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ON FREE GROUPS OF THE VARIETY $AN_2 \wedge N_2A$

BY

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Introduction. Let R be a commutative ring with unity and let $M(R)$ be the multiplicative group of 4×4 triangular matrices (a_{ij}) over R, where a_{11} is a unit element of R and $a_{ii}=1$ for $i=2, 3, 4$. If $\mathbf{V}(=\mathbf{AN}_2 \wedge \mathbf{N}_2\mathbf{A})$ denotes the variety of groups which are both abelian-by-class-2 and class-2-by-abelian, then it is routine to verify that $M(R) \in \mathbf{V}$. Here we prove the following,

THEOREM. *Let $F(\mathbf{V})$ denote the free group of finite or countable infinite rank of the variety $\mathbf{V}$. Then for a suitable choice of R, $F(\mathbf{V})$ is embedded in $M(R)$.*

Notation and preliminaries. Unless otherwise specified, we follow the notation of Hanna Neumann [2]. In particular, if $x, y, z, \ldots$ are elements of a group G, then we write $[x, y]=x^{-1}y^{-1}xy$; $[x, y, z]=[[x, y], z]$; $[x, y; u, v]=[[x, y], [u, v]]$. We write $\gamma_m(G)$ for the m-th term of the lower central series of G and $\gamma_m\gamma_n(G)$ for $\gamma_m(\gamma_n(G))$. Let F_m be the free group of rank m freely generated by $x_1, \ldots, x_m$; and let $H=\gamma_2\gamma_3(F_m)\cdot\gamma_3\gamma_2(F_m)$. If $w \in \gamma_2\gamma_2(F_m)$, then w can be written as

$$w = \prod_{1\le i<j\le m} [u(i,j), [x_i, x_j]] \bmod H,$$

where $u(i,j) \in \gamma_2(F_m)$ and contains no factor $[x_i, x_j]^{\pm 1}$.

Let $P=\{(i,j) \mid 1\le i<j\le m\}$ and define $(i,j)<(k,l)$ if either $i<k$ or if $i=k$ and $j<l$. Using P as an index set we can rewrite $w \in \gamma_2\gamma_2(F_m)$ as

$$w = \prod_{(i,j)=(1,2)}^{(m-1,m)} [u(i,j), [x_i, x_j]] \bmod H, \tag{1}$$

where

$$u(i,j) = \prod_{(r,s)>(i,j)} [x_r, x_s]^{\delta(r,s)}v(i,j)$$

with $\delta(r, s) \in Z$ and $v(i,j) \in \gamma_3(F_m)$.

Proof of the theorem. Let ZG denote the integral group ring of the free abelian group G freely generated by $\mathbf{x}_1, \mathbf{x}_2, \ldots$; and denote R to be the polynomial ring $ZG[\Lambda]$, where $\Lambda=\{\lambda_{i,i-1}^{(k)} \mid i=2, 3, 4;\ k=1, 2, \ldots\}$ is the set of independent and

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commuting indeterminates which also commute with every element of ZG. For each $k=1, 2, \ldots$; let

$$(2)\qquad \langle x_k\rangle = \begin{bmatrix} \mathbf{x}_k & 0 & 0 & 0 \\ \lambda_{21}^{(k)} & 1 & 0 & 0 \\ 0 & \lambda_{32}^{(k)} & 1 & 0 \\ 0 & 0 & \lambda_{43}^{(k)} & 1 \end{bmatrix}.$$

Consider the multiplicative subgroup $M^*(R)$ of $M(R)$ generated by $\langle x_k\rangle$'s for $k=1, 2, \ldots$. In what follows we shall show that $M^*(R)$ is isomorphic to $F/\gamma_2\gamma_3(F)\cdot\gamma_3\gamma_2(F)$. For this purpose we take $x_1, x_2, \ldots$ as a free set of generators for the free group F and define the natural homomorphism φ of F onto $M^*(R)$ by $x_k\varphi=\langle x_k\rangle$. We proceed to show that the kernel of φ is $\gamma_2\gamma_3(F)\cdot\gamma_3\gamma_2(F)$. If $w=x_{i_1}^{\varepsilon_1}\ldots x_{i_l}^{\varepsilon_l}(\epsilon_i\in\{1, -1\})$ is a word in F, then we define

$$(3)\qquad \langle w\rangle = \langle x_{i_1}\rangle^{\varepsilon_1}\ldots\langle x_{i_l}\rangle^{\varepsilon_l}.$$

To facilitate calculations in $M^*(R)$, we introduce mappings α_{ij} $(4\geq i\geq j\geq 1)$ of F into R be defining

$$(4)\qquad \alpha_{ij}(w) = ij\text{-entry of } \langle w\rangle.$$

Thus we have,

$$(5)\qquad \begin{aligned} &\alpha_{11}(w) = \mathbf{w},\ \alpha_{ii}(w) = 1 \quad \text{for } i = 2, 3, 4,\ \alpha_{i,i-1}(x_k) = \lambda_{i,i-1}^{(k)}, \\ &\alpha_{ij}(x_k) = 0 \quad \text{for } i-j\notin\{0, 1\}; \end{aligned}$$

and using matrix multiplication we compute

$$(6)\qquad \begin{aligned} \alpha_{21}[w_1, w_2] &= (-1+\mathbf{w}_2)\alpha_{21}(w_1)-(-1+\mathbf{w}_1)\alpha_{21}(w_2), \\ \alpha_{32}[w_1, w_2] &= 0 = \alpha_{43}[w_1, w_2], \\ \alpha_{31}[w_1, w_2] &= (-1+\mathbf{w}_2)\alpha_{31}(w_1)+(1-\mathbf{w}_1)\alpha_{31}(w_2)+(1-\mathbf{w}_2)\alpha_{32}(w_1)\alpha_{21}(w_1), \\ &\quad +(\mathbf{w}_1-1)\alpha_{32}(w_2)\alpha_{21}(w_2)+\mathbf{w}_1\alpha_{32}(w_1)\alpha_{21}(w_2) \\ &\quad -\mathbf{w}_2\alpha_{32}(w_2)\alpha_{21}(w_1), \\ \alpha_{42}[w_1, w_2] &= \alpha_{43}(w_1)\alpha_{32}(w_2)-\alpha_{43}(w_2)\alpha_{32}(w_1), \\ \alpha_{41}[u, v] &= \alpha_{42}(u)\alpha_{21}(v)-\alpha_{42}(v)\alpha_{21}(u) \quad \text{for } u, v\in\gamma_2(F). \end{aligned}$$

From (6), it follows in particular that $\alpha_{ij}(w)=0$ for all $w\in\gamma_2\gamma_3(F)\cdot\gamma_3\gamma_2(F)$ and as remarked in the introduction $\gamma_2\gamma_3(F)\cdot\gamma_3\gamma_2(F)$ is contained in the kernel of φ. Moreover, using (6) we note that

$$\alpha_{41}(w_1\ldots w_k) = \sum_{i=1}^{k}\alpha_{41}(w_i) \quad \text{for } w_1, \ldots, w_k\in\gamma_4(F);$$

$$(7)\qquad \alpha_{42}(w_1^{\varepsilon_1}\ldots w_k^{\varepsilon_k}) = \sum_{i=1}^{k}\epsilon_i\alpha_{42}(w_i) \quad \text{for } w_1, \ldots, w_k\in\gamma_2(F);$$

and

$$\alpha_{42}[x_r, x_s] = \lambda_{43}^{(r)}\lambda_{32}^{(s)}-\lambda_{43}^{(s)}\lambda_{32}^{(r)}.$$

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To complete the proof of the theorem we assume that w is a word in the kernel of φ (i.e. $\alpha_{ij}(w)=0$ for $4\geq i>j\geq 1$) and proceed to conclude that $w\in\gamma_2\gamma_3(F)\cdot\gamma_3\gamma_2(F)$. Since w involves only finitely many symbols, we may assume that $w\in F_m$, where F_m is freely generated by $x_1,\ldots,x_m$. Now $\alpha_{21}(w)=0$, together with the fact that the matrix $\begin{bmatrix}\mathbf{x}_k & 0\\ \lambda_{21}^{(k)} & 1\end{bmatrix}$ forms a part of the matrix $\langle x_k\rangle$ for $k=1,2,\ldots$; it follows by a well known theorem of Wilhelm Magnus [1] that $w\in\gamma_2\gamma_2(F_m)$ and by (1), we may assume that w can be written as

$$w=\prod_{(i,j)=(1,2)}^{(m-1,m)}[u(i,j),[x_i,x_j]]\bar{w},$$

where

$$u(i,j)=\prod_{(r,s)>(i,j)}[x_r,x_s]^{\delta(r,s)}v(i,j)$$

with $\delta(r,s)\in Z$, $v(i,j)\in\gamma_3(F_m)$ and $\bar{w}\in\gamma_2\gamma_3(F_m)\cdot\gamma_3\gamma_2(F_m)$.

Now, we have

$$0=\alpha_{41}(w)=\sum_{(i,j)=(1,2)}^{(m-1,m)}\alpha_{41}[u(i,j),[x_i,x_j]] \qquad \text{by (7)}$$

$$=\sum_{(i,j)=(1,2)}^{(m-1,m)}\{\alpha_{42}(u(i,j))\alpha_{21}[x_i,x_j]-\alpha_{42}[x_i,x_j]\alpha_{21}(u(i,j))\} \qquad \text{by (6)}$$

$$=\sum_{(i,j)=(1,2)}^{(m-1,m)}\left\{\left(\sum_{(r,s)>(i,j)}\delta(r,s)\alpha_{42}[x_r,x_s]\right)\alpha_{21}[x_i,x_j]-\alpha_{42}[x_i,x_j]\alpha_{21}(u(i,j))\right\} \qquad \text{by (7)}$$

(since $\alpha_{42}(v(i,j))=0$ by (6))

$$=\sum_{(i,j)=(1,2)}^{(m-1,m)}\left\{\left(\sum_{(r,s)>(i,j)}\delta(r,s)(\lambda_{43}^{(r)}\lambda_{32}^{(s)}-\lambda_{43}^{(s)}\lambda_{32}^{(r)})\right)\alpha_{21}[x_i,x_j]\right.$$

$$\left.-(\lambda_{43}^{(i)}\lambda_{32}^{(j)}-\lambda_{43}^{(j)}\lambda_{32}^{(i)})\alpha_{21}(u(i,j))\right\} \qquad \text{by (7)}$$

$$=\sum_{(i,j)=(1,2)}^{(m-1,m)}\lambda_{43}^{(i)}\lambda_{32}^{(j)}\mu(i,j)-\lambda_{43}^{(j)}\lambda_{32}^{(i)}v(i,j),$$

where

$$\mu(i,j),v(i,j)\in ZG[\lambda_{21}^{(k)},k=1,2,\ldots].$$

Since $\lambda_{ij}^{(k)}$'s are independent, it follows that $\mu(i,j)=0=v(i,j)$ for all $(i,j)\in P$. Let (i,j) be the least element of P for which $u(i,j)\notin\gamma_2\gamma_2(F_m)$. Then since $\mu(i,j)=-\alpha_{21}(u(i,j))$, we have $\alpha_{21}(u(i,j))=0$ which again by the theorem of Magnus implies that $u(i,j)\in\gamma_2\gamma_2(F_m)$, contrary to the choice of $u(i,j)$. Thus, each $u(i,j)$ in the representation of w lies in $\gamma_2\gamma_2(F_m)$ and it follows that $w\in\gamma_2\gamma_3(F)\cdot\gamma_3\gamma_2(F)$ as was required.

Remark. If $w=x_{i_1}^{\varepsilon_1}\ldots x_{i_l}^{\varepsilon_l}$ is an arbitrary word in F, then we can effectively compute $\alpha_{21}(w)$. Since R has a solvable word problem we can decide whether or not

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$\alpha_{21}(w)$ determines 0. If $\alpha_{21}(w)=0$, we effectively compute $\alpha_{41}(w)$ and decide whether or not $\alpha_{41}(w)=0$. If $\alpha_{21}(w)\neq 0$, then w is not in F'', and hence not in $\gamma_2\gamma_3(F)\cdot\gamma_3\gamma_2(F)$. If $\alpha_{21}(w)=0$ and if $\alpha_{41}(w)\neq 0$, then w is not in $\gamma_2\gamma_3(F)\cdot\gamma_3\gamma_2(F)$. Thus, as a consequence to the proof of the theorem, we have

COROLLARY. *$F/\gamma_2\gamma_3(F)\cdot\gamma_3\gamma_2(F)$ has a solvable word problem.*

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