HESSE'S THEOREM FOR A QUADRILATERAL WHOSE SIDES TOUCH A CONIC

William G. Brown

(received May 31, 1960)

1. Introduction. Hesse's theorem states that "if two pairs of opposite vertices of a quadrilateral are respectively conjugate with respect to a given polarity, then the remaining pair of vertices are also conjugate".

In the real projective plane there cannot exist such a quadrilateral, all four sides of which are self-conjugate [1, §5.54]. We shall show that such a quadrilateral exists in PG(2, 3), and that any geometry in which such a quadrilateral exists contains the configuration 13_4 of PG(2, 3). We shall thus provide a synthetic proof of Hesse's theorem for a quadrilateral of this type, which, together with [1, § 5.55], constitutes a complete proof of the theorem valid in general Desarguesian projective geometry. We shall also show analytically that a finite Desarguesian geometry which admits a Hessian quadrilateral all of whose sides touch a conic must be of type PG$(2, 3^n)$.

2. Example in PG(2, 3). Represent points and lines respectively by P_i, p_i $(i = 0, 1, \ldots, 12)$ with the rule that P_i, p_j are incident if and only if

$$i + j \equiv 0,\ 1,\ 3,\ \text{or } 9 \pmod{13}.$$

The table of incidences is

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
3	4	5	6	7	8	9	10	11	12	0	1	2
9	10	11	12	0	1	2	3	4	5	6	7	8
0	12	11	10	9	8	7	6	5	4	3	2	1

Canad. Math. Bull. vol. 3, no. 3, September 1960

https://doi.org/10.4153/CMB-1960-027-0 Published online by Cambridge University Press

Then the polarity $(P_4P_{10}P_{12})$ (P_0p_0) determines a conic such that the quadrilateral $p_0p_7p_8p_{11}$ has all four sides self-conjugate. Hesse's theorem evidently holds for this quadrilateral and this polarity.

3. THEOREM. Let $P_1P_3P_5P_2P_6P_9$ be a given quadrilateral whose sides $P_1P_3P_9$, $P_2P_6P_9$, $P_2P_3P_5$, $P_1P_5P_6$ contain their respective poles P_0, P_7, P_{11}, and P_8. Suppose P_1, P_2 conjugate; P_3, P_6 conjugate. Then P_5 and P_9 are conjugate.

Proof. The given quadrilateral has the same diagonal triangle as the quadrangle $P_0P_7P_{11}P_8$. We thus obtain the table

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4		6	7	8		10	11	12	0
3	4	5	6	7	8	9	10	11	12	0		
9	10	11	12	0	1	2		4	5		7	8
0	12	11	10	9	8	7	6	5	4	3	2	1
*	**	*	**		*	*			**			

where the columns marked with a single asterisk define the quadrilateral, those marked with a double asterisk define the diagonal triangle $P_4P_{10}P_{12}$, and the remaining columns are due to our last result.

Our initial hypothesis gives the further relations

7		11	
8	11		0
	0	1	2
3	6	7	8

When these last relations are combined with the previous table, the result, except for two missing entries, is the incidence table of PG(2, 3) exhibited earlier. The gaps are filled by applying Desargues' Theorem. Since triangles $P_{10}P_{11}P_3$, $P_2P_{12}P_9$ are perspective from P_1, therefore P_5, P_7 and P_0

are collinear; since triangles $P_5P_6P_2$, $P_{12}P_0P_1$ are perspective from P_{10}, therefore P_9, P_{11}, and P_8 are collinear. Thus the geometry contains the 13_4 of PG(2, 3), wherein the quadrilateral $P_1P_3P_5P_2P_6P_9$ has already been shown to satisfy Hesse's theorem.

We note that O'Hara and Ward's proof of Hesse's theorem [2, § 6.25] is also valid in general Desarguesian projective geometry.

4. We prove analytically that such a quadrilateral can exist only in a geometry of type $PG(2, 3^n)$, provided the geometry is finite.

Consider the quadrilateral

$$x_1 \pm x_2 \pm x_3 = 0 .$$

Any conic inscribed therein must be of the form

$$\sum C_i X_i^2 = 0$$

where

$$\sum C_i = 0 \qquad \text{(dual of [1, § 12.78]).}$$

In point coordinates this is

$$\sum \frac{x_i^2}{C_i} = 0 .$$

Since opposite vertices are conjugate, $C_1 = C_2 = C_3$. Hence $3C_1 = 0$. Hence $3 = 0$. Thus the geometry is of type $PG(2, 3^n)$.

REFERENCES

1. H.S.M. Coxeter, The Real Projective Plane, second edition, (Cambridge, 1955).

2. C.W. O'Hara & D.R. Ward, An Introduction to Projective Geometry, (Oxford, 1937).

University of Toronto

https://doi.org/10.4153/CMB-1960-027-0 Published online by Cambridge University Press