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MINIMAL PENCIL REALIZATIONS OF RATIONAL MATRIX FUNCTIONS WITH SYMMETRIES

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ABSTRACT. A theory of minimal realizations of rational matrix functions $W(\lambda)$ in the "pencil" form $W(\lambda) = C(\lambda A_1 - A_2)^{-1}B$ is developed. In particular, properties of the pencil $\lambda A_1 - A_2$ are discussed when $W(\lambda)$ is hermitian on the real line, and when $W(\lambda)$ is hermitian on the unit circle.

1**. Introduction.** The modern theory of systems and control relies on detailed knowledge of the properties of rational matrix functions; namely, $r \times n$ matrix valued functions $W(\lambda)$ where entries are scalar rational functions. Much of this detailed information can be obtained from "realizations" of $W(\lambda)$. In this note we follow Wimmer ([10] and [11]), and choose to define a realization as a representation of the form

$$W(\lambda) = C(\lambda A_1 - A_2)^{-1}B \tag{1.1}$$

where $\lambda A_1 - A_2$ is a *regular* pencil of square matrices over $\mathbb{C}$, *i.e.* $\det(\lambda A_1 - A_2) \not\equiv 0$. Representations of the form (1.1) with $A_1 = I$ (so that $\|W(\lambda)\| \longrightarrow 0$ as $|\lambda| \longrightarrow \infty$) have been intensively studied and, to avoid confusion, a representation of the form (1.1) is termed a *p-realization* (for "pencil" realization). When $\|W(\lambda)\| \longrightarrow 0$ as $|\lambda| \longrightarrow \infty$, $W(\lambda)$ is said to be *strictly proper*. A *p*-realization (1.1) is said to be *minimal* if A_1, A_2 have the smallest possible size.

The main purpose of this note is to reveal what symmetries are implied for a minimal *p*-realization when:

(a) $W(\lambda)$ is hermitian on the real line.

(b) $W(\lambda)$ is hermitian on the unit circle.

Both these questions are of considerable practical significance and, although this short paper contains new and interesting results, it is written so that the non-expert can appreciate the arguments. The answer to question (a) is elegant and is the main result of this work. A regular pencil $\lambda A_1 - A_2$ is said to be *H-selfadjoint* (or *H*-unitary), if there is a nonsingular hermitian matrix H for which $A_1^* H A_2$ is hermitian (for which $A_1^* H A_1 = A_2^* H A_2$). Such pencils have been studied recently in [7] and [6] and some details of canonical forms can be found in the latter work.

It is not difficult to see that, if A_1 and A_2 are hermitian (when the pencil $\lambda A_1 - A_2$ is said to be hermitian) then there is an H in which $\lambda A_1 - A_2$ is H-selfadjoint. Furthermore,

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every H-selfadjoint pencil is strictly equivalent to an hermitian pencil (Theorem 2.1 of [6]). Similarly, if $\lambda A_1 - A_2$ is H-unitary then $\lambda A_1 - A_2$ and $\lambda A_2^* - A_1^*$ are strictly equivalent (Theorem 3.2 of [6]).

The following theorems will be established.

THEOREM 1.1. *If $W(\lambda)$ is hermitian on the real line, then the regular pencil of any minimal p-realization is H-selfadjoint in some H.*

THEOREM 1.2. *If $W(\lambda)$ is hermitian on the unit circle then the regular pencil of a minimal p-realization is H-unitary in some H if and only if $W(\lambda)$ is strictly proper.*

In many cases, (but not exclusively) p-realizations are of interest because they admit a singular matrix A_1. (Otherwise they can be reduced to the classical case by writing $(\lambda A_1 - A_2)^{-1} = (\lambda I - A_1^{-1}A_2)^{-1}A_1^{-1}$.) They also admit the association of $W(\lambda)$ with a singular differential system

$$A_1\dot{x}(t) = A_2x(t) + Bu(t), \quad y(t) = Cx(t), \tag{1.2}$$

sometimes known as a *descriptor* system. This aspect of the theory has driven several earlier investigations. In most of the earlier papers on descriptor systems (see [3] and [8], for example) preliminary simplifying transformations are applied to the system and have the effect of obscuring symmetries that may be present. We therefore eschew this approach in favour of a direct approach consistent with that of Zhou et al. [12] and of the broad generalizations contained in the recent work of Alpay and Dym [1].

2. Minimal p-realizations.

The usual approach to the realization of a regular rational matrix function $W(\lambda)$ is to write

$$W(\lambda) = W_1(\lambda) + W_2(\lambda) \tag{2.1}$$

where $W_1(\lambda)$ is strictly proper and $W_2(\lambda)$ is a polynomial matrix function. Because $W_1(\lambda)$ is strictly proper it is well-known that there is a realization

$$W_1(\lambda) = C(\lambda I - A)^{-1}B, \tag{2.2}$$

(see Theorem 7.1.2 of [5], for example). Furthermore, this realization is minimal if and only if (A, B) is a full-range pair, and (C, A) is a null-kernel pair. That is, if A has size n,

$$\sum_{j=0}^{n-1} \operatorname{Im}(A^jB) = \mathbb{C}^n, \quad \bigcap_{j=0}^{n-1} \operatorname{Ker}(CA^j) = \{0\},$$

(see Sections 2.7 and 2.8 of [5]). The terms "full-range" and "null-kernel" were coined for the work [5] and are more frequently replaced by "controllable" and "observable", respectively. The latter terms originate with underlying differential systems (as in (1.2)) and, as the notions of controllability and observability are more sophisticated in the context of a singular systems (see [3] and [12], for example), we stay with the mathematically motivated terminology.

The argument used in the following theorem is natural and can be traced back to Rosenbrock [9] in 1974 (see also Wimmer [10]). Since the construction is important for the sequel we reproduce the argument here.

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THEOREM 2.1. *Every $r \times n$ rational matrix function has a p-realization.*

PROOF. Let $W(\lambda)$ be a rational matrix function, decompose $W(\lambda)$ as in (2.1), and note the realization (2.2) of $W_1(\lambda)$.

The function $\hat{W}(\lambda) := \lambda^{-1}W_2(\lambda^{-1})$ is also strictly proper rational and so has a realization

$$\hat{W}(\lambda) = -E(\lambda I - N)^{-1}F \tag{2.3}$$

(Theorem 7.1.2 of [5]). Furthermore, N is nilpotent, otherwise $\hat{W}(\lambda)$ (and hence $W_2(\lambda)$), would have a finite nonzero pole. It follows that

$$W_2(\lambda) = E(\lambda N - I)^{-1}F \tag{2.4}$$

and, combining (2.2) and (2.4),

$$W(\lambda) = [\,C \quad E\,]\begin{bmatrix} \lambda I - A & 0 \\ 0 & \lambda N - I \end{bmatrix}^{-1}\begin{bmatrix} B \\ F \end{bmatrix} \tag{2.5}$$

■

Note that we have proved a little more than is required by the theorem statement; namely, that there is a p-realization of the special form (2.5), in which N is nilpotent.

THEOREM 2.2 (*cf.* THEOREM 3.1 OF [1]). *Let $W(\lambda)$ be an $n \times n$ rational matrix function for which $W(\lambda_0)$ is nonsingular, $\lambda_0 \in \mathbb{C}$. Then a p-realization $W(\lambda) = C(\lambda A_1 - A_2)^{-1}B$ is minimal if and only (A_1K, B) is a full-range pair and (C, KA_1) is a null-kernel pair, where $K = (\lambda_0 A_1 - A_2)^{-1}$.*

PROOF. Define $\tilde{W}(\lambda)$ by:

$$\begin{aligned} \tilde{W}(\lambda) &= \lambda^{-1}W(\lambda_0 + \lambda^{-1}) = \lambda^{-1}C\{(\lambda_0 + \lambda^{-1})A_1 - A_2\}^{-1}B \\ &= \lambda^{-1}C(\lambda^{-1}A_1 + K^{-1})^{-1}B = C(A_1 + \lambda K^{-1})^{-1}B \\ &= CK(\lambda I + A_1K)^{-1}B, \end{aligned} \tag{2.6}$$

so that $\tilde{W}(\lambda)$ is strictly proper. Clearly, the p-realization for $W(\lambda)$ is minimal if and only if the realization (2.6) is minimal for $\tilde{W}(\lambda)$, *i.e.* (using the classical result above) if and only if (A_1K, B) is a full-range pair and (CK, A_1K) is a null-kernel pair. However, the latter condition is easily seen to be equivalent to the statement that (C, KA_1) is a null-kernel pair. ■

LEMMA 2.3. *Let $W(\lambda)$ be an $n \times n$ rational matrix function with* $\det W(\lambda_0) \neq 0$ *for some $\lambda_0 \in \mathbb{C}$ and define $W_1(\lambda)$, $W_2(\lambda)$ as in equation (2.1). Then the realizations*

$$W_1(\lambda) = C(\lambda I - A)^{-1}B, \quad \lambda^{-1}W_2(\lambda) = -E(\lambda I - N)^{-1}F \tag{2.7}$$

(of equations (2.2) and (2.3)) are minimal if and only if the realization (2.5) for $W(\lambda)$ is minimal.

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PROOF. Apply Theory 2.2 to the p-realization (2.5) of $W(\lambda)$, so that

$$A_1 = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad A_2 = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}.$$

Then the matrix K of Theorem 2.2 takes the form

$$K = \begin{bmatrix} (\lambda_0 I - A)^{-1} & 0 \\ 0 & (\lambda_0 N - I)^{-1} \end{bmatrix},$$

and $(A_1K, [\begin{smallmatrix} B \\ F \end{smallmatrix}])$ is clearly a full-range pair if and only if both

$$\big((\lambda_0 I - A)^{-1}, B\big), \quad \big(N(\lambda_0 N - I)^{-1}, F\big)$$

are full-range pairs.

Now the minimality of the realizations (2.7) implies that (A, B) and (N, F) are full-range pairs. Furthermore, since A_1 and $f(A_1)$ (where $f(\lambda) = (\lambda_0 - \lambda)^{-1}$) have the same invariant subspaces (see Theorem 2.11.3 of [5]), it follows that $\big((\lambda_0 I - A)^{-1}, B\big)$ is a full-range pair. Using the function $g(\lambda) = \lambda(\lambda_0\lambda - 1)^{-1}$ in a similar way, it follows that $\big(N(\lambda_0 N - I)^{-1}, F\big)$ is a full-range pair. The full-range property follows for $(A_1K, [\begin{smallmatrix} B \\ F \end{smallmatrix}])$.

Similarly, it follows that when the realizations (2.7) are minimal, the pair $([\,C \quad E\,], KA_1)$ is null-kernel. The minimality of realization (2.5) now follows from Theorem 2.2.

The converse statement is apparent. ■

LEMMA 2.4. *Let $W(\lambda)$ be an $n \times n$ rational matrix function with $W(\lambda_0)$ nonsingular and two minimal realizations,*

$$W(\lambda) = C(\lambda A_1 - A_2)^{-1} B = \hat{C}(\lambda \hat{A}_1 - \hat{A}_2)^{-1} \hat{B}.$$

Then $\lambda A_1 - A_2$ and $\lambda \hat{A}_1 - \hat{A}_2$ are strictly equivalent.

PROOF. A minimal realization can be used to reconstruct the strictly proper and polynomial parts of $W(\lambda)$ (see the technique described in Proposition A.3 of [6], and Theorem 6.3.3 of [7], for example). Then the zero and pole structures of $W_1(\lambda)$, $W_2(\lambda)$ uniquely define the elementary divisors of both $\lambda A_1 - A_2$ and $\lambda \hat{A}_1 - \hat{A}_2$ (including those at infinity). The strict equivalence then follows from the Kronecker theory (see Section A.5 of [5], for example). ■

3. **The size of minimal p-realizations.** Consider the decomposition (2.1). The *local degree* $\delta(W_2, \infty)$ of $W_2(\lambda)$ (and of $W(\lambda)$) *at infinity* is defined to be the local degree at zero of $W_2(\lambda^{-1})$ (see Section 4.1 of [2]). Thus, if

(3.1) $$W_2(\lambda^{-1}) = \sum_{j=-q}^{\infty} \lambda^j W_j, \quad W_{-q} \neq 0,$$

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in a neighbourhood of the origin, then

$$\delta(W_2,\infty) = \operatorname{rank}\begin{bmatrix} W_{-q} & \cdots & W_{-2} & W_{-1} \\ 0 & \ddots & & W_{-2} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & W_{-q} \end{bmatrix}.$$

As long as $W_2(\lambda) \not\equiv 0$ the function $\hat{W}(\lambda) = \lambda^{-1}W_2(\lambda^{-1})$ introduced in the construction of Theorem 2.1 will obviously have higher local degree at infinity than W_2, namely,

$$\delta(\hat{W},\infty) = \operatorname{rank}\begin{bmatrix} W_{-q} & \cdots & W_{-1} & W_0 \\ 0 & \ddots & & W_{-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & W_{-q} \end{bmatrix}.$$

It follows that the size of a minimal p-realization will exceed the McMillan degree of $W(\lambda)$ by

$$\delta(\hat{W},\infty) - \delta(W_2,\infty).$$

(The McMillan degree is the sum of all local degrees, finite or infinite, see Section 4.2 of [2].) In particular, if $W_2(\lambda)$ is finite at infinity ($q = 0$ in (3.1)), then the size of a p-realization exceeds the McMillan degree of $W(\lambda)$ by rank W_0.

EXAMPLE 1. Let $W(\lambda) = \lambda + \lambda^{-1}$. Then the McMillan degree of $W(\lambda)$ is two. The procedure of Theorem 2.1 produces a minimal p-realization

$$W(\lambda) = [\,1 \quad 1 \quad 0\,]\left(\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}. \tag{3.2}$$

The minimality is readily confirmed using Theorem 2.2.

EXAMPLE 2. Let

$$W(\lambda) = \begin{bmatrix} \lambda^{-1} + 1 + \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \tag{3.3}$$

with McMillan degree three, and singular polynomial part $W_2(\lambda)$. A minimal p-realization has size four. For example

$$W(\lambda) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\left(\lambda\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

is a minimal p-realization obtained by the methods of this section.

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4. **Rational functions hermitian on the real line.** The first theorem of this section is due to Wimmer [11] and is a major step in the proof of Theorem 1.1. Our proof is by different methods and is included in the interests of a self-contained presentation

THEOREM 4.1. *If $W(\lambda)$ is a rational matrix function which is hermitian on the real line then there is a self-adjoint minimal p-realization for $W(\lambda)$, i.e. there exist hermitian A_1 and A_2 and a matrix B such that*

$$W(\lambda) = B^*(\lambda A_1 - A_2)^{-1}B \tag{4.1}$$

is a minimal p-realization.

PROOF. It is clear that in the decomposition (2.1), $W_1(\lambda)$ and $W_2(\lambda)$ will be hermitian on the real line if and only if $W(\lambda)$ has this property. Then Theorem II.3.1 of [4] shows that there are minimal realizations (2.2) and (2.4) with N nilpotent, and unique nonsingular hermitian matrixes H_1 and H_2 such that

$$A^* = H_1AH_1^{-1}, \quad B^* = CH_1^{-1}, \tag{4.2}$$

$$N^* = H_2NH_2^{-1}, \quad F^* = EH_2^{-1}. \tag{4.3}$$

Writing $H = \operatorname{diag}[H_1, H_2]$ and using Lemma 2.3 we obtain the minimal p-realization

$$\begin{aligned} W(\lambda) &= [\,C \quad E\,]\begin{bmatrix} \lambda I - A & 0 \\ 0 & \lambda N - I \end{bmatrix}^{-1}\begin{bmatrix} B \\ F \end{bmatrix}, \\ &= [\,B^* \quad F^*\,]\begin{bmatrix} \lambda H_1^{-1} - AH_1^{-1} & 0 \\ 0 & \lambda NH_2^{-1} - H_2^{-1} \end{bmatrix}^{-1}\begin{bmatrix} B \\ F \end{bmatrix}. \end{aligned} \tag{4.4}$$

It follows from (4.2) and (4.3) that AH_1^{-1} and NH_2^{-1} are hermitian and so (4.4) has the symmetry required. ∎

Note that Wimmer also proves in [11] that a hermitian minimal p-realization of the form (4.1) is unique up to a natural congruence transformation.

Theorem 1.1 is now readily proved. Let

$$W(\lambda) = C(\lambda A_1 - A_2)^{-1}B \tag{4.5}$$

be an arbitrary minimal p-realization and

$$W(\lambda) = E^*(\lambda S_1 - S_2)E \tag{4.6}$$

be a self-adjoint minimal p-realization. Define $H_0 = (\lambda_0 S_1 - S_2)^{-1}$ where the real λ_0 is chosen so that the inverse exists. Then $\lambda_0 S_1 - S_2$ is self-adjoint with respect to H_0. To see this, write

$$S_1 = S_1H_0H_0^{-1} = S_1H_0(\lambda_0 S_1 - S_2) = \lambda_0 S_1H_0S_1 - S_1H_0S_2.$$

Since S_1 and $\lambda_0 S_1H_0S_1$ are hermitian, so is $S_1H_0S_2 = S_1^*H_0S$, as required.

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Now Lemma 2.4 shows that there exist nonsingular X and Y such that

$$\lambda A_1 - A_2 = Y^{-1}(\lambda S_1 - S_2)X.$$

It is easily verified that $\lambda A_1 - A_2$ is self-adjoint with respect to $Y^* H_0 Y$. ∎

A converse for Theorem 1.1 is:

THEOREM 4.2. *Let $\lambda A_1 - A_2$ be an $n \times n$ H-selfadjoint pencil and $\mu A_1 - A_2$ be nonsingular where $\mu \in \mathbb{R}$. Let B be an $n \times m$ matrix and define $C = B^* H(\mu A_1 - A_2)$. Then the function*

$$W(\lambda) = C(\lambda A_1 - A_2)^{-1} B$$

is hermitian on the real line.

PROOF. Define $T = (\mu A_1^* - A_2^*)H$ so that $C = B^* T^*$ and

$$W(\lambda) = B^* T^* (\lambda A_1 - A_2)^{-1} B = B^* \Big(\lambda A_1 (T^*)^{-1} - A_2 (T^*)^{-1}\Big)^{-1} B.$$

Now verify that TA_1 and TA_2 (and hence $A_1(T^*)^{-1}$, $A_2(T^*)^{-1}$) are hermitian.

EXAMPLE 3. The function $W(\lambda)$ of Example 2 is hermitian on the real line. The regular pencil of (3.3) is H-selfadjoint where

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

∎

5. **Rational functions hermitian on the unit circle.** The property $W(\lambda)^* = W(\bar{\lambda}^{-1})$ ensures that the poles of $W(\lambda)$ are located in the complex plane symmetrically with respect to the unit circle. This applies also to the pair of points 0 and ∞. Furthermore, if there are poles at these two points then, by definition, they will have the same algebraic multiplicities, say ρ.

Now our construction of minimal p-realizations shows that when $\rho > 0$ the function $(\lambda A_1 - A_2)^{-1}$ has pole multiplicities $\rho(\infty) > \rho$ and $\rho(0) = \rho$ (see the discussion of Section 3). However, if $\lambda A_1 - A_2$ is an H-unitary pencil the pole multiplicities of $(\lambda A_1 - A_2)^{-1}$ at zero and infinity must agree (Theorem 4.2 of [6]). Consequently, if $W(\lambda)^* = W(\bar{\lambda}^{-1})$, and $W(\lambda)$ has poles at 0 and ∞, then the pencil of a minimal p-realization cannot be H-unitary for any H.

The last statement means that the proof of Theorem 1.2 can be confined to the case when $W(\lambda)^* = W(\bar{\lambda}^{-1})$ and $W(\lambda)$ is strictly proper. But this is a relatively well-known case and there is a realization

$$W(\lambda) = C(\lambda I - A)^{-1} B. \tag{5.1}$$

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and a nonsingular hermitian H for which $A^*HA = H$ and $C = iB^*HA$ (see the argument of Section 6.6 of [7], for example). In particular, the pencil $\lambda I - A$ is H-unitary. This completes the proof of Theorem 1.2.

EXAMPLE 4. Notice that the function $W(\lambda) = \lambda + \lambda^{-1}$ of Example 1 has the property that $W(\lambda)^* = W(\bar{\lambda}^{-1})$. An easy direct computation verifies that the pencil of the minimal p-realization (3.2) is not H-unitary for any H.

EXAMPLE 5. The function $W(\lambda) = i\lambda/(\lambda + i)^2$ is strictly proper, is hermitian on the unit circle, and has a minimal p-realization

$$W(\lambda) = [\,-i \quad 0\,]\begin{bmatrix} \lambda + i & 1 \\ 0 & \lambda + i \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus $A = \begin{bmatrix} -i & -1 \\ 0 & -i \end{bmatrix}$ in (5.1). If $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ it is easily verified that $A^*HA = H$, *i.e.* $\lambda I - A$ is H-unitary.

REMARK. Our analysis in this note has been over the complex numbers. However, it has been proved in Theorem 7.1 of [6] that if the entries of $W(\lambda)$ are real polynomials (and $W(\lambda)$ is hermitian on the unit circle), then the pole multiplicities of $W(\lambda)$ at ± 1 (if any) are constrained by the symmetries so that the number of partial pole multiplicities of even order is even. The partial pole multiplicities of odd order are not constrained.

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REFERENCES

1. D. Alpay and H. Dym, *On a new class of realization formulas and their application*. Linear Algebra Appl. **241–243**(1996), 3–84.
2. H. Bart, I. Gohberg and M. A. Kaashoek, *Minimal Factorization of Matrix and Operator Functions*. Birkhäuser Verlag, Basel, 1979.
3. D. Cobb, *Controllability, observability, and duality in singular systems*. IEEE Trans. Automat. Control **AC-29**(1984), 1076–1082.
4. I. Gohberg, P. Lancaster and L. Rodman, *Matrices and Indefinite Scalar Products*. Birkhäuser Verlag, Basel, 1983.
5. ———, *Invariant Subspaces of Matrices with Applications*. John Wiley, New York, 1986.
6. I. Krupnik and P. Lancaster, *H-selfadjoint and H-unitary matrix pencils*. SIAM J. Matrix Anal. Appl. **19**(1998), to appear.
7. P. Lancaster and L. Rodman, *Algebraic Riccati Equations*. Oxford Univ. Press, 1995.
8. R. Nikoukhah, A. S. Willsky and B. C. Levy, *Reachability, observability, and minimality for shift-invariant two-point boundary-value descriptor systems*. Circuits Systems Signal Process. **8**(1989), 313–340.
9. H. H. Rosenbrock, *Structural properties of linear dynamical systems*. Internat. J. Control **20**(1974), 191–202.
10. H. K. Wimmer, *The structure of nonsingular polynomial matrices*. Math. Systems Theory **14**(1981), 367–379.

https://doi.org/10.4153/CMB-1998-027-8 Published online by Cambridge University Press

11. ______, *Indefinite inner-product spaces associated with hermitian polynomial matrices*. Linear Algebra Appl. **50**(1983), 609–619.

12. Z. Zhou, M. A. Shayman and T.-J. Tarn, *Singular systems: a new approach in the time domain*. IEEE Trans. Automat. Control **32**(1987), 42–50.

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