Canad. Math. Bull. Vol. 38 (4), 1995 pp. 390–395

A NOTE ON GEOMETRIC FACTORIALITY

S. M. BHATWADEKAR AND K. P. RUSSELL

ABSTRACT. Let k be a perfect field such that $\bar{k}$ is solvable over k. We show that a smooth, affine, factorial surface birationally dominated by affine 2-space $\mathbb{A}^2_k$ is geometrically factorial and hence isomorphic to $\mathbb{A}^2_k$. The result is useful in the study of subalgebras of polynomial algebras. The condition of solvability would be unnecessary if a question we pose on integral representations of finite groups has a positive answer.

1. **Introduction.** Let k be a field and A a regular factorial, affine k-algebra. Suppose $A \subset k[Z,T]$, the polynomial algebra in two variables over k. If k is algebraically closed and $k(Z,T)$ is a separable extension of the quotient field K of A, then by a famous result of Fujita and Miyanishi-Sugie, A is itself a polynomial algebra over k ([F] and [M-S], see also [R-1] for the case when $\operatorname{char} k > 0$). This result fails when k is not algebraically closed (see [B-D], Example 4.4 and 4.1 below). On the other hand, in counterexamples known to us, $[k(Z,T) : K] > 1$ and moreover, for perfect k, Russell ([R-2], Theorem 1.3) has shown that when $k[Z,T]$ is a simple (as ring) birational extension of A, then again A is a polynomial algebra over k. We therefore raise

QUESTION 1. Let k be a perfect field and A a regular, affine factorial, birational subalgebra of $k[Z,T]$. Is A a polynomial algebra over k?

We were motivated to study this question by considering regular, factorial affine k-algebras B such that

$$k[X] \subset B \subset k[X,Z,T].$$

It is then natural to ask whether B is a polynomial algebra and, if yes, whether X is a variable in B. This obviously is true if $\dim B = 1$, and has been shown to hold if $\dim B = 2$ by Russell and Sathaye ([R-S]). If $\dim B = 3$, it is not difficult to give counterexamples to the first part of the question (see [B-D], Example 4.4 and 4.2 below), even if k is algebraically closed. A first step in studying this situation will be to consider the ring extensions

$$k(X) \subset B \bigotimes_{k[X]} k(X) \subset k(X)[Z,T].$$

In case the extension $k[X,Z,T]/B$ is birational, an affirmative answer to Question 1 would imply that B is "generically" polynomial over $k[X]$ if $\operatorname{char} k = 0$, a result of interest even if we assume to begin with that B is polynomial over k.

This work was done when the first author was visiting the mathematics department of McGill University, Montreal. He gratefully acknowledges the warm hospitality and financial support received from McGill University and CICMA during his stay.

Received by the editors June 6, 1994.

AMS subject classification: Primary: 13F20; secondary: 14M05, 20C05.

The key to answering Question 1 is to ascertain that factoriality of A is preserved when the base field k is extended to L, where L/k is a finite Galois extension. We show this (see Proposition 3.4) in case L/k is solvable with the help of a result on integral representations (Proposition 2.2). If the condition of solvability could be removed there, Question 1 would be answered positively in general.

2. **A result from the representation theory.** Let G be a finite group and M a finite $\mathbf{Z}[G]$-module. For any subgroup $H \subset G$, we put $\mathrm{Inv}_H(M) = \{m \in M \mid hm = m \, \forall h \in H\}$. M is said to be a *permutation module* for G if M is free over $\mathbf{Z}$ with a basis S permuted by G. We then call S a permutable basis for M. M is said to be *transitive* if G is transitive on S. It is clear that any permutation module for G is a direct sum of transitive ones, corresponding to the decomposition of S into G-orbits.

LEMMA 2.1. *Let G be a finite group and let M be a transitive permutation left $\mathbf{Z}[G]$-module. Let H be a normal subgroup of G. Then* Inv_H *is a transitive permutation $\mathbf{Z}[G/H]$-module.*

PROOF. Let S be a transitively permutable basis of M and let $S_1, \ldots, S_t$ be the all distinct H-orbits of S. Then since H is normal in G and S is a transitively permutable basis (for G) it follows that any two distinct H-orbits have the same number of elements and given two orbits S_i, S_j there exists $g \in G$ such that $g \cdot S_i = S_j$.

Let $\omega_i = \sum_{v \in S_i} v \in M$, $1 \leq i \leq t$. Then $\mathrm{Inv}_H(M) = \bigoplus_{i=1}^t \mathbf{Z}\omega_i$ and given ω_i, ω_j there exists $g \in G$ such that $g \cdot \omega_i = \omega_j$.

Thus $\mathrm{Inv}_H(M)$ is a transitive permutation $\mathbf{Z}[G/H]$-module.

PROPOSITION 2.2. *Let G be a finite solvable group. Let F be a permutation $\mathbf{Z}[G]$-module and let M and N be $\mathbf{Z}[G]$-submodules of F such that $F = M \oplus N$. Furthermore, assume M is also a permutation $\mathbf{Z}[G]$-module. Then* $\mathrm{Inv}_G(N) = 0 \Rightarrow N = 0$.

PROOF. Let H be a normal subgroup of G. Since every permutation $\mathbf{Z}[G]$-module is a direct sum of transitive permutation modules, it follows from Lemma 2.1 that $\mathrm{Inv}_H(F)$ and $\mathrm{Inv}_H(M)$ are permutation $\mathbf{Z}[G/H]$-modules. Moreover, $\mathrm{Inv}_H(F) = \mathrm{Inv}_H(M) \oplus \mathrm{Inv}_H(N)$ and $\mathrm{Inv}_{G/H}\big(\mathrm{Inv}_H(N)\big) = \mathrm{Inv}_G(N)$. Therefore, as F and M are obviously permutation $\mathbf{Z}[H]$-modules, it is enough to prove the result when G is simple. But as G is solvable, this means that it is enough to prove the result when G is a cyclic group of prime order.

So we assume $|G| = p$, p a prime integer. Let g be a generator of G and let I be the ideal of $\mathbf{Z}[G]$ (note that $\mathbf{Z}[G]$ is commutative) generated by the element $g - 1$.

Let $F = \bigoplus_{i=1}^n F_i$ be a direct sum decomposition of F into transitive permutation $\mathbf{Z}[G]$-submodules of F. Since G is cyclic of order p, up to isomorphism $\mathbf{Z}[G]$ has only two transitive permutation modules viz. $\mathbf{Z}[G]$ (as a module) and $\mathbf{Z}$ (with the trivial G-module structure). Therefore it follows that $\mathrm{Inv}_G(F_i) \approx F_i/IF_i = \mathbf{Z}$ and hence $\mathrm{Inv}_G(F) \approx F/IF$. Similarly $\mathrm{Inv}_G(M) \approx M/IM$.

Now $F = M \oplus N$ and $\mathrm{Inv}_G(N) = 0$. So we see that $N/IN = 0$, *i.e.* $IN = N$. Since I is the principal ideal of $\mathbf{Z}[G]$ generated $g-1$, we get $(g-1)N = N$ and hence $(g-1)^pN = N$. But g is an element of G of order p. Therefore $(g-1)^pN = N$ implies that $pN = N$.

As N is a submodule of F and F is a free abelian group (since F is a permutation $\mathbf{Z}[G]$-module), $pN = N$ implies $N = 0$.

REMARK 2.3. Let $F = \oplus_{i=1}^{s} F_i$, where $F_1, \ldots, F_s$ are transitive permutation modules of rank $r_1, \ldots, r_s$. Then $s = \mathrm{rank}\big(\mathrm{Inv}_G(F)\big)$ and $H^0(G,F) \simeq \oplus_{i=1}^{s} \mathbf{Z}/\tilde{r}_i\mathbf{Z}$, where $\tilde{r}_i = |G|/r_i$. (Here $H^0(G,M) = \mathrm{Inv}_G(M)/\,\mathrm{Trace}(M)$; see [L]). Moreover, $H^0(G,N) = 0$ in the situation of Proposition 2.2. So Proposition 2.2 holds for arbitrary finite G in case F, or M, is transitive. It is therefore reasonable to ask

QUESTION 2. Does Proposition 2.2 remain true without the assumption that G is solvable?

3. Factorial surfaces dominated by $\mathbb{A}^2$.

LEMMA 3.1. *Let k be a field and let L/k be a finite separable extension. Let X be a smooth, quasi-projective scheme over k. Let $x \in X$ be a closed point of X and let $\pi: \tilde{X} \longrightarrow X$ be the blowing up of X with the center x (this will be referred to as monoidal transformation). Then the canonical map: $\pi_L: \tilde{X}_L \longrightarrow X_L$ (obtained by base change) is the blowing up of X_L with centre $p^{-1}(x)$ where $p: X_L \longrightarrow X$ is the canonical morphism.*

PROOF. Without loss of generality, we can assume that X is affine, say $X = \mathrm{Spec}(A)$. Let m be the maximal ideal of A corresponding to the closed point x. Let $B = A \otimes_k L$ and let $I = mB$. Then, since L is separable over k, I is the defining ideal of the closed subset $p^{-1}(x)$ of $\mathrm{Spec}(B)$. Now the result follows from the definition of blowing up and the following isomorphisms of L-algebras:

$$B \oplus I \oplus I^2 \cdots \approx (A \oplus m \oplus m^2 \cdots) \bigotimes_A B = (A \oplus m \oplus m^2 \cdots) \bigotimes_k L.$$

LEMMA 3.2. *Let k be a field and let L/k be a finite Galois extension with Galois group G. Let X be a smooth, geometrically integral, quasi-projective scheme over k. Then X_L is smooth and integral. The group G acts on the class group $\mathrm{Cl}(X_L)$ inducing a (left) $\mathbf{Z}[G]$-module structure. Moreover* $\mathrm{rank}\big(\mathrm{Cl}(X)\big) = \mathrm{rank}\,\mathrm{Inv}_G\big(\mathrm{Cl}(X_L)\big)$.

PROOF. It is obvious that X_L is smooth, integral and G acts (in a canonical manner) on $\mathrm{Cl}(X_L)$.

Let $p: X_L \longrightarrow X$ be the canonical morphism. Let C be an irreducible closed subset of X of codimension one and let $C'_1, \ldots, C'_n$ be the irreducible components of $p^{-1}(C)$. Then the codimension of C'_i in X_L is 1 for $1 \leq i \leq n$ and $p^*(C) = \sum_{i=1}^{n} C'_i$ (as L/k is separable), where $p^*: \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(X_L)$ is the group homomorphism induced by p. It is easy to see that $p^*\big(\mathrm{Cl}(X)\big) \subset \mathrm{Inv}_G\big(\mathrm{Cl}(X_L)\big)$.

Since p is a finite morphism and X, X_L are smooth, there exists a group homomorphism $p_*: \mathrm{Cl}(X_L) \longrightarrow \mathrm{Cl}(X)$ such that $p_*p^* =$ multiplication by the integer $|G|$. This gives the equality

$$\mathrm{rank}\,\mathrm{Cl}(X) = \mathrm{rank}\big(p^*\,\mathrm{Cl}(X)\big).$$

Let $\mathrm{Tr}\colon \mathrm{Cl}(X_L) \longrightarrow \mathrm{Cl}(X_L)$ be the trace homomorphism defined by $\mathrm{Tr}(c) = \sum_{g \in G} g \cdot c$. Then it is easy to see that $\mathrm{Im}(\mathrm{Tr}) \subset \mathrm{Inv}_G\big(\mathrm{Cl}(X_L)\big)$ and for $v \in \mathrm{Inv}_G\big(\mathrm{Cl}(X_L)\big)$, $\mathrm{Tr}(v) = |G|v$. Therefore we get the equality

$$\mathrm{rank}\big(\mathrm{Im}(\mathrm{Tr})\big) = \mathrm{rank}\big(\mathrm{Inv}_G \mathrm{Cl}(X_L)\big).$$

Since $p^* \mathrm{Cl}(X) \subset \mathrm{Inv}_G \mathrm{Cl}(X_L)$, to prove the result it is enough to show the inclusion $\mathrm{Im}(\mathrm{Tr}) \subset p^* \mathrm{Cl}(X)$.

Let C' be an irreducible closed subset of X_L of codimension 1. Let $H = \{g \mid g \in G, g(C') = C'\}$ be the stabilizer of C' and let $p(C') = C$. Then we have $\mathrm{Tr}(C') = |H| p^*(C)$. Thus we have $\mathrm{Im}(\mathrm{Tr}) \subset p^* \mathrm{Cl}(X) \subset \mathrm{Inv}_G\big(\mathrm{Cl}(X_L)\big)$. Therefore, by both of the equalities above, we have

$$\mathrm{rank}\big(\mathrm{Cl}(X)\big) = \mathrm{rank}\, \mathrm{Inv}_G\big(\mathrm{Cl}(X_L)\big).$$

LEMMA 3.3. *Let k be a field and let X be a smooth, integral, quasi-projective scheme over k. Let V be an affine open subscheme of X such that $\mathrm{Cl}(V) = 0$ and $k^* =$ the group of units in $\Gamma(V)$, the ring of regular functions on V. Let $C_1, \ldots, C_n$ be the irreducible components of the closed set $X - V$. Then the codimension of C_i in X is 1 for $1 \leq i \leq n$ and $\mathrm{Cl}(X)$ is a free abelian group with basis $\{C_1, C_2, \ldots, C_n\}$.*

PROOF. Since X is quasi-projective, integral and V is affine, it is clear that the codimension of C_i in X is 1 for $1 \leq i \leq n$.

Since $\mathrm{Cl}(V) = 0$, $\mathrm{Cl}(X)$ is generated by $C_1, \ldots, C_n$. So it is enough to show that they are linearly independent.

Suppose $0 = \sum_{i=1}^n n_i C_i$ in $\mathrm{Cl}(X)$, where the n_i are integers. This means that there exists a non zero element f of $k(X)$ (the function field of X) such that $(f) = \sum_{i=1}^n n_i C_i$, where f is the principal divisor defined by f on X. Since $C_i \cap V = 0$ for $1 \leq i \leq n$, f and $1/f$ are regular on V and therefore $f \in k^*$ by assumption. But then $(f) = 0$. Therefore $n_i = 0$ for $1 \leq i \leq n$ and we are through.

PROPOSITION 3.4. *Let k be a perfect field and A a regular, factorial, birational subalgebra of $k[Z, T]$. Let L/k be a finite Galois extension. If the Galois group $G = G(L/k)$ is solvable, then $A \otimes_k L$ is factorial.*

PROOF. Let $X = \mathrm{Spec}(A)$ and $\mathbb{A}_k^2 = \mathrm{Spec}\, k[Z, T]$. Since A is a birational subring of $k[Z, T]$, we obtain a birational morphism $f\colon \mathbb{A}_k^2 \longrightarrow X$. Then by Lemma 3.1 (and well known results on "Resolution of Singularities of Surfaces") it is clear that there exists a sequence of monoidal transformations

$$X_n \xrightarrow{\pi_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X$$

and a morphism $g\colon \mathbb{A}_k^2 \longrightarrow X_n$ such that g is an open immersion and $\pi_1 \circ \pi_2 \circ \cdots \pi_n \circ g = f$.

Put $Y = X_n$ and $\pi = \pi_1 \circ \pi_2 \circ \cdots \pi_n$. Then $\pi \circ g = f$ and hence we get a commutative triangle

$$\begin{array}{ccc} \mathbb{A}_L^2 & \xrightarrow{g_L} & Y_L \\ f_L \searrow & & \swarrow \pi_L \\ & X_L & \end{array}$$

with the following properties:

(1) g_L is an open immersion and $g_L(\mathbb{A}^2_L) = V_L$ where $g(\mathbb{A}^2_k) = V$.

Let $p\colon Y_L \longrightarrow Y$ denote the canonical map.

(2) Let C' be an irreducible closed subset of Y_L of codimension 1. Then C' is an irreducible component of $Y_L - V_L$ if and only if $p(C')$ is an irreducible component of $Y - V$.

(3) Let E' be an irreducible closed subset of Y_L of codimension 1. Then $\pi_L(E') =$ is a (closed) point if and only if $(\pi \circ p)(E') =$ is a (closed) point.

It is easy to establish properties (1), (2) and (3) (with the help of Lemma 3.1) and these will not be proved.

Let S be the set of all irreducible components of $Y_L - V_L$. Then since $V_L \simeq \mathbb{A}^2_L$, by Lemma 3.3, $\mathrm{Cl}(Y_L)$ is a free abelian group with S as a basis. Moreover by property (2) it follows that $\mathrm{Cl}(Y_L)$ is a permutation $\mathbf{Z}[G]$-module with S as a permutable basis.

Let T be the set of all irreducible closed subsets E' of Y_L such that $\pi_L(E')$ is a point. Then by property (3) it follows that G permutes the elements of T. Moreover, as Y is obtained from X by a sequence of monoidal transformations, it follows by Lemma 3.1 that the subgroup M of $\mathrm{Cl}(Y_L)$ generated by the elements of T is a free abelian group with basis T. Thus M is a permutation $\mathbf{Z}[G]$-module. Furthermore $\mathrm{Cl}(Y_L) = \mathrm{Cl}(X_L) \oplus M$ as $\mathbf{Z}[G]$-modules.

Since A is factorial, $\mathrm{Cl}(X) = 0$. Hence by Lemma 3.2, as $\mathrm{Cl}(X_L)$ is a free abelian group (being a direct summand of the permutation module $\mathrm{Cl}(Y_L)$), we have $\mathrm{Inv}_G\big(\mathrm{Cl}(X_L)\big) = 0$. Therefore, as G is solvable, by Proposition 2.2 we have $\mathrm{Cl}(X_L) = 0$, showing that $A \otimes_k L$ is factorial.

Let A be as in Proposition 3.4. Then there exists a finite Galois extension L/k such that, in the notation of the proof of Proposition 3.4, all fundamental points of π_L are rational over L (equivalently, all exceptional curves in Y_L are absolutely irreducible) and all irreducible components of $Y_L - \mathbb{A}^2_L$ are absolutely irreducible. Then $\mathrm{Aut}(\bar{k}/L)$ acts trivially on $\mathrm{Cl}(Y_{\bar{k}})$. If $G = G(L/k)$ is solvable, it therefore follows from Proposition 3.4 that $A \otimes_k \bar{k}$ is factorial. We will say that $f\colon \mathbb{A}^2 \longrightarrow X$ is "split" by L/k.

THEOREM 3.5. *Let k be a perfect field and $f\colon \mathbb{A}^2_k \longrightarrow X$ a birational morphism, where X is a smooth, factorial, affine surface. If f is "split" by a solvable Galois extension L/k, in particular if* $\mathrm{Gal}(\bar{k}/k)$ *is solvable, then X is isomorphic to $\mathbb{A}^2$ over k.*

PROOF. $X_{\bar{k}}$ is smooth and, by Proposition 3.4 above, factorial. By [F] and [M-S], $X_{\bar{k}} = \mathbb{A}^2_{\bar{k}}$. By the triviality of separable forms of $\mathbb{A}^2_k$ ([K], Theorem 3), $X \simeq \mathbb{A}^2_k$.

4. **Some examples.**

4.1. Let $k = \mathbb{R}$ and $A = \mathbb{R}[x, y, v]/xy - v^2 - 1$. Then A is factorial and $A \subset \mathbb{R}[Z, T]$ with $x = Z^2 + 1, y = 1 + 2ZT + (Z^2 + 1)T^2, v = Z + (Z^2 + 1)T$ (see [B-D] Example 4.4 for a more elaborate version). This extension is not birational and one of the starting points of our investigation was the question whether A can be birationally embedded in $\mathbb{R}[Z, T]$. By Theorem 3.5, this is not possible. (Note that $A \otimes_{\mathbb{R}} \mathbb{C}$ is not factorial).

4.2 . Let k be a field of characteristic 0, algebraically closed to fix the ideas. We are interested in affine, regular factorial k-algebras B such that

$$k[X] \subset B \subset k[X, Z, T]$$

and the extension $k[X, Z, T]/B$ is birational. As an example consider $B = k[x, v, t, s]$ with $st - xv = 1$. Then B is as above with $X = x$, $Z = \frac{s-1}{x}$, $T = \frac{t-1}{x}$. B is not polynomial over k, but $B \otimes_{k[x]} k(x)$ is over $k(x)$. Should Proposition 2.2 be true even for non-solvable G, we would know that this holds in general for B as above. Under the assumption that B is itself polynomial over k, we would have proved that X is "generically" a variable in B. It is of course much conjectured, but not yet proved, that then X is in fact a variable in B.

REFERENCES

[B-D] S. M. Bhatwadekar and A. Dutta, *On residual variables and stably polynomial algebras*, Comm. Algebra, to appear.

[F] T. Fujita, *On Zariski problem*, Proc. Japan Acad. (A) **55**(1979), 106–110.

[L] S. Lang, *Rapport sur le cohomologie des groupes*, W. A. Benjamin, New York, Amsterdam, 1966.

[K] T. Kambayashi, *On the absence of non-trivial separable forms of the affine plane*, J. Algebra **35**(1975), 449–456.

[M-S] M. Miyanishi and T. Sugie, *Affine surfaces containing cylinderlike open sets*, J. Math. Kyoto Univ. **20**(1980), 11–42.

[R-1] K. P. Russell, *On affine-ruled rational surfaces*, Math. Ann. **255**(1981), 287–302.

[R-2] ———, *Simple birational extensions of two dimensional rational domains*, Compositio Math. **33**(1976), 197–208.

[R-S] K. P. Russell and A. Sathaye, *On finding and cancelling variables in* $k[X, Y, Z]$, J. Algebra **57**(1979), 153–166.

Tata Institute of Fundamental Research
Bombay
India

Department of Mathematics and Statistics and CICMA
McGill University
Montreal, Quebec

https://doi.org/10.4153/CMB-1995-057-0 Published online by Cambridge University Press