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The John–Nirenberg Inequality for the Regularized BLO Space on Non-homogeneous Metric Measure Spaces

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Abstract. Let $(\mathcal{X}, d, \mu)$ be a metric measure space satisfying the geometrically doubling condition and the upper doubling condition. In this paper, the authors establish the John-Nirenberg inequality for the regularized BLO space $\widetilde{\mathrm{RBLO}}(\mu)$.

1 Introduction

In the classical Euclidean space (the Euclidean space equipped with the Lebesgue measure), the John–Nirenberg inequality for the space $\mathrm{BMO}(\mathbb{R}^n)$ established by John and Nirenberg [12] examines the rate of logarithmic growth of functions in $\mathrm{BMO}(\mathbb{R}^n)$; see, for instance [5, p. 123]. In 2001, Tolsa [15] introduced the regularized BMO space $\mathrm{RBMO}(\mu)$ for non-doubling measures and established a version of John–Nirenberg inequality suitable for the space $\mathrm{RBMO}(\mu)$. In [9], Hytönen also established the John–Nirenberg inequality for the space $\mathrm{RBMO}(\mu)$ on non-homogeneous metric measure spaces. On the other hand, Coifman and Rochberg [2] introduced the space $\mathrm{BLO}(\mathbb{R}^n)$ as a subspace of $\mathrm{BMO}(\mathbb{R}^n)$. We mention that the first author and his coauthors constructed a nonnegative function in $\mathrm{BMO}(\mathbb{R}^n)$ but not in $\mathrm{BLO}(\mathbb{R}^n)$ in [14]. Recently, Wang et al. [16] established the John–Nirenberg inequality for the space $\mathrm{BLO}^p(\mathbb{R}^n)$ with $0 < p \leq 1$ and proved the equivalence between the $\mathrm{BLO}^p(\mathbb{R}^n)$ spaces for $p \in (0, \infty)$. Moreover, Jiang [11] and Lin and Yang [13], respectively, introduced the space $\mathrm{RBLO}(\mu)$ for non-doubling measures and the space $\mathrm{RBLO}(\mu)$ on non-homogeneous metric measure spaces. We refer the reader to the monograph [18] for more developments on harmonic analysis for non-doubling measures.

The aim of this paper is to establish the John–Nirenberg inequality for the regularized BLO space, $\widetilde{\mathrm{RBLO}}(\mu)$, via the discrete coefficient on non-homogeneous metric measure spaces. To state our main result, we first recall some necessary notation and notions. The following notion of geometrically doubling can be found in [3, pp. 66–67] and is also known as *metrically doubling* (see [8, p. 81]).

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Definition 1.1 A metric space $(\mathcal{X}, d)$ is said to be *geometrically doubling* if there exists some $N_0 \in \mathbb{N}^+ := \{1, 2, \dots\}$ such that for any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

The following definition of upper doubling was originally introduced by Hytönen [9].

Definition 1.2 A metric measure space $(\mathcal{X}, d, \mu)$ is said to be *upper doubling* if μ is a Borel measure on $\mathcal{X}$ and there exist a *dominating function* $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant $C_{(\lambda)}$, depending on λ, such that for each $x \in \mathcal{X}$, $r \mapsto \lambda(x, r)$ is non-decreasing and for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \le \lambda(x, r) \le C_{(\lambda)} \lambda(x, r/2).$$

Remark 1.3 (i) If $\lambda(x, r) := \mu(B(x, r))$ for any $x \in \mathcal{X}$ and $r \in (0, \infty)$, then the upper doubling space $(\mathcal{X}, d, \mu)$ is just the space of homogeneous type in the sense of Coifman and Weiss [3, 4]; if $(\mathcal{X}, d, \mu) = (\mathbb{R}^n, |\cdot|, \mu)$ and, for any $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, $\lambda(x, r) := Cr^\kappa$ with C being a positive constant, then $(\mathcal{X}, d, \mu)$ is just the n-dimensional Euclidean space equipped with the non-doubling measure only satisfying the polynomial growth condition introduced by Tolsa [15].

(ii) It was proved in [10] that there exists another dominating function $\widetilde{\lambda}$ such that $\widetilde{\lambda} \le \lambda$, $C_{(\widetilde{\lambda})} \le C_{(\lambda)}$ and, for any $x, y \in \mathcal{X}$ with $d(x, y) \le r$,

$$\widetilde{\lambda}(x, r) \le C_{(\widetilde{\lambda})} \widetilde{\lambda}(y, r). \tag{1.1}$$

If a metric measure space $(\mathcal{X}, d, \mu)$ is both upper doubling and geometrically doubling, then it is simply called a *non-homogeneous metric measure space*. By Remark 1.3, we always assume that the dominating function λ satisfies (1.1). In the whole paper, for any ball $B \subseteq \mathcal{X}$, we denote its center and radius by c_B and r_B, respectively, and, moreover, for any $\rho \in (0, \infty)$, we denote the ball $B(c_B, \rho r_B)$ by ρB. When we speak of a ball B in $(\mathcal{X}, d, \mu)$, it is understood that it comes with a fixed center and radius, although these, in general, are not uniquely determined by B as a set; see [8, pp. 1–2]. In other words, for any two balls $B, S \subseteq \mathcal{X}$, if $B = S$, then $c_B = c_S$ and $r_B = r_S$. From this, we deduce that if $B \subseteq S$, then $r_B \le 2r_S$, which guarantees that the definition of the following discrete coefficient $\widetilde{K}^{(\rho)}_{B,S}$ makes sense; see [6, pp. 314–315] for some details. We mention that the discrete coefficient $\widetilde{K}^{(\rho)}_{B,S}$ was introduced by Bui and Duong [1] as an analogue of the quantity introduced by Tolsa [15] in the setting of non-doubling measures; see also [6, 7].

Definition 1.4 For any $\rho \in (1, \infty)$ and any two balls $B \subset S \subset \mathcal{X}$, let

$$\widetilde{K}^{(\rho)}_{B,S} := 1 + \sum_{k=-\lfloor \log_\rho 2 \rfloor}^{N^{(\rho)}_{B,S}} \frac{\mu(\rho^k B)}{\lambda(c_B, \rho^k r_B)}.$$

Here and hereafter, for any $a \in \mathbb{R}$, $\lfloor a \rfloor$ represents the greatest integer which is not larger than a, and $N_{B,S}^{(\rho)}$ is the smallest integer satisfying $\rho^{N_{B,S}^{(\rho)}} r_B \geq r_S$.

Remark 1.5 Hytönen [9] introduced a continuous version, $K_{B,S}$, of the coefficient $\widetilde{K}_{B,S}^{(\rho)}$ as follows: for any two balls $B \subset S \subset \mathcal{X}$, let

$$K_{B,S} := 1 + \int_{(2S)\setminus B} \frac{1}{\lambda(c_B, d(x, c_B))} d\mu(x).$$

Obviously, $K_{B,S} \leq C\widetilde{K}_{B,S}^{(\rho)}$ with C being a positive constant independent of the balls B and S. On $(\mathbb{R}^n, |\cdot|, \mu)$ with μ satisfying the polynomial growth condition, $K_{B,S}$ and $\widetilde{K}_{B,S}^{(\rho)}$ are equivalent, namely, $C_1 K_{B,S} \leq \widetilde{K}_{B,S}^{(\rho)} \leq C_2 K_{B,S}$ with C_1, C_2 being positive constants independent of the balls B and S, but $K_{B,S}$ and $\widetilde{K}_{B,S}^{(\rho)}$ are usually not equivalent on $(\mathcal{X}, d, \mu)$; see [7] for more details.

Before we recall the definition of the regularized BLO space $\widetilde{\mathrm{RBLO}}(\mu)$, we also need the following notion of an (α, β)-doubling ball introduced in [9].

Definition 1.6 Let $\alpha, \beta \in (1, \infty)$. A ball $B \subset \mathcal{X}$ is said to be (α, β)-*doubling* if $\mu(\alpha B) \leq \beta \mu(B)$.

In $(\mathcal{X}, d, \mu)$, if β is large enough, then, for any $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$, there exists some $j \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N}^+$ such that $\alpha^j B$ is (α, β)-doubling, and, for μ-a.e. $x \in \mathcal{X}$, there exist arbitrary small (α, β)-doubling balls centered at x with the radii of the form $\alpha^{-j} r$ for $j \in \mathbb{N}^+$ and any preassigned number $r \in (0, \infty)$; see [9, Lemmas 3.2 and 3.3] for more details. In what follows, let $\nu := \log_2 C_{(\lambda)}$ and $n_0 := \log_2 N_0$, where N_0 is as in Definition 1.1. Throughout the paper, for any $\alpha \in (1, \infty)$ and ball B, the smallest (α, β_α)-doubling ball of the form $\alpha^j B$ with $j \in \mathbb{Z}_+$ is denoted by $\widetilde{B}^\alpha$, where

$$\beta_\alpha := \max\{\alpha^{n_0}, \alpha^\nu\} + 30^{n_0} + 30^\nu = \alpha^{\max\{n_0, \nu\}} + 30^{n_0} + 30^\nu; \tag{1.2}$$

see [10] for the details.

The following regularized BLO space $\widetilde{\mathrm{RBLO}}(\mu)$ was introduced in [17].

Definition 1.7 Let $\eta, \rho \in (1, \infty)$, and let β_ρ be as in (1.2). A real-valued function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the space $\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)$ if there exists a nonnegative constant C such that for all balls B,

$$\frac{1}{\mu(\eta B)} \int_B \Big[f(y) - \operatorname*{essinf}_{\widetilde{B}^\rho} f\Big] d\mu(y) \leq C,$$

and that for all (ρ, β_ρ)-doubling balls $B \subset S$,

$$\operatorname*{essinf}_B f - \operatorname*{essinf}_S f \leq C\widetilde{K}_{B,S}^{(\rho)}.$$

Moreover, the $\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)$ norm of f is defined to be the minimal constant C as above and denoted by $\|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)}$.

Remark 1.8 (i) If we replace $\widetilde{K}^{(\rho)}_{B,S}$ by $K_{B,S}$ in Definition 1.7, then $\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)$ becomes the space $\mathrm{RBLO}_{\eta,\rho}(\mu)$ in [13].

(ii) In [17, Remark 2.6(i)], Yang et al. pointed out that $\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)$ is independent of the choices of the constants $\eta,\rho\in(1,\infty)$, and, moreover, there is an equivalent norm for $\|\cdot\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}}(\mu)$ as follows. Let $\eta,\rho\in(1,\infty)$. Suppose that for any given $f\in L^1_{\mathrm{loc}}(\mu)$, there exist a nonnegative constant $\widetilde{C}$ and a real number f_B for any ball B such that for all balls B,

$$\frac{1}{\mu(\eta B)}\int_B[f(y)-f_B]\,d\mu(y)\le\widetilde{C}, \tag{1.3}$$

that for all balls $B\subset S$,

$$|f_B-f_S|\le\widetilde{C}\widetilde{K}^{(\rho)}_{B,S}, \tag{1.4}$$

and that for all balls B,

$$f_B\le\operatorname*{ess\,inf}_B f. \tag{1.5}$$

Define the norm $\|f\|_{*,\eta,\rho}:=\inf\{\widetilde{C}\}$, where the infimum is taken over all the nonnegative constants $\widetilde{C}$ as above. Then the norm $\|\cdot\|_{*,\eta,\rho}$ is independent of the choice of the constant $\eta\in(1,\infty)$; namely, for any fixed $\rho\in(1,\infty)$, let $\eta_1>\eta_2>1$, then

$$\|\cdot\|_{*,\eta_1,\rho}\le\|\cdot\|_{*,\eta_2,\rho}\le C_{(\eta_1,\eta_2,\rho)}\|\cdot\|_{*,\eta_1,\rho}, \tag{1.6}$$

where $C_{(\eta_1,\eta_2,\rho)}$ is a positive constant, depending on η_1, η_2 and ρ. Moreover, the norms $\|\cdot\|_{*,\eta,\rho}$ and $\|\cdot\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)}$ are equivalent; namely,

$$C_{(\rho)}\|\cdot\|_{*,\eta,\rho}\le\|\cdot\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)}\le C_{(\eta,\rho)}\|\cdot\|_{*,\eta,\rho}, \tag{1.7}$$

where $C_{(\rho)}$ and $C_{(\eta,\rho)}$ are positive constants, depending on ρ and η,ρ, respectively.

We now give the main result of this paper.

Theorem 1.9 *Let $(\mathcal{X},d,\mu)$ be a non-homogeneous metric measure space. Then, for any $\eta\in(2,\infty)$, there exists a positive constant c such that for any $f\in\widetilde{\mathrm{RBLO}}(\mu)$, any ball $B_0=B(x_0,r)$ and any $t\in(0,\infty)$,*

$$\mu\big(\{x\in B_0:[f(x)-f_{B_0}]>t\}\big)\le 2\mu(\eta B_0)e^{-ct/\|f\|_{\widetilde{\mathrm{RBLO}}(\mu)}}, \tag{1.8}$$

where f_{B_0} is as in Remark 1.8(ii) *with B replaced by B_0.*

In Section 2, we mainly give the proof of Theorem 1.9. Hytönen [9, p. 487] pointed out that only the basic covering lemma rather than the Besicovitch covering theorem is available in the present setting; we have to borrow some ideas from the proof of [9, Proposition 6.1] to prove Theorem 1.9. Though the proof of Theorem 1.9 follows essentially the same method used in the Hytönen work on $\mathrm{RBMO}(\mu)$ in [9], we make a more detailed and effective discussion; see, for instance, the proof of (2.4). Moreover, we obtain that the range of η in Theorem 1.9 is sharp by the present method (see Remark 2.7), and it is unclear whether the John–Nirenberg inequality (1.8) for the space $\widetilde{\mathrm{RBLO}}(\mu)$ holds true for $\eta\in(1,2]$.

We now make some conventions on notation. Throughout the whole paper, we denote by C, $\widetilde{C}$, or c a positive constant, which is independent of the main parameters, but they may vary from line to line. Constants with subscripts, such as C_1 and c_1, do not change in different occurrences. Moreover, we use $C_{(\alpha)}$, $c_{(\alpha)}$, or $\widetilde{c}_{(\alpha)}$ to denote a positive constant depending on the parameter α. For any ball B and $f \in L^1_{\rm loc}(\mathcal{X})$, $m_B(f)$ stands for the mean of f over the ball B, namely, $m_B(f) := \frac{1}{\mu(B)}\int_B f(y)\,d\mu(y)$.

2 The Proof of Theorem 1.9

To prove Theorem 1.9, we first recall some useful properties of $\widetilde{K}^{(\rho)}_{B,S}$ that were proved in [6].

Lemma 2.1 *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space. Let $\rho \in (1, \infty)$ and $\widetilde{K}^{(\rho)}_{B,S}$ be as in Definition 1.4.*

(i) *There exists a positive constant $c_{(\rho)}$, depending on ρ, such that for all balls $B \subset R \subset S$, $\widetilde{K}^{(\rho)}_{B,R} \le c_{(\rho)}\widetilde{K}^{(\rho)}_{B,S}$.*
(ii) *For any $\alpha \in [1, \infty)$, there exists a positive constant $c_{(\alpha,\rho)}$, depending on α and ρ, such that for all balls $B \subset S$ with $r_S \le \alpha r_B$, $\widetilde{K}^{(\rho)}_{B,S} \le c_{(\alpha,\rho)}$.*
(iii) *For any $\alpha \in [1, \infty)$, there exists a positive constant $\widetilde{c}_{(\alpha,\rho)}$, depending on α and ρ, such that for all balls B, $\widetilde{K}^{(\rho)}_{B,\widetilde{B}^{\alpha}} \le \widetilde{c}_{(\alpha,\rho)}$.*
(iv) *There exists a positive constant $\widetilde{c}_{(\rho)}$, depending on ρ, such that for all balls $B \subset R \subset S$, $\widetilde{K}^{(\rho)}_{B,S} \le \widetilde{K}^{(\rho)}_{B,R} + \widetilde{c}_{(\rho)}\widetilde{K}^{(\rho)}_{R,S}$.*

The following basic covering lemma, which can be called the $5r$-covering lemma, is a simple corollary of [8, Theorem 1.2] and [9, Lemma 2.5].

Lemma 2.2 *Let $(\mathcal{X}, d)$ be a geometrically doubling metric space. Then every family $\mathcal{F}$ of balls of uniformly bounded diameter contains an at most countable disjointed subfamily $\mathcal{G}$ such that*

$$\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B\in\mathcal{G}} 5B.$$

The following lemma is a special case of [9, Corollary 3.6].

Lemma 2.3 *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space. Let $\rho \in [5, \infty)$. Then, for any $f \in L^1_{\rm loc}(\mu)$ and μ-a.e. $x \in \mathcal{X}$,*

$$f(x) = \lim_{\substack{B\downarrow x \\ (\rho,\beta_\rho)-doubling}} m_B(f),$$

where the limit is along the decreasing family of all (ρ, β_ρ)-doubling balls containing x, ordered by set inclusion.

Before proving Theorem 1.9, we also need the following lemmas.

Lemma 2.4 *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space. Then, for any $f \in \widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)$ and any two balls B_1, B_2 satisfying*

$$d(c_{B_1}, c_{B_2}) \le c_1 \max\{r_{B_1}, r_{B_2}\} \le c_2 \min\{r_{B_1}, r_{B_2}\} \tag{2.1}$$

with c_1 and c_2 being positive constants independent of the balls B_1 and B_2,

$$|f_{B_1} - f_{B_2}| \le C_{(c_1,c_2,\eta,\rho)} \|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)},$$

where $C_{(c_1,c_2,\eta,\rho)}$ is a positive constant, depending on c_1, c_2, η and ρ, and f_{B_i} is as in Remark 1.8(ii) *with B replaced by B_i, $i = 1, 2$.*

Proof From (2.1), we see that there exist some positive constants m and M, depending on c_1 and c_2, such that

$$B_1 \cup B_2 \subseteq mB_1 \quad \text{and} \quad 2mB_1 \subseteq MB_2.$$

This, together with (ii) and (i) of Lemma 2.1, the fact that (1.3) through (1.5) hold with $\widetilde{C} = \|f\|_{*,\eta,\rho}$ and (1.7), shows that

$$\begin{aligned}|f_{B_1} - f_{B_2}| &\le |f_{B_1} - f_{mB_1}| + |f_{mB_1} - f_{B_2}| \\ &\le \widetilde{K}^{(\rho)}_{B_1,mB_1} \|f\|_{*,\eta,\rho} + \widetilde{K}^{(\rho)}_{B_2,mB_1} \|f\|_{*,\eta,\rho} \\ &\le \frac{c_{(m,\rho)}}{C_{(\rho)}} \|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)} + \frac{c_{(\rho)}}{C_{(\rho)}} \widetilde{K}^{(\rho)}_{B_2,MB_2} \|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)} \\ &\le \frac{c_{(m,\rho)} + c_{(\rho)} c_{(M,\rho)}}{C_{(\rho)}} \|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)}.\end{aligned}$$

By choosing $C_{(c_1,c_2,\eta,\rho)} := [c_{(m,\rho)} + c_{(\rho)} c_{(M,\rho)}]/C_{(\rho)}$, we finish the proof of Lemma 2.4. ∎

Lemma 2.5 *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space. Let $\rho \in (1, \infty)$. Then, for any $f \in \widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)$ and any (ρ, β_ρ)-doubling ball B,*

$$m_B(f) - f_B \le C_{(\rho,\beta_\rho)} \|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)},$$

where $C_{(\rho,\beta_\rho)}$ is a positive constant, depending on ρ and β_ρ, and f_B is as in Remark 1.8(ii).

Proof From the property of the (ρ, β_ρ)-doubling ball, the fact that (1.3) holds with $\widetilde{C} = \|f\|_{*,\eta,\rho}$ and (1.7), it follows that

$$\begin{aligned}m_B(f) - f_B &= \frac{\mu(\rho B)}{\mu(B)} \frac{1}{\mu(\rho B)} \int_B [f(x) - f_B]\, d\mu(x) \\ &\le \beta_\rho \|f\|_* \le \frac{\beta_\rho}{C_{(\rho)}} \|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)}.\end{aligned}$$

By choosing $C_{(\rho,\beta_\rho)} := \beta_\rho / C_{(\rho)}$, we finish the proof of Lemma 2.5. ∎

Lemma 2.6 *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space. Let $\alpha, \rho \in (1, \infty)$. Then, for any $f \in \widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)$ and any ball B,*

$$f_B - f_{\widetilde{B}^\alpha} \le C_{(\alpha,\rho)} \|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)},$$

where $C_{(\alpha,\rho)}$ is a positive constant, depending on α and ρ, and f_B and $f_{\widetilde{B}^\alpha}$ are as in Remark 1.8(ii).

Proof From the fact that (1.4) holds with $\widetilde{C} = \|f\|_{*,\eta,\rho}$, Lemma 2.1(iii) and (1.7), it follows that

$$f_B - f_{\widetilde{B}^\alpha} \le \widetilde{K}^{(\rho)}_{B,\widetilde{B}^\alpha}\|f\|_{*,\eta,\rho} \le \frac{\widetilde{c}_{(\alpha,\rho)}}{C_{(\rho)}}\|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)}.$$

By choosing $C_{(\alpha,\rho)} := \widetilde{c}_{(\alpha,\rho)}/C_{(\rho)}$, we finish the proof of Lemma 2.6. ∎

Now we turn to prove Theorem 1.9.

Proof of Theorem 1.9. Let $\alpha := 5\eta$ and let $K := 2C^*\|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)}$, where

$$C^* := \beta_\alpha(C_3 + C_4 + C_5 + C_6 + C_7 + C_8 + C_9C_{10} + C_9C_{11})$$

with C_3 through C_{11} will be chosen later. Now we consider the following two cases of $t \in (0,\infty)$.

Case (I) $t \in (0, 2K)$. In this case, by choosing $c \in (0, \frac{\ln 2}{4C^*}]$, we have

$$e^{ct/\|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)}} \le e^{2K\ln 2/(4C^*\|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)})} \le 2,$$

which implies that (1.8) holds.

Case (II) $t \in [2K,\infty)$. In this case, we first let $\eta \in (4,\infty)$. For any $x \in B_0$ such that $f(x) - f_{B_0} > t$, let B_x be the biggest (α,β_α)-doubling ball with center x and radius $\alpha^{-i}r$ for some $i \in \mathbb{Z}_+$ such that

$$B_x \subseteq \sqrt{\eta}B_0 \quad \text{and} \quad f_{B_x} - f_{B_0} > K. \tag{2.2}$$

In fact, since $\eta \in (4,\infty)$, we have that for any $x \in B_0$ and any $i \in \mathbb{Z}_+$, $B(x,\alpha^{-i}r) \subset \sqrt{\eta}B_0$. On the other hand, by Lemma 2.3, we see that for any $x \in B_0$ with $f(x) - f_{B_0} > t \ge 2K$, there exist arbitrarily small (α,β_α)-doubling balls $B^* = B(x,\alpha^{-i}r)$ with $i \in \mathbb{Z}_+$ such that $m_{B^*}(f) - f_{B_0} > 2K$. It then follows from Lemma 2.5 that

$$f_{B^*} - f_{B_0} = m_{B^*}(f) - f_{B_0} - [m_{B^*}(f) - f_{B^*}] > 2K - C_3\|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)} > K,$$

where $C_3 := C_{(\rho,\beta_\rho)}$, which implies that for any $x \in B_0$, the ball B_x satisfying (2.2) exists. From the fact that $m_{B_x}(f) \ge \operatorname{essinf}_{B_x} f \ge f_{B_x}$, we further conclude that

$$\frac{1}{\mu(B_x)}\int_{B_x}[f(x) - f_{B_0}]\,d\mu(x) = m_{B_x}(f) - f_{B_x} + f_{B_x} - f_{B_0} > K. \tag{2.3}$$

Moreover, we claim that

$$K < f_{B_x} - f_{B_0} \le \frac{3}{2}K. \tag{2.4}$$

To prove (2.4), we consider the following two cases.

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Case (i) $f_{\widetilde{\alpha B_x}^\alpha} - f_{B_0} \le K$. In this case, by Lemmas 2.4 and 2.6, we have

$$\begin{aligned} K < f_{B_x} - f_{B_0} &= f_{B_x} - f_{\alpha B_x} + f_{\alpha B_x} - f_{\widetilde{\alpha B_x}^\alpha} + f_{\widetilde{\alpha B_x}^\alpha} - f_{B_0} \\ &\le |f_{B_x} - f_{\alpha B_x}| + C_{(\alpha,\rho)} \|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)} + K \\ &\le (C_4 + C_5)\|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)} + K \le \frac{3}{2}K, \end{aligned}$$

where $C_4 := C_{(1,\alpha,\eta,\rho)}$ and $C_5 := C_{(\alpha,\rho)}$;

Case (ii) $f_{\widetilde{\alpha B_x}^\alpha} - f_{B_0} > K$. In this case, notice that $\widetilde{\alpha B_x}^\alpha$ is the (α, β_α)-doubling ball of the form $\alpha^i B_x$ with $i \in \mathbb{N}^+$ and B_x is the biggest (α, β_α)-doubling ball of the form $B(x, \alpha^{-i} r)$ with $i \in \mathbb{Z}_+$. From the maximality of B_x, we deduce that $B(x,r) \subset \widetilde{\alpha B_x}^\alpha$. Let A_x be the smallest ball of the form $B(x, \alpha^i r)$, $i \in \mathbb{N}^+$, satisfying $B(x, \alpha^i r) \not\subset \sqrt{\eta} B_0$. Notice that $r_{B_0} = r$, $B(x,r) \subset \sqrt{\eta} B_0$ and $B(x, \alpha^i r) \not\subset \sqrt{\eta} B_0$ for any $i \ge \lfloor \log_\alpha(\sqrt{\eta} - 1)\rfloor + 1 =: i_0$, we have $r \le r_{A_x} \le \alpha^{i_0} r$, which implies that $B_x \subset A_x \subset \alpha^{i_0} \widetilde{\alpha B_x}^\alpha$. It then follows from Lemmas 2.4 and 2.1 that

$$|f_{A_x} - f_{B_0}| \le C_6 \|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)} \le K,$$

where $C_6 := C_{(1,\alpha^{i_0},\eta,\rho)}$ and

$$\begin{aligned} \widetilde{K}^{(\rho)}_{B_x,A_x} &\le c_{(\rho)} \widetilde{K}^{(\rho)}_{B_x,\alpha^{i_0}\widetilde{\alpha B_x}^\alpha} \le c_{(\rho)} \big[\widetilde{K}^{(\rho)}_{B_x,\widetilde{\alpha B_x}^\alpha} + \widetilde{c}_{(\rho)} \widetilde{K}^{(\rho)}_{\widetilde{\alpha B_x}^\alpha, \alpha^{i_0}\widetilde{\alpha B_x}^\alpha}\big] \\ &\le c_{(\rho)} \big[\widetilde{K}^{(\rho)}_{B_x,\alpha B_x} + \widetilde{c}_{(\rho)} \widetilde{K}^{(\rho)}_{\alpha B_x,\widetilde{\alpha B_x}^\alpha} + \widetilde{c}_{(\rho)} c_{(\alpha,\rho)}\big] \\ &\le c_{(\rho)} \big[c_{(\alpha,\rho)} + \widetilde{c}_{(\rho)} \widetilde{c}_{(\alpha,\rho)} + \widetilde{c}_{(\rho)} c_{(\alpha,\rho)}\big]. \end{aligned}$$

The above estimates, together with the fact that (1.4) holds with $\widetilde{C} = \|f\|_{*,\eta,\rho}$ and (1.7), shows that

$$\begin{aligned} K < f_{B_x} - f_{B_0} &= f_{B_x} - f_{A_x} + f_{A_x} - f_{B_0} \\ &\le \widetilde{K}^{(\rho)}_{B_x,A_x} \|f\|_{*,\eta,\rho} + |f_{A_x} - f_{B_0}| \\ &\le \frac{c_{(\rho)}\big[c_{(\alpha,\rho)} + \widetilde{c}_{(\rho)}\widetilde{c}_{(\alpha,\rho)} + \widetilde{c}_{(\rho)} c_{(\alpha,\rho)}\big]}{C_{(\rho)}} \|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)} + K \\ &= C_7 \|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)} + K \le \frac{3}{2}K, \end{aligned}$$

where $C_7 := c_{(\rho)}\big[c_{(\alpha,\rho)} + \widetilde{c}_{(\rho)}\widetilde{c}_{(\alpha,\rho)} + \widetilde{c}_{(\rho)} c_{(\alpha,\rho)}\big]/C_{(\rho)}$. This completes the proof of our claim.

By Lemma 2.2, there exists a disjointed subfamily $\{B_j := B_{x_j}\}_{j\in J}$ such that

$$\Big[\bigcup_{x\in B_0} B_x\Big] \subset \Big[\bigcup_{j\in J} 5B_j\Big]. \tag{2.5}$$

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Writing $A_j := 5B_j$. If $x \in B_0$ and $f(x) - f_{B_0} > nK$ with $n \in \mathbb{N}^+$, then $x \in A_j$ for some $j \in J$, and hence by (2.4) and Lemma 2.4,

$$\begin{aligned} f(x) - f_{A_j} &= f(x) - f_{B_0} + f_{B_0} - f_{B_j} + f_{B_j} - f_{A_j} \\ &\geq [f(x) - f_{B_0}] - [f_{B_j} - f_{B_0}] - |f_{B_j} - f_{A_j}| \\ &\geq nK - \frac{3}{2}K - C_8\|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)} > (n-2)K, \end{aligned}$$

where $C_8 := C_{(1,5,\eta,\rho)}$. This, together with (2.5), implies that for $n \in [2,\infty)$,

$$\begin{aligned} &\{x \in B_0 : [f(x) - f_{B_0}] > nK\} \\ &\quad\subseteq \bigcup_{x \in B_0 : f(x) - f_{B_0} > nK} \{y \in B_x : [f(y) - f_{B_0}] > nK\} \\ &\quad\subseteq \bigcup_{j\in J} \{y \in A_j : [f(y) - f_{A_j}] > (n-2)K\}. \end{aligned}$$

Meanwhile, by Remark 1.8(ii), (1.6), and (1.7), we see that

$$\|\cdot\|_{\widetilde{\mathrm{RBLO}}_{\sqrt{\eta},\rho}(\mu)} \leq C_9\|\cdot\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)}, \tag{2.6}$$

where $C_9 := C_{(\sqrt{\eta},\sqrt{\eta},\rho)}$ is a constant from (1.6). From the fact that the balls $B_j = B_{x_j}$ are (α,β_α)-doubling, disjoint, and contained in $\sqrt{\eta}B_0$, (2.3), (1.5), (1.6), (1.7), Lemma 2.4, and (2.6), we deduce that

$$\begin{aligned} \sum_{j\in J}\mu(\eta A_j) &= \sum_{j\in J}\mu(\alpha B_j) \leq \beta_\alpha \sum_{j\in J}\mu(B_j) \\ &\leq \frac{\beta_\alpha}{K}\sum_{j\in J}\int_{B_j}[f(x) - f_{B_0}]\,d\mu(x) \\ &\leq \frac{\beta_\alpha}{K}\sum_{j\in J}\int_{B_j}[(f(x) - f_{\sqrt{\eta}B_0}) + |f_{\sqrt{\eta}B_0} - f_{B_0}|]\,d\mu(x) \\ &\leq \frac{\beta_\alpha}{K}\int_{\sqrt{\eta}B_0}[(f(x) - f_{\sqrt{\eta}B_0}) + |f_{\sqrt{\eta}B_0} - f_{B_0}|]\,d\mu(x) \\ &\leq \frac{\beta_\alpha}{K}\Big[\mu(\sqrt{\eta}\cdot\sqrt{\eta}B_0)\|f\|_{*,\sqrt{\eta},\rho} + C_{10}\mu(\sqrt{\eta}B_0)\|f\|_{\widetilde{\mathrm{RBLO}}_{\sqrt{\eta},\rho}(\mu)}\Big] \\ &\leq \frac{\beta_\alpha}{K}\Big[C_{11}\mu(\eta B_0)\|f\|_{\widetilde{\mathrm{RBLO}}_{\sqrt{\eta},\rho}(\mu)} + C_{10}\mu(\eta B_0)\|f\|_{\widetilde{\mathrm{RBLO}}_{\sqrt{\eta},\rho}(\mu)}\Big] \\ &\leq \frac{\beta_\alpha}{K}[C_9C_{11} + C_9C_{10}]\,\mu(\eta B_0)\|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)} \leq \frac{1}{2}\mu(\eta B_0), \end{aligned} \tag{2.7}$$

where $C_{10} := C_{(1,\sqrt{\eta},\eta,\rho)}$ and $C_{11} := \frac{1}{C_{(\rho)}}$.

Now from iterating with the balls A_j in place of B_0, we obtain

$$\begin{aligned} &\{x \in B_0 : [f(x) - f_{B_0}] > 2nK\} \\ &\quad\subseteq \bigcup_{j_1}\{x \in A_{j_1} : [f(x) - f_{A_{j_1}}] > 2(n-1)K\} \\ &\quad\subseteq \bigcup_{j_1,j_2}\{x \in A_{j_1,j_2} : [f(x) - f_{A_{j_1,j_2}}] > 2(n-2)K\} \\ &\quad\subseteq \cdots \subseteq \bigcup_{j_1,j_2,\ldots,j_n}\{x \in A_{j_1,j_2,\ldots,j_n} : [f(x) - f_{A_{j_1,j_2,\ldots,j_n}}] > 0\}, \end{aligned}$$

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and then

$$
\begin{aligned}
\mu\big(\{x \in B_0 : [f(x) - f_{B_0}] > 2nK\}\big) &\le \sum_{j_1,\dots,j_{n-1},j_n} \mu(A_{j_1,\dots,j_{n-1},j_n}) \\
&\le \sum_{j_1,\dots,j_{n-1}} \sum_{j_n} \mu(\eta A_{j_1,\dots,j_{n-1},j_n}) \\
&\le \sum_{j_1,\dots,j_{n-1}} \frac{1}{2}\mu(\eta A_{j_1,\dots,j_{n-1}}) \\
&\le \cdots \le \frac{1}{2^n}\mu(\eta B_0).
\end{aligned}
$$

Choosing $n \in \mathbb{N}^+$ such that $2nK \le t < 2(n+1)K$. It then follows from the above estimates that

$$
\begin{aligned}
\mu\big(\{x \in B_0 : [f(x) - f_{B_0}] > t\}\big) &\le \mu\big(\{x \in B_0 : [f(x) - f_{B_0}] > 2nK\}\big) \\
&\le 2^{-n}\mu(\eta B_0) \\
&\le 2^{-(2K)^{-1}t+1}\mu(\eta B_0) \\
&= 2e^{-ct/\|f\|_{\widetilde{\mathrm{RBLO}}_{\eta,\rho}(\mu)}}\mu(\eta B_0).
\end{aligned}
$$

Since $-n \le 1 - \frac{t}{2K}$, by choosing $c \in (0, \frac{\ln 2}{4C^*}]$, we see that (1.8) holds.

For $\eta \in (2,\infty)$, let $\gamma \in (0,1)$ such that $\eta^\gamma > 2$. In this case, we can find a ball B_z be the biggest (α, β_α)-doubling ball with center z and radius $\alpha^{-i}r$ for some $i \in \mathbb{Z}_+$, such that $B_z \subseteq \eta^\gamma B_0$ and $f_{B_z} - f_{B_0} > K$, and the rest of the proof is completely analogous to the above. Hence, we finish the proof of Theorem 1.9. ∎

Remark 2.7 (i) In the proof of (2.7), we need that $\eta^\gamma < \eta$, which implies that $\gamma \in (0,1)$.

(ii) The range of η in Theorem 1.9, namely, $\eta \in (2,\infty)$, is sharp by the present method. In fact, in the proof of Theorem 1.9, we need to find the biggest (α, β_α)-doubling ball B_x such that $B_x \subseteq \eta^\gamma B_0$, which implies that $r_{B_x} + r_{B_0} \le 2r_{B_0} \le \eta^\gamma r_{B_0}$ with $\gamma \in (0,1)$. By letting $\gamma \to 1$, we have that $\eta \ge 2$. Thus, $\eta \in (2,\infty)$. However, it is unclear that whether the John–Nirenberg inequality (1.7) for the space $\widetilde{\mathrm{RBLO}}(\mu)$ holds true for $\eta \in (1,2]$.

Corollary 2.8 *Let $(\mathcal{X}, d, \mu)$ be a metric measure space of non-homogeneous type. Then, for every $\eta \in (2,\infty)$ and $p \in [1,\infty)$, there exists a positive constant C such that for any $f \in \widetilde{\mathrm{RBLO}}(\mu)$ and all balls B,*

$$
\Big[\frac{1}{\mu(\eta B)}\int_B [f(x) - f_B]^p \, d\mu(x)\Big]^{1/p} \le C\|f\|_{\widetilde{\mathrm{RBLO}}(\mu)},
$$

where f_B is as in Remark 1.8(ii).

Proof From the situation of the corollary and Theorem 1.9, we conclude that

$$\begin{aligned}
&\frac{1}{\mu(\eta B)}\int_B [f(x)-f_B]^p\,d\mu(x)\\
&\quad=\frac{p}{\mu(\eta B)}\int_0^\infty t^{p-1}\mu(\{x\in B:[f(x)-f_B]>t\})dt\\
&\quad\le 2p\int_0^\infty t^{p-1}e^{-ct/\|f\|_{\widetilde{\mathrm{RBLO}}(\mu)}}dt\\
&\quad=\frac{2p\Gamma(p)}{c^p}\|f\|^p_{\widetilde{\mathrm{RBLO}}(\mu)},
\end{aligned}$$

which completes the proof of Corollary 2.8. ∎

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