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MULTIPARAMETER SPECTRAL THEORY OF SINGULAR DIFFERENTIAL OPERATORS

by B. P. RYNNE

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1. Introduction

In this paper we investigate certain aspects of the multiparameter spectral theory of systems of singular ordinary differential operators. Such systems arise in various contexts. For instance, separation of variables for a partial differential equation on an unbounded domain leads to a multiparameter system of ordinary differential equations, some of which are defined on unbounded intervals. The spectral theory of systems of regular differential operators has been studied in many recent papers, e.g. [**1**, **3**, **6**, **9**, **19**, **21**], but the singular case has not received so much attention. Some references for the singular case are [**7**, **8**, **10**, **13**, **14**, **18**, **20**], in addition general multiparameter spectral theory for self adjoint operators is discussed in [**3**, **9**, **19**].

The questions considered in this paper relate to the geometry of the spectrum of the system as a subset of $\mathbb{R}^k$. We prove various results on the structure and location of certain subsets of the spectrum. In addition we consider some aspects of the oscillation theory of the eigenfunctions of the differential operators.

In Section 2 we introduce some notation and definitions and describe the main assumptions that will be imposed on the multiparameter systems considered. In particular we will assume throughout the paper that the system is "right definite" in a sense specified below. This condition is standard in multiparameter spectral theory, see [**19**]. Also in Section 2 we discuss certain cones in $\mathbb{R}^k$ which will be used later to discuss the distribution of various points of the spectrum.

An abstract multiparameter system satisfying the conditions of Section 2 is considered in Section 3 and the location and structure of the spectrum is investigated. The methods used are mostly variational and are similar in spirit to those of [**3**].

In Section 4 a particular class of singular differential operators is defined and it is shown that the resulting multiparameter system satisfies the hypotheses of the previous section. In addition the detailed structure of the operators enables us to deduce more precise information concerning the spectrum than was available in Section 3. We also consider the oscillation properties of the eigenfunctions of the differential system and relate these to a subset of the spectrum defined in Section 3.

The operators considered in Section 4 are singular in the sense that their intervals of definition are semi-infinite (of the form $[a,\infty)$). Other types of singularity could also be considered using similar methods, e.g. intervals of the form $(-\infty,\infty)$ or singular coefficients in the differential operators. In addition a combination of regular and singular differential operators could be considered.

Finally, in Section 5, we consider the question of the number of eigenvalues of the multiparameter system. In the regular case there are necessarily an infinite number of isolated eigenvalues of finite multiplicity having no finite point of accumulation. However, in the singular case, this need not be true. We give sufficient conditions which, when satisfied, determine whether the number of isolated eigenvalues of finite multiplicity is finite or infinite.

2. Notation and definitions

Suppose we are given infinite dimensional Hilbert spaces H_r, $r=1,\dots,k$, and let U_r denote the set $\{u_r \in H_r : \|u_r\| = 1\}$. Let U be the set

$$U = \left\{ (u_1,\dots,u_k) \in \bigoplus_{r=1}^{k} H_r : \|u_r\| = 1, r = 1,\dots,k \right\}.$$

Now suppose that we have self adjoint linear operators $T_r, V_{rs}, r,s = 1,\dots,k$, such that:

(i) $V_{rs}: H_r \to H_r$ is bounded, $r,s=1,\dots,k$;

(ii) $T_r: D(T_r) \subset H_r \to H_r$ is bounded below, $r=1,\dots,k$.

For any $u_r \in U_r$ let $v_{rs}(u_r) = (V_{rs}u_r, u_r)$, and define the matrix

$$V(u) = [v_{rs}(u_r)], \qquad u \in U.$$

We assume that there exists $\delta > 0$ such that

$$\det V(u) \geqq \delta, \qquad \text{for all} \quad u \in U. \tag{2.1}$$

This condition is known as right definiteness and is a standard assumption in multiparameter theory.

For any vectors $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{R}^k$ the notation $\boldsymbol{\lambda} \cdot \boldsymbol{\mu}$ will be used for the usual inner product in $\mathbb{R}^k$, i.e. $\boldsymbol{\lambda} \cdot \boldsymbol{\mu} = \sum_{s=1}^{k} \lambda_s \mu_s$, and if M is a $k \times k$ matrix $M\lambda$ denotes the usual multiplication of a matrix and a vector. Also, if S is a topological space and $A \subset S$ then the notation $\bar{A}$ denotes the topological closure of the set A in S.

We now introduce some subsets of $\mathbb{R}^k$ which will be useful in the discussion of the geometry of the spectrum. Note that a partial order $\leqq$ can be defined on $\mathbb{R}^k$ by

$$\boldsymbol{\lambda} \leqq \boldsymbol{\mu} \Leftrightarrow \lambda_r \leqq \mu_r, \quad \mathrm{r} = 1,\dots,\mathrm{k}, \qquad \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{R}^k.$$

Definition 2.1. Let $\mathbf{0}$ denote the vector $(0,\dots,0) \in \mathbb{R}^k$, and define the sets

$$C = \{\boldsymbol{\lambda} \in \mathbb{R}^k : V(u)\boldsymbol{\lambda} \leqq \mathbf{0} \text{ for some } u \in U\},$$

$$C^+ = \{\boldsymbol{\lambda} \in \mathbb{R}^k : \text{there exists } W \in \overline{V(U)} \text{ such that } W\boldsymbol{\lambda} \leqq \mathbf{0}\},$$

where $V(U)$ denotes the set $\{V(u) : u \in U\}$.

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It is obvious that $C \subset C^+$ and the sets C, C^+, are cones (a set $A \subset \mathbb{R}^k$ is said to be a cone if $\mathbf{a} \in A \Rightarrow \alpha\mathbf{a} \in A$ for all numbers $\alpha \geqq 0$). These cones have been used in several papers e.g. [1, 3] and a detailed discussion of their geometry is contained in [5]. In particular Theorem 4.4 and Corollary 4.6 of [5] shows that the right definiteness condition (2.1) implies that $C^+ = \bar{C}$ and that C^+ does not contain a line.

Lemma 2.2. *For any $\lambda, \omega \in \mathbb{R}^k$ the set $A = (\lambda + C^+) \cap (\omega - C^+)$ is compact.*

Proof. Note that the set A may be empty. We regard the empty set as compact. The set A is closed since C^+ is closed. Now suppose that A is unbounded and let μ^n, $n = 1, 2, \ldots$, be a sequence of points in A with $\|\mu^n\| \to \infty$ as $n \to \infty$. Define $v^n = (1/\|\mu^n\|)\mu^n$ and let $v^n \to v$ as $n \to \infty$ (by choosing a subsequence if necessary). Since C^+ is a closed cone it follows that $v \in C^+$ and $v \in -C^+$. Thus v and $-v$ belong to C^+. This implies that C^+ contains a line, which contradicts Theorem 4.4 of [5], and hence the set A must be bounded. □

Finally we define a cone D by

Definition 2.3.

$$D = \{\lambda \in \mathbb{R}^k: \text{there exists } \zeta < 0 \text{ such that } V(u)\lambda \leqq (\zeta, \ldots, \zeta), \text{ for all } u \in U\}.$$

It is shown in [4] that (2.1) implies that D is not empty. It is obvious that $D \subset C$.

The multiparameter spectrum of the system of operators $[T_r, V_{rs}]$, will now be defined, together with various subsets of the spectrum. If T is a linear operator in a Hilbert space H we let $\sigma(T)$ denote the usual spectrum of T.

Definition 2.4. Define the operators $W_r(\lambda)$ in the spaces H_r, $r = 1, \ldots, k$, by

$$W_r(\lambda) = T_r + \sum_{s=1}^{k} \lambda_s V_{rs}, \; D(W_r(\lambda)) = D(T_r), \qquad \lambda \in \mathbb{C}^k.$$

The spectrum, σ, of the multiparameter system $[T_r, V_{rs}]$ consists of the set of points $\lambda \in \mathbb{C}^k$ such that

$$0 \in \sigma(W_r(\lambda)), \; r = 1, \ldots, k.$$

This definition of the spectrum coincides with the definition via spectral measures used in [9, 19] (see [19, Theorem 4.7]).

It is well known that the right definiteness assumption (2.1) implies that $\sigma \subset \mathbb{R}^k$, therefore we need only consider the operators $W_r(\lambda)$ for values of $\lambda \in \mathbb{R}^k$. For these values of λ the operators $W_r(\lambda)$ are self adjoint. The spectrum can now be partitioned into various subsets. First we consider the case of a single operator.

Definition 2.5. Let T be a self adjoint operator in a Hilbert space H. A point $\lambda\in\sigma(T)$ is said to be an eigenvalue of T if there exists a non zero vector $u\in H$ such that $Tu=\lambda u$. The vector u is an eigenvector. The multiplicity of the operator T at λ is defined to be $m(T,\lambda)=\operatorname{Dim} N(T-\lambda I)$, where $N(T-\lambda I)$ denotes the null space of the operator $T-\lambda I$. If λ is not an eigenvalue the multiplicity of T at λ is zero.

A point $\lambda\in\sigma(T)$ is said to be in the essential spectrum of T, $\sigma_e(T)$, if there exists a sequence of unit vectors $u^n\in H$ such that

$$\|(T-\lambda I)u^n\|\to 0 \quad \text{as } n\to\infty,$$

but u^n has no strongly convergent subsequence.

A point $\lambda\in\sigma(T)$ is said to be in the point spectrum of T, $\sigma_p(T)$, if λ is an eigenvalue of T and $\lambda\notin\sigma_e(T)$.

Note that there are several definitions of the essential spectrum of an operator in the literature (see [**12**, **15**]), not all of which are equivalent to each other. Our definition of the essential spectrum is based on [**15**, p. 14] and contains the essential spectrum of Dunford and Schwartz, together with any eigenvalues of infinite multiplicity. Also our definition of the point spectrum is smaller than the usual definition in that it does not contain eigenvalues which lie in the essential spectrum.

We now consider a similar partition of the multiparameter spectrum.

Definition 2.6. A point $\boldsymbol{\lambda}\in\sigma$ is said to be an eigenvalue of the multiparameter system $[T_r, V_{rs}]$ if 0 is an eigenvalue of each of the operators $W_r(\boldsymbol{\lambda})$, $r=1,\dots,k$. The multiplicity of the system at $\boldsymbol{\lambda}$ is defined to be the vector $\mathbf{m}(\boldsymbol{\lambda})=(m_1(\boldsymbol{\lambda}),\dots,m_k(\boldsymbol{\lambda}))$, where $m_r(\boldsymbol{\lambda})=m(W_r(\boldsymbol{\lambda}),0)$ is the multiplicity of the operator $W_r(\boldsymbol{\lambda})$ at 0. We say that an eigenvalue $\boldsymbol{\lambda}$ has finite multiplicity if and only if $m_r(\boldsymbol{\lambda})$ is finite for each r.

The essential spectrum, σ_e, of the multiparameter system, $[T_r, V_{rs}]$, consists of the set of points $\boldsymbol{\lambda}\in\sigma$ for which

$$0\in\sigma_e(W_r(\boldsymbol{\lambda})), \qquad r=1,\dots,k.$$

Similarly $\boldsymbol{\lambda}\in\sigma$ belongs to the point spectrum σ_p if

$$0\in\sigma_p(W_r(\boldsymbol{\lambda})), \qquad r=1,\dots,k.$$

Finally $\boldsymbol{\lambda}\in\sigma$ is said to be in the mixed spectrum, σ_m, if $\boldsymbol{\lambda}\in\sigma\backslash\{\sigma_e\cup\sigma_p\}$.

The above definitions of the subsets of the multiparameter spectrum rely on the operators $W_r(\boldsymbol{\lambda})$ rather than the spectral measure of the system as defined in [**9**] or [**19**]. It can be shown that the set σ_p as defined above coincides with the set of isolated points in σ whose associated spectral projections have finite dimensional ranges.

3. Geometry of the spectrum

Having defined various subsets of the spectrum above we now consider the geometrical structure of these subsets. Firstly we describe certain variational results for self adjoint operators which will be required below.

Suppose that the operator T, defined in the infinite dimensional Hilbert space H, is self adjoint and bounded below. Let

$$\rho^e(T) = \inf\{\lambda \in \mathbb{R}: \lambda \in \sigma_e(T)\},$$

(if $\sigma_e(T)$ is empty then we put $\rho^e(T) = \infty$), and let $\sigma'_p(T) \subset \sigma_p(T)$ be the set of eigenvalues $\lambda \in \sigma_p(T)$ for which $\lambda < \rho^e(T)$. The set $\sigma'_p(T)$ has the form

$$\sigma'_p(T) = \{\rho^i(T): i = 0, \ldots, N(T)\},$$

where $\rho^i(T)$ is an increasing sequence of numbers and we assume that the number of occurrences of an eigenvalue λ in the sequence $\rho^i(T)$ is equal to the multiplicity of the eigenvalue. The number $N(T)$ may be a finite integer or infinity. If $N(T) = \infty$ then $\lim_{i \to \infty} \rho^i(T) = \rho^e(T)$.

Using the minimax principle, [22], we can define the numbers

$$\gamma^i(T) = \sup_{E^i} \left\{ \inf_u \{(Tu, u): u \in D(T) \cap E^{i\perp}, \|u\| = 1\} \right\}, \quad i = 0, 1, \ldots,$$

where $E^i \subset H$ is an arbitrary i dimensional linear subspace of H. It can easily be shown, using the definition of the essential spectrum, that for $i \leqq N(T)$, $\gamma^i(T) = \rho^i(T)$, while for $i > N(T)$, $\gamma^i(T) = \rho^e(T)$ (if $N(T) = \infty$ then $\gamma^i(T) = \rho^i(T)$ for all $i \geqq 0$). Thus applying the minimax principle to the operator T yields the set σ'_p and the point $\rho^e(T)$.

Now suppose that B is a bounded, self adjoint operator on H and let

$$b^+ = \sup\{(Bu, u): \|u\| = 1\}, \; b^- = \inf\{(Bu, u): \|u\| = 1\}.$$

Applying the minimax principle to the self adjoint operator $S = T + B$ produces the numbers $\gamma^i(S)$. It can be seen from the definition of the γ^i that we have

$$\gamma^i(T) + b^- \leqq \gamma^i(S) \leqq \gamma^i(T) + b^+, \qquad i = 0, 1, \ldots. \tag{3.1}$$

Thus the numbers $\gamma^i(S)$ depend continuously on the perturbation B in the sense that the $\gamma^i(S)$ can be made arbitrarily close the $\gamma^i(T)$, uniformly in i, by choosing $\|B\|$ small enough.

These results can now be applied to the multiparameter operators $W_r(\lambda)$ defined in Section 2. We use the notation $\rho^e_r = \rho^e(T_r)$, $\rho^e_r(\lambda) = \rho^e(W_r(\lambda))$, $\gamma^i_r(\lambda) = \gamma^i(W_r(\lambda))$, for each $\lambda \in \mathbb{R}^k$. Since the operators V_{rs} are bounded it follows from the above remarks that the $\gamma^i_r(\lambda)$ are continuous functions of $\lambda \in \mathbb{R}^k$ for all i and r.

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Definition 3.1. The subset σ'_p of the multiparameter point spectrum is defined to be

$$\sigma'_p = \{\lambda \in \sigma_p : \rho^e_r(\lambda) > 0,\ r = 1, \ldots, k\}.$$

The above variational results provide information about the portion of the spectrum of T below $\rho^e(T)$, i.e. below the lowest point of the essential spectrum. For certain types of operators this information is sufficient to determine the location of the complete spectrum of the operator.

Definition 3.2. The self adjoint operator T is said to be of type *SD* if $\sigma_e(T) = [\rho^e(T), \infty)$.

The multiparameter system $[T_r, V_{rs}]$ is said to be of type *SD* if, for each $\lambda \in \mathbb{R}^k$, the operators $W_r(\lambda)$, $r = 1, \ldots, k$, are of type *SD*.

If the operator T is of type *SD* then obviously $\sigma'_p(T) = \sigma_p(T)$ and a knowledge of the set $\sigma'_p(T)$ and $\rho^e(T)$ (as provided by the minimax principle) completely determines the location of $\sigma(T)$. In the case of a multiparameter system of type *SD* we also have $\sigma'_p = \sigma_p$. The singular differential operators which we will encounter in Section 4 will be shown to be of type *SD*.

This completes the preliminary discussion and we may now begin our consideration of the multiparameter spectrum. For ease of presentation we deal with the point spectrum and essential spectrum first and then briefly consider the mixed spectrum in a similar manner.

Let $\mathbb{N}$ denote the set of positive integers $i \geqq 0$ and define a multi-index to be a vector of the form $\mathbf{i} = (i_1, \ldots, i_k) \in \mathbb{N}^k$. Since the points $\gamma^i_r(\lambda)$ are defined via the minimax principle the following theorem can be proved in the same manner as Theorem 2 of [3].

Theorem 3.3. *Corresponding to each multi-index* $\mathbf{i} \in \mathbb{N}^k$, *there is a unique point* $\lambda^{\mathbf{i}} \in \sigma$ *such that* $\gamma^{i_r}_r(\lambda^{\mathbf{i}}) = 0$, $r = 1, \ldots, k$.

We define the set $\Lambda = \{\lambda^{\mathbf{i}} : \mathbf{i} \in \mathbb{N}^k\}$. In view of the above discussion it can be seen that the points $\lambda^{\mathbf{i}} \in \Lambda$ need not be multiparameter eigenvalues of finite multiplicity as they are in [3] where essential spectrum does not occur. Also there may be points in σ_p which are not in the set Λ. However the following result holds.

Theorem 3.4.

$$\sigma'_p \subset \Lambda.$$

Proof. Suppose that $\lambda \in \sigma'_p$. Then, by definition, $0 \in \sigma'_p(W_r(\lambda))$ for each r, so it follows from the above variational results that $\gamma^{i_r}_r(\lambda) = 0$ for some i_r. Hence $\lambda = \lambda^{\mathbf{i}}$ for some multi-index $\mathbf{i}$. □

We now consider the location, in $\mathbb{R}^k$, of part of the spectrum.

Theorem 3.5. *Let* $\mathbf{i}, \mathbf{j}$ *be multi-indices with* $\mathbf{i} \leqq \mathbf{j}$. *Then* $\lambda^{\mathbf{j}} \in \lambda^{\mathbf{i}} + C^+$. *If* λ *is such that* $\rho_r^e(\lambda) \leqq 0$ *for all* r *then* $\lambda \in \lambda^{\mathbf{i}} + C^+$ *for all* $\mathbf{i} \in \mathbb{N}^k$.

Proof. Choose arbitrary multi-indices $\mathbf{i}, \mathbf{j}$ with $\mathbf{i} \leqq \mathbf{j}$ and let ε_n be a sequence of numbers such that $\varepsilon_n > 0$ and $\varepsilon_n \to 0$. Given any integer n it follows from the minimax principle and the definition of $\lambda^{\mathbf{i}}$ that there exist subspaces $E_r^{i_r} \subset H_r$ such that $\dim E_r^{i_r} = i_r$ and

$$\inf\{(W_r(\lambda^{\mathbf{i}})u_r, u_r) : u_r \in U_r \cap D(T_r) \cap E_r^{i_r \perp}\} \geqq -\varepsilon_n.$$

Now let

$$\delta_r = \inf\{(W_r(\lambda^{\mathbf{j}})u_r, u_r) : u_r \in U_r \cap D(T_r) \cap E_r^{i_r \perp}\}.$$

Since $\mathbf{j} \geqq \mathbf{i}$ it again follows from the minimax principle, and the definition of $\lambda^{\mathbf{j}}$, that $\delta_r \leqq 0$. Hence we may choose $u_r^n \in U_r \cap D(T_r) \cap E_r^{i_r \perp}$ such that $(W_r(\lambda^{\mathbf{j}})u_r^n, u_r^n) \leqq \varepsilon_n$. Thus

$$(W_r(\lambda^{\mathbf{j}})u_r^n, u_r^n) - (W_r(\lambda^{\mathbf{i}})u_r^n, u_r^n) \leqq 2\varepsilon_n$$

and so

$$V(u^n)(\lambda^{\mathbf{j}} - \lambda^{\mathbf{i}}) \leqq 2\boldsymbol{\varepsilon}_n, \ \boldsymbol{\varepsilon}_n = (\varepsilon_n, \ldots, \varepsilon_n) \in \mathbb{R}^k. \tag{3.2}$$

Repeating this construction for each n gives a sequence $u^n \in U$. Now let the matrix M be an accumulation point of the sequence of matrices $V(u^n)$ (M exists since the V_{rs} are bounded and so the set $\{V(u^n)\}$ is bounded). Inequality (3.2) shows that $M(\lambda^{\mathbf{j}} - \lambda^{\mathbf{i}}) \leqq \mathbf{0}$ and hence $\lambda^{\mathbf{j}} - \lambda^{\mathbf{i}} \in C^+$. The second part of the theorem is proved similarly. □

An immediate consequence of this theorem and the previous one is that $\sigma_p \cup \sigma_e \subset \lambda^{\mathbf{0}} + C^+$.

Theorem 3.5 shows that the points $\lambda^{\mathbf{i}} \in \Lambda$ are ordered in a similar manner to those of Binding and Browne in [3], using the cone C^+ instead of C. The cone C could have been used in Theorem 3.3 if we restricted attention to points $\lambda^{\mathbf{i}}, \lambda^{\mathbf{j}} \in \sigma'_p$, however at present we have no guarantee that σ_p is not empty. This question will be considered in Section 5.

The geometry of σ_e will now be considered in more detail. To ensure that σ_e is non-empty we require the following assumption.

Condition F. The number ρ_r^e is finite for each $r = 1, \ldots, k$.

Note that the minimax principle and (3.1) implies that Condition F holds if and only if $\rho_r^e(\lambda)$ is finite for all $\lambda \in \mathbb{R}^k$. In particular Condition F is a necessary condition for σ_e to be non-empty.

Lemma 3.6. *If Condition F holds then there exists a point* $\lambda^* \in \mathbb{R}^k$ *such that* $\rho_r^e(\lambda^*) \leqq 0$, $r = 1, \ldots, k$. *For any such point,* λ^*, *and any multi-index* $\mathbf{i} \in \mathbb{N}^k$, $\lambda^* \in \lambda^{\mathbf{i}} + C^+$.

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Proof. Choose an element $\boldsymbol{\mu} \in D$ and consider the operators

$$W_r(\alpha\boldsymbol{\mu}) = T_r + \alpha \sum_{s=1}^{k} \mu_s V_{rs}, \ \alpha \in \mathbb{R}, \ r = 1, \ldots, k.$$

By Definition 2.2 the operator $\sum_{s=1}^{k} \mu_s V_{rs}$ is negative definite so it follows from Condition F and (3.1) that, for large enough α, $\boldsymbol{\lambda}^* \in \alpha\boldsymbol{\mu}$ is an appropriate choice. The second assertion follows from Theorem 3.5. □

Lemma 3.7. *If Condition F holds then* Λ *is contained in the compact set* $A = (\boldsymbol{\lambda}^0 + C^+) \cap (\boldsymbol{\lambda}^* - C^+)$, *where* $\boldsymbol{\lambda}^*$ *is as in Lemma 3.6.*

Proof. The result follows from Lemma 2.2, Theorem 3.5 and Lemma 3.6. □

Note that the set A in the above lemma is non-empty since $\boldsymbol{\lambda}^* \in \boldsymbol{\lambda}^0 + C^+$, by Lemma 3.6. A similar remark applies to other sets, to be defined below, having the same form as A.

Now, for any integer t, let $\hat{\mathbf{t}}$ denote the multi-index $(t, \ldots, t)$.

Lemma 3.8. *If Condition F holds then the limit* $\lim_{t\to\infty} \boldsymbol{\lambda}^{\hat{\mathbf{t}}}$ *exists and is finite.*

Proof. It follows from the compactness of the set A in Lemma 3.7 that the set $\{\boldsymbol{\lambda}^{\hat{\mathbf{t}}} : t \in \mathbb{N}\} \subset A$ must have an accumulation point $\boldsymbol{\mu} \in A$. Theorem 3.5 and the closedness of C^+ shows that $\boldsymbol{\lambda}^{\mathbf{i}} \in \boldsymbol{\mu} - C^+$ for all $\mathbf{i} \in \mathbb{N}^k$. Now suppose that the increasing sequence t_n, $n = 1, 2, \ldots$, is such that $\boldsymbol{\lambda}^{\hat{\mathbf{t}}_n} \to \boldsymbol{\mu}$ as $n \to \infty$. Then it can be shown, as in the proof of Lemma 2.2, that the sets $A_n = (\boldsymbol{\lambda}^{\hat{\mathbf{t}}_n} + C^+) \cap (\boldsymbol{\mu} - C^+)$ converge to the point $\boldsymbol{\mu}$. Thus $\boldsymbol{\lambda}^{\hat{\mathbf{t}}} \to \boldsymbol{\mu}$ as $t \to \infty$, since $\boldsymbol{\lambda}^{\hat{\mathbf{t}}} \in A_n$ for all $t \geqq t_n$.

We will use the notation $\boldsymbol{\lambda}^e = \lim_{t\to\infty} \boldsymbol{\lambda}^{\hat{\mathbf{t}}}$. It should be observed that in this limiting process the multi-indices $\hat{\mathbf{t}}$ tend to infinity along the line generated by the multi-index $(1, \ldots, 1)$. However it can easily be shown, in a similar manner to the proof of Lemma 3.8, that the same limit, $\boldsymbol{\lambda}^e$, is obtained if any other sequence of multi-indices is chosen, so long as each of the sequences of the components of the multi-indices tend to infinity.

Theorem 3.9. *If Condition F holds then*

$$\boldsymbol{\lambda}^e \in \sigma_e,$$

$$\sigma_e \subset \boldsymbol{\lambda}^e + C^+,$$

$$\Lambda \subset (\boldsymbol{\lambda}^0 + C^+) \cap (\boldsymbol{\lambda}^e - C^+),$$

and the set $(\boldsymbol{\lambda}^0 + C^+) \cap (\boldsymbol{\lambda}^e - C^+)$ *is compact. In addition, if the multiparameter system is of type SD, then*

$$\sigma_e \supset \boldsymbol{\lambda}^e + D.$$

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Proof. Consider the operator $W_r(\lambda^e)$ for any r. By definition the $(t+1)$'th eigenvalue of $W_r(\lambda^{\hat{\mathbf{t}}})$ is zero and $\lambda^{\hat{\mathbf{t}}} \to \lambda^e$ as $t \to \infty$. So, by the remarks following (3.1), it can be seen that $\gamma^t(W_r(\lambda^e)) \to 0$. Thus $0 \in \sigma_e(W_r(\lambda^e))$, and so $\lambda^e \in \sigma_e$. If $\lambda \in \sigma_e$ then $\lambda \in \lambda^{\hat{\mathbf{t}}} + C^+$ for all t and, since C^+ is closed, this implies that $\lambda \in \lambda^e + C^+$, thus $\sigma_e \subset \lambda^e + C^+$. Now, for any multi-index $\mathbf{j}$, it follows from Theorem 3.5 that $\lambda^{\hat{\mathbf{t}}} \in \lambda^{\mathbf{j}} + C^+$ for all t large enough, so letting $t \to \infty$ gives $\lambda^e \in \lambda^{\mathbf{j}} + C^+$. Hence Theorem 3.5 shows that $\Lambda \subset A = (\lambda^0 + C^+) \cap (\lambda^e - C^+)$. The compactness of the set A follows from Lemma 2.2. Finally, for any $\mu \in D$, we have $\rho_r^e(\lambda^e + \mu) \leqq \rho_r^e(\lambda^e) \leqq 0$, $r = 1, \ldots, k$, (using (3.1) and the definition of D) so, if the system is of type SD, we must have $0 \in \sigma_e(W_r(\lambda^e + \mu))$. □

We now consider the mixed spectrum of the system and attempt to obtain results analogous to those obtained above for this set. For simplicity of notation we will only consider the subset of σ_m given by

$$\sigma_m^n = \{\lambda \in \mathbb{R}^k : 0 \in \sigma_p(W_r(\lambda)),\ r = 1, \ldots, n,\ 0 \in \sigma_e(W_r(\lambda)),\ r = n+1, \ldots, k\},$$

where n is an integer, $n < k$. The notation $\sigma_m^n(\tilde{\mathbf{i}})$ will denote the set

$$\sigma_m^n(\tilde{\mathbf{i}}) = \{\lambda \in \mathbb{R}^k : \gamma_r^{\tilde{i}_r}(\lambda) = 0,\ r = 1, \ldots, n,\ 0 \in \sigma_e(W_r(\lambda)),\ r = n+1, \ldots, k\},$$

where $\tilde{\mathbf{i}} = (\tilde{i}_1, \ldots, \tilde{i}_n) \in \mathbb{N}^n$.

The following results are similar to the above results for the essential spectrum and can be proved in the same way. If $\tilde{\mathbf{i}} \in \mathbb{N}^n$ and $\mathbf{j} \in \mathbb{N}^k$ then $\mathbf{j} \leqq \tilde{\mathbf{i}}$ means $j_r \leqq \tilde{i}_r$, $r = 1, \ldots, n$.

Theorem 3.10. *Let $\tilde{\mathbf{i}}, \mathbf{j}$ be multi-indices with $\mathbf{j} \leqq \tilde{\mathbf{i}}$ and let $\lambda \in \sigma_m^n(\tilde{\mathbf{i}})$. Then $\lambda \in \lambda^{\mathbf{j}} + C^+$. In particular if $\mathbf{j} = (\tilde{\mathbf{i}}, \mathbf{j}')$, where $\mathbf{j}' \in \mathbb{N}^{k-n}$, then $\lambda \in \lambda^{\mathbf{j}} + C^+$ for all $\mathbf{j}' \in \mathbb{N}^{k-n}$.*

This theorem, together with the previous results, proves the following corollary.

Corollary 3.11. $\sigma \subset \lambda^0 + C^+$.

In considering the mixed spectrum σ_m^n we only require the following finiteness condition rather than Condition F.

Condition F′. $\rho_r^e < \infty$, $r = n+1, \ldots, k$.

As in the case of the essential spectrum this condition is necessary for σ_m^n to be non-empty.

Lemma 3.12. *Suppose that Condition F′ holds and choose any $\tilde{\mathbf{i}} \in \mathbb{N}^n$. Then there exists a point $\lambda^*(\tilde{\mathbf{i}}) \in \mathbb{R}^k$ such that $\gamma_r^{\tilde{i}_r}(\lambda^*(\tilde{\mathbf{i}})) \leqq 0$, $r = 1, \ldots, n$, $\rho_r^e(\lambda^*(\tilde{\mathbf{i}})) \leqq 0$, $r = n+1, \ldots, k$. For any such point, $\lambda^*(\tilde{\mathbf{i}})$, and any multi-index $\mathbf{j} \leqq \tilde{\mathbf{i}}$, $\lambda^*(\tilde{\mathbf{i}}) \in \lambda^{\mathbf{j}} + C^+$.*

For any $\tilde{\mathbf{i}} \in \mathbb{N}^n$ and any integer, t, let $(\tilde{\mathbf{i}}, \hat{\mathbf{t}})$ denote the multi-index $(\tilde{i}_1, \ldots, \tilde{i}_n, t, \ldots, t) \in \mathbb{N}^k$.

Lemma 3.13. *If Condition F′ holds then, for any* $\tilde{\mathbf{i}} \in \mathbb{N}^n$, *the limit* $\boldsymbol{\lambda}^m(\tilde{\mathbf{i}}) = \lim_{t\to\infty} \boldsymbol{\lambda}^{(\tilde{\mathbf{i}},\mathbf{t})}$ *exists and is finite.*

Theorem 3.14. *If Condition F′ holds then, for all* $\tilde{\mathbf{i}} \in \mathbb{N}^n$,

$$\boldsymbol{\lambda}^m(\tilde{\mathbf{i}}) \in \sigma_m^n(\tilde{\mathbf{i}}),$$

$$\sigma_m^n(\tilde{\mathbf{i}}) \subset \boldsymbol{\lambda}^m(\tilde{\mathbf{i}}) + C^+,$$

$$\{\boldsymbol{\lambda}^{(\tilde{\mathbf{i}},\mathbf{j}')} \in \Lambda : \mathbf{j}' \in \mathbb{N}^{k-n}\} \subset (\boldsymbol{\lambda}^0 + C^+) \cap (\boldsymbol{\lambda}^m(\tilde{\mathbf{i}}) - C^+),$$

and the set $(\boldsymbol{\lambda}^0 + C^+) \cap (\boldsymbol{\lambda}^m(\tilde{\mathbf{i}}) - C^+)$ *is compact.*

We now show that the set $\sigma_m^n(\tilde{\mathbf{i}})$ is a subset of a $k-n$ dimensional continuum in the sense of the following theorem.

Theorem 3.15. *If Condition F′ holds then, for each* $\tilde{\mathbf{i}} \in \mathbb{N}^n$, *there exists a continuous function* $\boldsymbol{\lambda}^{\tilde{\mathbf{i}}} : \mathbb{R}^{k-n} \to \mathbb{R}^k$, *such that* $\sigma_m^n(\tilde{\mathbf{i}})$ *is a subset of the range of* $\boldsymbol{\lambda}^{\tilde{\mathbf{i}}}$ *in* $\mathbb{R}^k$. *If the system is of type SD then there exists a continuous function* $\boldsymbol{\psi}^{\tilde{\mathbf{i}}} : \mathbb{R}^{k-n} \to \mathbb{R}^{k-n}$ *such that*

$$\sigma_m^n(\tilde{\mathbf{i}}) = \{\boldsymbol{\lambda} \in \mathbb{R}^k : \boldsymbol{\lambda} = \boldsymbol{\lambda}^{\tilde{\mathbf{i}}}(\boldsymbol{\mu}) \text{ for some } \boldsymbol{\mu} \in \mathbb{R}^{k-n} \text{ and } \boldsymbol{\psi}^{\tilde{\mathbf{i}}}(\boldsymbol{\mu}) \leqq \mathbf{0}\}.$$

Proof. Theorem 1 of [3] shows that we may make an invertible transformation $\boldsymbol{\lambda} \to \tilde{\boldsymbol{\lambda}}$, and a corresponding transformation of the array $[V_{rs}] \to [\tilde{V}_{rs}]$, such that the sub array $[\tilde{V}_{rs}]_{r,s=1}^n$ is right definite. Now writing $\boldsymbol{\eta} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$, $\boldsymbol{\mu} = (\tilde{\lambda}_{n+1}, \ldots, \tilde{\lambda}_k)$, we may consider the first n equations of the transformed system as a perturbed multiparameter system with $\boldsymbol{\eta}$ as the spectral parameter and $\boldsymbol{\mu}$ a perturbation parameter. By the construction of the above transformation this multiparameter system is right definite for each $\boldsymbol{\mu}$ and the perturbation depends continuously (in norm) on $\boldsymbol{\mu}$. It follows that, for each $\boldsymbol{\mu}$, the points $\boldsymbol{\eta}^{\tilde{\mathbf{i}}}(\boldsymbol{\mu})$ can be defined as above, via the minimax principle, for each $\tilde{\mathbf{i}} \in \mathbb{N}^n$. In addition the proof of Theorem 2 of [3] shows that the functions $\boldsymbol{\eta}^{\tilde{\mathbf{i}}}(\boldsymbol{\mu})$ are continuous functions of $\boldsymbol{\mu}$.

Now, by definition of $\sigma_m^n(\tilde{\mathbf{i}})$, a necessary condition for $\boldsymbol{\lambda} \in \sigma_m^n(\tilde{\mathbf{i}})$ is that $\tilde{\boldsymbol{\lambda}}$ is of the form $(\boldsymbol{\eta}^{\tilde{\mathbf{i}}}(\boldsymbol{\mu}), \boldsymbol{\mu})$ for some $\boldsymbol{\mu}$ and $\rho_r^e(\boldsymbol{\lambda}) \leqq 0$ for $r = n+1, \ldots, k$. This proves the first part of the theorem. If the system is of type *SD* then this condition is also sufficient thus, since $\rho_r^e(\boldsymbol{\lambda})$ is a continuous function of $\boldsymbol{\lambda}$, the theorem is proved. □

The above discussion of the set $\sigma_m^n(\tilde{\mathbf{i}})$ suffers from a similar problem to that of the discussion of σ_p' and Λ, namely that 0 need not belong to the point spectrum of the operators $W_r(\boldsymbol{\lambda}^{\tilde{\mathbf{i}}})$, $r = 1, \ldots, n$. Thus the sets $\sigma_m^n(\tilde{\mathbf{i}})$ need not actually be subsets of σ_m^n, or even of σ_m. Also there may be points in σ_m^n that are not in any $\sigma_m^n(\tilde{\mathbf{i}})$ in the same way that σ_p' need not equal σ_p. If the system is of type *SD* then the latter problem does not arise, however the former problem remains. A particular situation in which this problem does not arise is when it is known that the operators $W_r(\boldsymbol{\lambda})$, $r = 1, \ldots, n$, have an infinite

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number of points in the point spectrum below the essential spectrum. In this case it follows that 0 must belong to the point spectrum of the operators $W_r(\lambda^{\bar{\mathbf{i}}})$, $r=1,\dots,n$, see the proof of Theorem 3.4. If the operators T_r have compact resolvents for $r=1,\dots,n$, then this situation holds. In Section 5 we describe sufficient conditions for this to be so for a class of singular differential operators which do not have compact resolvents. These operators will also be shown to be of type SD so neither of the above problems occurs when the conditions of Section 5 are satisfied.

Since the perturbation used in the proof of Theorem 3.15 depends linearly on μ, and so is analytic, the continuity assertions of the theorem can be improved upon in certain circumstances. For instance suppose that the null spaces are necessarily at most 1 dimensional (this is true for the differential operators discussed below). Then analytic perturbation theory of multiparameter eigenvalue problems shows that if, for some μ, the point $\eta^{\bar{\mathbf{i}}}(\mu)$ is an eigenvalue of the perturbed subsystem used in the proof of Theorem 3.15 then the function $\eta^{\bar{\mathbf{i}}}$ is analytic near μ (see [17]). Thus, if it is known that $\sigma_m^n(\bar{\mathbf{i}})\subset\sigma_m^n$ (see the above remarks), then the functions $\lambda^{\bar{\mathbf{i}}}$ of Theorem 3.15 are analytic.

4. Singular differential operators

In this section we describe an important class of multiparameter systems for which the abstract theory of the previous section is applicable. In addition more precise information is available in some respects.

Consider the following multiparameter system of differential equations

$$-\frac{d}{dx_r}\left(p_r(x_r)\frac{d}{dx_r}u_r(x_r)\right)+q_r(x_r)u_r(x_r)+\sum_{s=1}^{k}\lambda_s v_{rs}(x_r)u_r(x_r)=0,$$

$$r=1,\dots,k, \qquad (4.1)$$

on the open intervals $x_r\in I_r=(a_r,\infty)$, where a_r is a finite number, $r=1,\dots,k$. We assume that the functions p_r, q_r, v_{rs}, $r,s=1,\dots,k$, are real valued and continuous on the closed intervals $\bar{I}_r$ and the v_{rs} are bounded. In addition suppose that the functions p_r are continuously differentiable and there exists $\alpha>0$ such that $p_r(x_r)\geqq\alpha$ for all $x_r\in I_r$, $r=1,\dots,k$.

The one parameter spectral theory of singular differential operators of this type is studied extensively in e.g. [**2**, **12**, **15**, **16**]. Other types of singularity could also be considered in a similar manner, as could the case where some of the operators are singular and some regular. Examples III and IV of [**20**] are of the above form. Some aspects of the multiparameter spectral theory of differential operators defined on intervals of the form $(-\infty,\infty)$ and having periodic coefficients are considered in [**10**, **18**].

We remark that most of the results on differential operators that we require are contained in [**12**], however in [**12**] the coefficient functions are assumed to be C^∞ so we usually refer to other sources.

Letting $L^2(I_r)$ denote the Hilbert space of square integrable functions defined on I_r we define bounded self adjoint linear operators $V_{rs}: L^2(I_r)\to L^2(I_r)$ by

$$(V_{rs}u_r)(x_r)=v_{rs}(x_r)u_r(x_r),\ u_r\in L^2(I_r),\quad r=1,\dots,k.$$

Now let $C_0^\infty(I_r)$ be the set of infinitely differentiable functions with compact support in I_r and define the operators $\tilde{T}_r: C_0^\infty(I_r)\to L^2(I_r)$ by

$$(\tilde{T}_r u_r)(x_r) = -\frac{d}{dx_r}\left(p_r(x_r)\frac{d}{dx_r}u_r(x_r)\right) + q_r(x_r)u_r(x_r),\ u_r\in C_0^\infty(I_r),\quad r=1,\dots,k.$$

The operators $\tilde{T}_r$ are symmetric linear operators in the spaces $L^2(I_r)$ (see [**15**, p. 22]). We can now choose self adjoint extensions T_r of the operators $\tilde{T}_r$ (see [**15**, p. 24]), which leads to the Hilbert space realization of the system (4.1)

$$\left(T_r + \sum_{s=1}^{k} \lambda_s V_{rs}\right)u_r = 0. \tag{4.2}$$

In order to apply the theory of Section 3 we require the definiteness condition

$$\det[v_{rs}(x_r)] \geqq \beta, \text{ for all } x_r\in I_r, \quad r=1,\dots,k, \tag{4.3}$$

for some $\beta>0$. This condition corresponds to the abstract right definiteness condition (2.1). In addition to the above assumptions we suppose that the following limits exist and are finite

$$p_r^\infty = \lim_{x_r\to\infty} p_r(x_r),\ q_r^\infty = \lim_{x_r\to\infty} q_r(x_r),\ v_{rs}^\infty = \lim_{x_r\to\infty} v_{rs}(x_r), \quad r,s=1,\dots,k.$$

Now let V^∞ be the matrix with components $(V^\infty)_{rs}=v_{rs}^\infty$, and define $\mathbf{q}^\infty=(q_1^\infty,\dots,q_k^\infty)$, $\mathbf{v}_r^\infty=(v_{r1}^\infty,\dots,v_{rk}^\infty)$, $r=1,\dots,k$. The definiteness assumption (4.3) implies that the matrix V^∞ is non-singular.

The operators $W_r(\lambda)$ defined in Section 2 are now differential operators of the form

$$(W_r(\lambda)u_r)(x_r) = -\frac{d}{dx_r}\left(p_r(x_r)\frac{d}{dx_r}u_r(x_r)\right) + \left(q_r(x_r) + \sum_{s=1}^{k}\lambda_s v_{rs}(x_r)\right)u_r(x_r),$$

$$u_r\in D(T_r),\quad r=1,\dots,k. \tag{4.4}$$

It is shown in Theorem 2.9 of [**15**] that, for each $\boldsymbol{\lambda}\in\mathbb{R}^k$, a differential operator of this form, with the above hypotheses on the coefficient functions, is bounded below and is of type SD. Thus the multiparameter system (4.2) is of type SD and the results of Section 3 are applicable to this system. In addition we have

$$\rho_r^e(\boldsymbol{\lambda}) = q_r^\infty + \boldsymbol{\lambda}\cdot\mathbf{v}_r^\infty, \tag{4.5}$$

see Theorem 2.9 of [**15**]. This information enables us to improve Theorem 3.9 on the location of σ_e and σ_p.

Definition 4.1. Define the cone

$$C^\infty = \{\boldsymbol{\lambda}\in\mathbb{R}^k : V^\infty\boldsymbol{\lambda}\leqq \mathbf{0}\}.$$

The cone C^∞ is convex and the non-singularity of the matrix V^∞ implies that C^∞ is contained in a half plane in $\mathbb{R}^k$. It is obvious that $C^\infty \subset C^+$. Let $\boldsymbol{\omega} = -(V^\infty)^{-1}\mathbf{q}^\infty$. It follows from (4.5) that $\rho_r^e(\boldsymbol{\omega}) = 0$, for each r, and hence Theorem 3.5 shows that $\boldsymbol{\omega} \in \boldsymbol{\lambda}^{\mathbf{i}} + C^+$ for all $\mathbf{i} \in \mathbb{N}^k$.

Theorem 4.2. *Under the above hypotheses the spectrum of the multiparameter system* (4.2) *satisfies*

$$\sigma_e = \boldsymbol{\omega} + C^\infty,$$

$$\sigma_p \subset (\boldsymbol{\lambda}^0 + C^+) \cap \operatorname{int}(\boldsymbol{\omega} - C^\infty),$$

$$\Lambda \subset (\boldsymbol{\lambda}^0 + C^+) \cap (\boldsymbol{\omega} - C^\infty),$$

and the set $(\boldsymbol{\lambda}^0 + C^+) \cap (\boldsymbol{\omega} - C^\infty)$ *is compact* (int *denotes topological interior*). *Also* $\boldsymbol{\omega} = \boldsymbol{\lambda}^e$, *where* $\boldsymbol{\lambda}^e$ *is as in Section 3. Finally*

$$\sigma_m \subset \mathbb{R}^k \backslash \{(\boldsymbol{\omega} + C^\infty) \cup \operatorname{int}(\boldsymbol{\omega} - C^\infty)\}.$$

Proof. The spectral structure of the operators $W_r(\boldsymbol{\lambda})$ implies that a point $\boldsymbol{\lambda} \in \mathbb{R}^k$ belongs to σ_e if and only if $\rho_r^e(\boldsymbol{\lambda}) \leqq 0$ for each $r = 1, \ldots, k$. Thus, by the above remarks, $\boldsymbol{\lambda} \in \sigma_e \Leftrightarrow q_r^\infty + \boldsymbol{\lambda} \cdot \mathbf{v}_r^\infty \leqq 0$, $r = 1, \ldots, k$, or, equivalently, $\mathbf{q}^\infty + V^\infty \boldsymbol{\lambda} \leqq \mathbf{0}$. Equality holds in this relation only when $\boldsymbol{\lambda} = \boldsymbol{\omega}$, thus the first statement now follows from Definition 4.1. Similarly $\boldsymbol{\lambda} \in \sigma_p \Rightarrow q_r^\infty + \boldsymbol{\lambda} \cdot \mathbf{v}_r^\infty > 0$, $r = 1, \ldots, k$, which, together with Theorem 3.5, proves the second statement in a similar manner to the first. The third statement follows similarly. The compactness result follows from Lemma 2.2. These results, together with Theorem 3.9 and the definition of $\boldsymbol{\lambda}^e$, show that $\boldsymbol{\lambda}^e \in (\boldsymbol{\omega} - C^\infty) \cap (\boldsymbol{\omega} + C^\infty)$ and so, since C^∞ does not contain a line, we have $\boldsymbol{\lambda}^e = \boldsymbol{\omega}$. The final result is proved in a similar manner to the proof of the first three statements. □

Note that the remarks following Lemma 3.8 concerning the limiting process also hold in the present situation. Also Theorem 4.2 shows that any accumulation point of the point spectrum σ_p must belong to the boundary of the set $\boldsymbol{\omega} - C^\infty$.

In the construction of the self adjoint multiparameter system (4.2) we chose self adjoint extensions T_r of the symmetric operators $\tilde{T}_r$. In general these extensions are not unique so it is natural to ask what effect the choice of the extension has on the spectrum of the multiparameter system. The following discussion of this question does not depend on the operators being differential operators and applies to any system of the form (4.2) where the T_r are self adjoint extensions of symmetric operators $\tilde{T}_r$. However we assume that the operators $\tilde{T}_r$ are closed. This assumption is not restrictive since any symmetric operator is closable. For each r let (d_r^+, d_r^-) denote the deficiency indices of the symmetric operators $\tilde{T}_r$ (see [**12**]). Since the operators $\tilde{T}_r$ possess self adjoint extensions if and only if $d_r^+ = d_r^-$ we assume that $d_r^+ = d_r^- = d_r$, say, $r = 1, \ldots, k$, and define a "deficiency vector" $\mathbf{d} \in \mathbb{N}^k$ for the multiparameter system by $\mathbf{d} = (d_1, \ldots, d_k)$. We assume that $d_r < \infty$ for each r.

If we let $\tilde{W}_r(\boldsymbol{\lambda}) = \tilde{T}_r + \sum_{s=1}^k \lambda_s V_{rs}$, $r = 1, \ldots, k$, then since the operators V_{rs} are bounded and self adjoint it follows from Theorem 6, p. 33 of [**16**], that the operators $\tilde{W}_r(\boldsymbol{\lambda})$ have deficiency indices (d_r, d_r) for all $\boldsymbol{\lambda} \in \mathbb{R}^k$.

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Theorem 4.3. *The essential spectrum* σ_e *of* (4.2) *is independent of the choice of the self adjoint extensions* T_r.

Proof. Theorem 2.2 of [**15**] shows that the essential spectrum $\sigma_e(W_r(\lambda))$ is independent of the extension T_r, for each $\lambda \in \mathbb{R}^k$ and $r = 1, \ldots, k$. Therefore the set $\sigma_e = \{\lambda : 0 \in \sigma_e(W_r(\lambda)), r = 1, \ldots, k\}$ is independent of the extensions chosen. □

In the following theorem the essential spectrum of a symmetric operator is defined in the same way as the essential spectrum of a self adjoint operator was defined above.

Theorem 4.4. *Let* $\mathbf{d}$ *be the deficiency vector of the multiparameter system* $[\tilde{T}_r, V_{rs}]$ *and let* $\mathbf{m}$ *be any multi-index satisfying* $\mathbf{m} \leqq \mathbf{d}$. *Given any* $\lambda \in \mathbb{R}^k$ *such that* $0 \notin \sigma_e(\tilde{W}_r(\lambda))$, $r = 1, \ldots, k$, *let* $\tilde{\mathbf{m}}(\lambda)$ *be the multiplicity of the system* $[\tilde{T}_r, V_{rs}]$ *at* λ. *Then there exists a self adjoint extension* T_r *of* $\tilde{T}_r$, *for each* r, *such that the resulting self adjoint multiparameter system has* $\lambda \in \sigma_p$ *with multiplicity* $\mathbf{m}(\lambda) = \tilde{\mathbf{m}}(\lambda) + \mathbf{m}$.

Proof. Apply the proof of Theorem 10, p. 1400 of [**12**], to the operators $\tilde{W}_r(\lambda)$. □

Returning to the system of differential operators it can be shown that $d_r = 1$ for each r (Theorem 5, p. 203 of [**16**]). Therefore Theorems 4.3, 4.4 are applicable to this system if we regard the operators $\tilde{T}_r$ as the closure of the operators defined above on the domains $C_0^\infty(I_r)$. In particular the proof of Theorem 4.2 shows that the set of points $\lambda \in \mathbb{R}^k$ such that $0 \notin \sigma_e(\tilde{W}_r(\lambda))$, $r = 1, \ldots, k$, is equal to $\mathrm{int}(\omega - C^\infty)$ so, for this system, Theorem 4.4 shows that the containment of σ_p in Theorem 4.2 is the best possible in general. Note also that the above hypotheses imply that $\tilde{\mathbf{m}}(\lambda) = \mathbf{0}$ (see [**12**; p. 1400]), and the operators $W_r(\lambda)$ have at most 1-dimensional kernels, (see Appendix II, Section 7 of [**2**]).

We now briefly consider the oscillation properties of the eigenfunctions of the multiparameter system. Recall that in the present case $\sigma_p \subset \Lambda$, however not all points $\lambda^{\mathbf{i}} \in \Lambda$ are necessarily in σ_p.

Theorem 4.5. *If* $\lambda^{\mathbf{i}} \in \sigma_p$ *then there exists an eigenfunction* $\mathbf{u}^{\mathbf{i}}(\mathbf{x}) = (u_1^{\mathbf{i}}(x_1), \ldots, u_k^{\mathbf{i}}(x_k))$ $(\mathbf{x} = (x_1, \ldots, x_k))$ *such that the function* $u_r^{\mathbf{i}}(x_r)$ *has exactly* i_r *zeros in* I_r. *The eigenfunction* $\mathbf{u}^{\mathbf{i}}$ *is unique up to scaling of the component functions.*

Proof. The proof is based on the oscillation properties of one-parameter singular differential operators (see [**12**] or [**11**, p. 255, Prob. 2]) and, using these properties, is similar to the proof of Theorem 3.1 of [**6**], which deals with the case of regular differential operators. □

5. Kneser conditions

In Section 3 the points $\lambda^{\mathbf{i}} \in \sigma$ were defined for all $\mathbf{i} \in \mathbb{N}^k$; however, these points need not be isolated eigenvalues of finite multiplicity. We now consider the multiparameter system of differential operators (4.2) and present some conditions which enable us to determine whether the system has a finite or infinite set of isolated eigenvalues. In particular we provide sufficient conditions to assert that the point spectrum of the system (4.2) consists of the set Λ. The discussion is based on the so called "Kneser conditions" used in the one parameter oscillation theory of singular differential operators (see [**15**, p. 199; or **12**, p. 1481]).

The conditions we are going to consider depend critically on the behaviour of the coefficients of the differential operators as $x_r \to \infty$. For the rest of this section we suppose that the system (4.1) satisfies all the assumptions of Section 4 and, in addition, we impose the following hypothesis on the functions v_{rs}.

Condition K. The functions $v_{rs}(x_r)$ tend to the finite limits v_{rs}^{∞} as $x_r \to \infty$ and satisfy

$$\lim_{x_r \to \infty} x_r^2(v_{rs}(x_r) - v_{rs}^{\infty}) = 0, \quad r, s = 1, \dots, k.$$

In the proofs of the following theorems we require the Sturm comparison theorem as stated in Lemma 3.5, p. 1462 of [**12**]. The proof of this lemma given in [**12**] is valid with the continuity conditions we have imposed on the coefficients of the operators $W_r(\lambda)$.

Theorem 5.1. *Suppose that the array of coefficients v_{rs} satisfies Condition K and suppose that the coefficients q_r in the operators T_r satisfy*

$$\lim_{x_r \to \infty} q_r(x_r) = q_r^{\infty}, \ \limsup_{x_r \to \infty} x_r^2(q_r(x_r) - q_r^{\infty}) < -1/4, \quad r = 1, \dots, k.$$

Then the point spectrum of the system (4.2) *is equal to the set* Λ.

Proof. It follows from the boundedness assumptions on the functions p_r and Sturm's comparison theorem that, for any $\lambda \in \mathbb{R}^k$, we may apply Kneser's theorem [**15**, p. 199] to the operators $W_r(\lambda)$. This theorem, together with Theorem 5.2 of [**15**], shows that the spectrum of the operator $W_r(\lambda)$ contains an infinite sequence of isolated eigenvalues of finite multiplicity converging to the lowest point of the essential spectrum. Thus if $\lambda = \lambda^{\mathbf{i}}$, for any $\mathbf{i}$, it follows from the definition of $\lambda^{\mathbf{i}}$, using the minimax principle, in Section 3 that 0 is the (i_r+1)th eigenvalue of $W_r(\lambda^{\mathbf{i}})$, i.e. $0 \in \sigma_p(W_r(\lambda^{\mathbf{i}}))$, so $\lambda^{\mathbf{i}} \in \sigma_p$. Thus $\Lambda \subset \sigma_p$ and the result now follows from Theorem 3.4. □

Lemma 5.2. *Suppose that Condition K holds and let $A \subset \mathbb{R}^k$ be a compact set. Define the functions*

$$h_r^A(x_r) = \inf_{\lambda \in A} \left\{ \sum_{s=1}^{k} \lambda_s v_{rs}(x_r) \right\}, \quad x_r \in \bar{I}_r, \quad r = 1, \dots, k.$$

The functions h_r^A are bounded, continuous functions on the intervals $\bar{I}_r$. In addition, if we define

$$h_r^{A\infty} = \inf_{\lambda \in A} \left\{ \sum_{s=1}^{k} \lambda_s v_{rs}^{\infty} \right\}, \quad r = 1, \dots, k,$$

then $\lim_{x_r\to\infty} h_r^A(x_r)=h_r^{A\infty}$ *and*

$$\lim_{x_r\to\infty} x_r^2(h_r^A(x_r)-h_r^{A\infty})=0, \quad r=1,\dots,k.$$

Proof. The result follows easily from the boundedness and continuity of the functions v_{rs} together with the above hypotheses on the behaviour of the functions as $x_r\to\infty$. □

Theorem 5.3. *Suppose that the array of coefficients* v_{rs} *satisfies Condition K and let the coefficients* q_r *satisfy*

$$\lim_{x_r\to\infty} q_r(x_r)=q_r^\infty,\ \liminf_{x_r\to\infty} x_r^2(q_r(x_r)-q_r^\infty)>-1/4, \quad r=1,\dots,k.$$

Then the point spectrum of the system (4.2) *consists of a finite number of points.*

Proof. Let A be the set $A=(\lambda^0+C^+)\cap(\omega-C^\infty)$. Theorem 4.2 shows that A is a compact set, therefore the functions h_r^A of Lemma 5.2 can be defined for this set. Now consider the differential equations

$$-\frac{d}{dx_r}\left(p_r(x_r)\frac{d}{dx_r}y_r(x_r)\right)+(q_r(x_r)+h_r^A(x_r))y_r(x_r)=0, \quad r=1,\dots,k. \tag{5.1}$$

Let n_r be the maximum number of zeros of any solution y_r of (5.1) in I_r and let $n_r(\lambda)$ be the maximum number of zeros of any solution u_r of (4.1) in I_r, for each $\lambda\in A$. By the construction of the functions h_r^A we have

$$h_r^A(x_r)\leqq\sum_{s=1}^k \lambda_s v_{rs}(x_r), \quad x_r\in I_r,$$

for all $\lambda\in A$. Therefore using the Sturm comparison theorem it follows that $n_r(\lambda)\leqq n_r+1$ for all $\lambda\in A$.

By the construction of the point ω and Definition 4.1 we see that

$$\inf_{\lambda\in A}\left\{\sum_{s=1}^k \lambda_s v_{rs}^\infty\right\}=\inf_{\lambda\in\omega-C^\infty}(V^\infty\lambda)_r=(V^\infty\omega)_r=-q_r^\infty$$

and hence, by Lemma 5.2, we have

$$\lim_{x_r\to\infty}[q_r(x_r)+h_r^A(x_r)]=q_r^\infty+\inf_{\lambda\in A}\left\{\sum_{s=1}^k \lambda_s v_{rs}^\infty\right\}=0, \quad r=1,\dots,k.$$

By Lemma 5.2 and the hypotheses on the q_r we also have

$$\liminf_{x_r\to\infty} x_r^2[q_r(x_r)+h_r^A(x_r)]>-1/4, \quad r=1,\dots,k.$$

Thus we can apply Kneser's theorem to the equations (5.1) to show that the numbers n_r must be finite for each $r=1,\dots,k$.

Now suppose that $\lambda^{\mathbf{i}}\in\sigma_p$ for some $\mathbf{i}\in\mathbb{N}^k$. From Theorem 4.2 it follows that $\lambda^{\mathbf{i}}\in A$. By Theorem 4.5, there exists an eigenfunction $\mathbf{u}^{\mathbf{i}}=(u_1^{\mathbf{i}},\dots,u_k^{\mathbf{i}})$ such that the function $u_r^{\mathbf{i}}$ has exactly i_r zeros in I_r so the above result shows that $i_r\leqq n_r+1$. Hence the number of indices $\mathbf{i}$ for which $\lambda^{\mathbf{i}}\in\sigma_p$ is finite and, since $\sigma_p\subset\Lambda$, this proves the theorem. □

We note that we could consider the situation where some of the coefficients q_r satisfy the hypotheses of Theorem 5.1 while others satisfy the hypotheses of Theorem 5.3. In this case there exists bounds on the values of i_r for which $\lambda^{\mathbf{i}}\in\sigma_p$ for some values of r. However, in general, this is not sufficient to decide whether the number of points in σ_p is finite or infinite.

The Kneser conditions can also be used to resolve the question of the existence of the mixed spectrum as discussed at the end of Section 4. If the operators T_r, $r=1,\dots,n$, satisfy the hypotheses of Theorem 5.1 then it follows from the definition of the sets $\sigma_m^n(\tilde{\mathbf{i}})$ and σ_m^n, together with the method of proof of Theorem 5.1, that $\sigma_m^n(\tilde{\mathbf{i}})\subset\sigma_m^n$. In addition the remarks following Theorem 4.4 show that the kernels of the operators $W_r(\lambda)$ are at most 1-dimensional so the discussion at the end of Section 3 is applicable to this system.

Finally we remark that the result of Theorem 5.1 can be obtained under other conditions. All that the proof required was that for each r the operators $W_r(\lambda)$ have an infinite sequence of eigenvalues in the point spectrum below the essential spectrum. If some of the operators T_r are regular, and so have compact resolvent, then the corresponding operators $W_r(\lambda)$ do not have any essential spectrum and the sequence of eigenvalues tends to infinity. Therefore no further conditions on these operators are necessary to obtain Theorem 5.1. This result also holds for "quasi-regular" differential operators (see [2] for a discussion of quasi-regular differential operators), so again we do not require further conditions on such operators. Another sufficient condition for the operators T_r to have the necessary spectral behaviour is that $\lim_{x_r\to\infty}q_r(x_r)=\infty$ (see [16]). In this case the operators again do not have any essential spectrum. In addition to these conditions, Chapter 6 of [15] contains other oscillation criteria which could have been used in the above discussion instead of Kneser's theorem to produce similar results.

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Department of Mathematics
University of York
Heslington
York YO1 5DD