4.1 we have

$$J(L^{(M)}[0,\infty)) = 2\bar{\beta}_M = 2\beta_M = 2; \quad J(L^{(N)}[0,\infty)) = \frac{1}{\bar{\alpha}_N} = \frac{1}{\alpha_N} = 2.$$

This result coincides with the fact that both the spaces $L^{(M)}[0, \infty)$ and $L^{(N)}[0, \infty)$ are nonreflexive.

EXAMPLE 3. Consider the N-function (see Gallardo [2])

$$\Phi_{p,r}(u) = |u|^p \ln^r(1 + |u|), \quad 1 \le p < \infty,\ 0 < r < \infty.$$

It is easy to check that $\phi_{p,r}(t)$, the right derivative of $\Phi_{p,r}(u)$, is convex when $1 \le p < \infty$, $2 \le r < \infty$. The index function

$$F_{\Phi_{p,r}}(t) = \frac{t\Phi'_{p,r}(t)}{\Phi_{p,r}(t)} = p + \frac{rt}{(1+t)\ln(1+t)}$$

is decreasing from $p + r$ to p on $[0, \infty)$ since

$$\frac{d}{dt}\Phi_{p,r}(t) = \frac{r[\ln(1+t) - t]}{(1+t)^2 \ln^2(1+t)} < 0.$$

So $C^0_{\Phi_{p,r}}(t) = \lim_{t\to 0} F_{\Phi_{p,r}}(t) = p + r$. According to (25) and Theorem 4.1 we have

$$J(l^{(\Phi_{p,r})}) = J(L^{(\Phi_{p,r})}[0,\infty)) = 2\beta^0_{\Phi_{p,r}} = 2 \cdot 2^{-1/(p+r)} = 2^{1-1/(p+r)}.$$

REMARK 2. The author studied the estimation of $J(L^{(\Phi)}[0, 1])$ in [11] and showed that:

- if $\phi(t)$ is concave then $1/\alpha_{\Phi[1,\infty)} \leq J(L^{(\Phi)}[0,1]) \leq 1/\bar{\alpha}_{\Phi}$;
- and if $\phi(t)$ is convex then $2\beta_{\Phi[1,\infty)} \leq J(L^{(\Phi)}[0,1]) \leq 2\bar{\beta}_{\Phi}$, where

$$\alpha_{\Phi[1,\infty)} = \inf_{u\in[1,\infty)} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \quad \beta_{\Phi[1,\infty)} = \sup_{u\in[1,\infty)} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}.$$

Consequently we have the nonsquare constants for the N-functions given in Example 2 and Example 3:

$$J(L^{(M)}[0,1]) = J(L^{(N)}[0,1]) = 2;$$

$$2^{1-1/p} \leq \frac{2\Phi_{p,r}^{-1}(1)}{\Phi_{p,r}^{-1}(2)} \leq J(L^{(\Phi_{p,r})}[0,1]) \leq 2^{1-1/(p+r)}.$$

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